ZERO SETS FOR CONTINUOUS FUNCTIONS[1]

R. C. BUCK

1. **Introduction.** Let D be a closed set in m-space, and let V be a linear space of real valued continuous functions defined on D. For any $f \in V$, let $Z(f)$ be its zero-set, the set of $p \in D$ with $f(p)=0$. If $m=1$, and V is a class of well behaved functions, then $Z(f)$ will be discrete except for f identically zero. This behavior depends strongly upon m. When $m \geqq 2$, then—except in special cases—one expects $Z(f)$ to be a nowhere dense set which is locally of dimension $m-1$.

In this note, we shall be concerned with certain distribution-properties of the zero-sets for functions of m variables which do not seem to have been explicitly stated before; as will be seen, they do not depend upon the nature of the functions in V but only upon the dimension of V. If dim $(V)=N$, then there exists a system of N nonempty disjoint relatively open sets $D_1, D_2, \cdots, D_N$ in D such that if f and g are in V, and if they agree at some point in each of the sets D_j, then $f=g$. Otherwise stated, if $f \in V$ and $Z(f)$ intersects each of the sets D_j, then $f \equiv 0$.

If D is the plane, and V the class of functions $f(x, y)=ax+by+c$, then the assertion is merely that there are three open sets D_j which cannot all be cut by a single straight line. Again, choose V as the space of polynomials in x, y, z of degree at most 2. If $f \in V$ but $f \not\equiv 0$ then $Z(f)$ is a quadric surface in 3-space. Any set of nine points in 3-space lies in some set $Z(f)$; most sets of ten points do not. If we choose a set of points $p_1, \cdots, p_{10}$ through which no quadric passes, and then surround each by a sufficiently small open sphere, we obtain ten open sets $D_1, \cdots, D_{10}$ which form a system of uniqueness domains for the space V.

2. **Uniqueness domains.** Let V be any subspace of $C[D]$ of dimension N, and let $\phi_1, \phi_2, \cdots, \phi_N$ be a basis for V.

THEOREM 1. *There exist disjoint nonempty sets* $D_1, D_2, \cdots, D_N$, *relatively open in* D, *and a constant* B *such that for any* $f \in V$, *and* $p \in D$, *and any choice of* $p_i \in \overline{D}_i$

$$(1) \qquad |f(p)| \leqq B\{|f(p_1)| + |f(p_2)| + \cdots + |f(p_N)|\}.$$

PROOF. Let S be the set of all points in N-space of the form $x=\langle \phi_1(p), \phi_2(p), \cdots, \phi_N(p)\rangle$ for $p \in D$. If S were to lie in a proper

Received by the editors August 10, 1959 and, in revised form, September 23, 1959.
[1] This work was carried out under OOR Contract DA-11-022-ORD-2059.

subspace of N-space, there would exist constants $b_1, b_2, \cdots, b_N$ such that $\sum b_j x_j = 0$ for all $x \in S$. This would imply that $\sum b_j \phi_j = 0$ so that the functions ϕ_j would not have been independent. We can therefore assume that S generates all of N-space. There then exist points $p_1, p_2, \cdots, p_N$ in D such that the N points $\langle \phi_1(p_i), \phi_2(p_i), \cdots, \phi_N(p_i) \rangle$, for $i = 1, 2, \cdots, N$, are linearly independent. Accordingly, the N by N matrix

$$\Delta = \begin{bmatrix} \phi_1(p_1) & \phi_2(p_1) & \cdots & \phi_N(p_1) \\ \phi_1(p_2) & \phi_2(p_2) & \cdots & \phi_N(p_2) \\ \vdots & \vdots & & \vdots \\ \phi_1(p_N) & \phi_2(p_N) & \cdots & \phi_N(p_N) \end{bmatrix}$$

is nonsingular.

For any point $p \in D$, let L_p be the functional defined on $C[D]$ by $L_p(f) = f(p)$. The functionals L_{p_i}, for $i = 1, 2, \cdots, N$, are linearly independent; for, if $\sum c_i L_{p_i} = 0$, then for each $j = 1, 2, \cdots, N$, we would have

$$0 = \sum c_i L_{p_i}(\phi_j) = \sum c_i \phi_j(p_i)$$

and by the nonsingularity of Δ, $c_1 = c_2 = \cdots = 0$. Since $\dim(V) = N$, its dual space V' is also of dimension N. Accordingly, $L_{p_1}, L_{p_2}, \cdots, L_{p_N}$ constitutes a basis for V'. In particular then, if $f \in V$ and $L_{p_i}(f) = f(p_i) = 0$ for $i = 1, 2, \cdots, N$, then $L(f) = 0$ for every $L \in V'$, and $f \equiv 0$. This argument rests upon the nonsingularity of Δ; since the functions $\phi_1, \phi_2, \cdots, \phi_N$ are continuous on D, $d(p_1, p_2, \cdots, p_N) = \det(\Delta)$ is a continuous function on $D^N = D \times D \times \cdots \times D$. In particular, we can choose nonempty open sets $D_1, D_2, \cdots, D_N$ in D so that $d(p_1, p_2, \cdots, p_N) \neq 0$ for $p_i \in D_i$. As before, the functionals $L_{p_1}, L_{p_2}, \cdots, L_{p_N}$ will form a basis for V' for any choices of the points p_i, with $p_1 \in D_1, p_2 \in D_2, \cdots, p_N \in D_N$. Hence, if $f \in V$, and $Z(f)$ touches each of the sets D_i, $i = 1, 2, \cdots, N$, then $f \equiv 0$. This is part of the statement (1). To obtain the more complete formulation, we observe that the sets D_i can be taken so that $\inf |d(p_1, \cdots, p_N)| = \delta > 0$, for all $p_i \in \overline{D}_i$, $i = 1, 2, \cdots, N$. Let $P = (p_1, p_2, \cdots, p_N)$, and let $C_j(r, P)$ be the unique numbers defined by

$$\sum_{j=1}^{N} C_j(r, P)\phi_j(p_k) = \begin{cases} 1 & \text{if } k = r, \\ 0 & \text{if } k \neq r \end{cases}$$

for $r = 1, 2, \cdots, N$, and any P in $D_1 \times D_2 \times \cdots \times D_N$. For each j and r, $C_j(r, P)$ is continuous in P, and since $|d(P)| \geq \delta$, there exists a constant B_0, determined by the functions $\{\phi_j\}$ and the sets D_i, such that

$$|C_j(r, P)| \leqq B_0 \qquad \text{all } j, r, p_i \in D_i.$$

Setting $\psi_r = \sum_{j=1}^N C_j(r, P)\phi_j$, we obtain a new basis for V such that for any $f \in V$,

$$f = \sum_{r=1}^{N} f(p_r)\psi_r. \tag{2}$$

Moreover, for any r,

$$\|\psi_r\| \leqq \sum_1^N |C_j(r, P)| \, \|\phi_j\| \leqq B_0 \sum_1^N \|\phi_j\| \leqq B$$

where we have used $\|g\|$ to denote $\sup_{q \in D} |g(q)|$, for any $g \in C[D]$. Note that B again is determined only by the $\{\phi_j\}$ and the D_i. Returning to (2), we see that

$$\|f\| \leqq B \sum_1^N |f(p_i)|$$

holding for any choice of $p_i \in D_i$, for $i = 1, 2, \cdots, N$.

The conclusion of Theorem 1 can be restated in an interesting way. Introduce a special functional M on $C[D]$:

$$M(f) = \sum_1^N \min_{p_i \in \bar{D}_i} |f(p_i)|.$$

If we were to replace "min" by "max," this would define a seminorm on $C[D]$. On the finite dimensional subspace V, it would in fact be a norm, and would therefore have to define there the unique locally convex linear topology. As defined above, however, M is not a seminorm, for it is not true in general that $M(f+g) \leqq M(f) + M(g)$. However, Theorem 1 implies that M still defines a Hausdorff topology on V; if $M(f) = 0$, then f must take the value zero somewhere in each of the sets $\bar{D}_i$, and $f \equiv 0$. The essence of the more general formulation (1) is that this topology again coincides with the unique linear topology in V, and there must exist a constant B such that $\|f\| \leqq BM(f)$, for all $f \in V$.

Another familiar instance of this general principle is the fact that in a finite dimensional function space, uniform convergence and L^1 (or L^p) convergence coincide. This is usually formulated as an inequality: there is a constant A such that

$$|f(p)| \leqq A \int_D |f|$$

for all $p \in D$, and $f \in V$. This follows at once from (1). Let δ be the smallest of the measures $\mu(D_i)$. Then,

$$\int_D |f| \geqq \sum_1^N \int_{D_i} |f| \geqq \sum_1^N \mu(D_i) \min_{p_i \in \bar{D}_i} |f(p_i)| \geqq \delta M(f) \geqq \frac{\delta}{B} \|f\|.$$

3. **Generalizations.** It is natural to ask if there are any other necessary restrictions upon the sets $\{D_i\}$ if they are to be uniqueness domains. This is answered by the next statement.

THEOREM 2. *If $D_1, D_2, \cdots, D_N$ is any collection of open disjoint nonempty subsets of D, then there is a subspace V of dimension N in $C[D]$ such that if $f \in V$, and if $Z(f)$ touches each of the sets D_i, then $f \equiv 0$.*

Choose functions ϕ_i so that $\phi_i(p)$ is the distance from p to the closed complement of D_i. Then, ϕ_i vanishes off D_i, but is strictly positive on D_i. Let V be the set of functions $f = \sum c_i \phi_i$. If f takes the value 0 at some point of D_i, then it follows that $c_i = 0$; accordingly, if $Z(f)$ contains at least one point of each D_i, then $f \equiv 0$.

From this, by a method suggested by the referee, other examples can be constructed. Let θ be any continuous mapping of D onto another set D', and let V' be an N dimensional subspace of $C[D']$. Suppose that $\{D_1', \cdots, D_N'\}$ is a collection of uniqueness domains in D', and let $D_i = \theta^{-1}(D_i')$. For any $\phi' \in V'$, define a function ϕ on D by $\phi(p) = \phi'(\theta p)$. Then, this yields a linear space $V \subset C[D]$ having $\{D_1, \cdots, D_N\}$ for uniqueness domains. In particular, the sets D_i can be selected so that their union exhausts all but a nowhere dense subset of D, and this can be achieved with many different choices of V.

This is no longer possible in general if we wish to retain the more precise inequality (1) given in Theorem 1. Suppose that we have a space V of dimension N, and a collection of sets $\{D_i\}_1^N$ such that for any $f \in V$,

$$\|f\| \leqq B \sum_1^N \inf_{p_i \in D_i} |f(p_i)|.$$

Suppose that there is a point q which is a common boundary point for all the sets D_i. Then $N = 1$. For, if $f \in V$ and $f(q) = 0$, then by the inequality and the continuity of f, $f \equiv 0$. Suppose ϕ_1 and ϕ_2 are two independent functions in V; set $f = \phi_2(q)\phi_1 - \phi_1(q)\phi_2$ and have $f(q) = 0$.

UNIVERSITY OF WISCONSIN AND
INSTITUTE FOR DEFENSE ANALYSES