

WEAK PARTITION RELATIONS

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ABSTRACT. The partition relation $\aleph_1 \rightarrow (\aleph_1)_{3,2}^2$, which was known to contradict the continuum hypothesis [1], is disproved without this hypothesis.

For cardinals $\kappa \geq \omega$ and $\lambda \geq 1$, let $P(\kappa, \lambda)$ be the following partition relation: For any mapping F of the set $[\kappa]^2$ of unordered pairs of elements of κ into λ , there is a set $B \subseteq \kappa$, of cardinality $|B| = \kappa$, such that the image of $[B]^2$ under F is not all of λ . Such a set B will be called *slightly homogeneous* for the partition F . Partition relations of this sort were studied by Erdős, Hájnal, and Rado [1, especially §18], who assumed, in most of their theorems, that the continuum hypothesis holds. We shall show that $P(\aleph_1, 3)$ can be disproved outright (in Zermelo-Frankel set theory ZF with the axiom of choice).

Notice that, for $\lambda \leq \mu$, $P(\kappa, \lambda)$ implies $P(\kappa, \mu)$. For, given a map $F: [\kappa]^2 \rightarrow \mu$, we let $G: [\kappa]^2 \rightarrow \lambda$ be the composite of F and a surjection $\mu \rightarrow \lambda$ and observe that any set which is slightly homogeneous for G is also slightly homogeneous for F . Thus, for a fixed κ , the partition relation $P(\kappa, \lambda)$ becomes weaker as λ increases. Using this fact, we easily obtain the following lemma by induction on m .

LEMMA 1. *If $n \leq m < \omega$ and if $P(\kappa, n)$, then, for any map $F: [\kappa]^2 \rightarrow m$, there is a $B \subseteq \kappa$ with $|B| = \kappa$ and $|F([B]^2)| < n$.*

When $\lambda > \kappa$, then $P(\kappa, \lambda)$ is trivially true, because κ itself is slightly homogeneous for any F . At the other extreme, $P(\kappa, 1)$ is trivially false.

For finite λ , $P(\kappa, \lambda)$ coincides with the partition relation $\kappa \rightarrow (\kappa)_{\lambda, \lambda-1}^2$ of Kleinberg [3]. In particular, $P(\kappa, 2)$ holds if and only if κ is (strongly) inaccessible and weakly compact [5, Theorems 8.3 and 9.4], so it is relatively consistent with ZF to assume that $P(\kappa, 2)$ holds only for $\kappa = \omega$. ($P(\omega, 2)$ is a form of Ramsey's theorem [6].) On the other hand, one can prove the existence of uncountable κ such that $P(\kappa, 3)$. Indeed, for singular κ , $P(\kappa, 3)$ holds if and only if κ is a strong limit cardinal and $P(\text{cf}(\kappa), 2)$, so, for example, $P(\beth_\omega, 3)$ holds.

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If we assume the generalized continuum hypothesis, then the question whether $P(\kappa, \lambda)$ holds or not is completely answered for successor cardinals by the following result of Erdős, Hájnal, and Rádo [1, Theorem 17].

THEOREM 0. *If $2^\kappa = \kappa^+$, then not $P(\kappa^+, \kappa^+)$.*

COROLLARY. *If $2^\kappa = \kappa^+$, then $P(\kappa^+, \lambda)$ if and only if $\lambda > \kappa^+$.*

We shall prove the following theorem, which becomes a special case of the above corollary if one assumes the continuum hypothesis.

THEOREM 1. *Not $P(\aleph_1, 3)$.*

This theorem has subsequently been improved by Galvin and Shelah [2] who have shown that $P(\aleph_1, 4)$ is also false. Their proof is quite similar to my proof of Theorem 1; in fact both proofs use the same partition. As far as I know, the consistency of $P(\aleph_1, 5)$ is still an open question.

The first (and longest) part of the proof of Theorem 1 will be the construction of three linearly ordered sets, each of cardinality \aleph_1 , such that no uncountable subset of any of the three is isomorphic or anti-isomorphic to a subset of any other. One of the three sets will be \aleph_1 with its standard ordering (as an ordinal). Before describing the others, we introduce some terminology.

DEFINITION. A subset X of a linearly ordered set Y is *pseudo-dense* in Y if it meets every half-open interval $[a, b)$ of Y .

LEMMA 2. *There is a linearly ordered set A of cardinality \aleph_1 such that*

(a) *There are no monotone (increasing or decreasing) sequences of length \aleph_1 in A .*

(b) *For every subset Y of A there is a countable subset X of Y which is pseudo-dense in Y .*

(c) *A is isomorphic to its dual A^* .*

PROOF. Notice that any subset Y of the real line has the property required in (b). The required X is obtained by choosing one element from each nonempty intersection $Y \cap (p, q)$ of Y with a rational interval, subject to the restriction that, if such an intersection has a largest element, then that largest element is to be the chosen one. If $[a, b)$ is an interval of Y and there is a $c \in Y$ strictly between a and b , then, for rationals p, q such that $a < p < c < q < b$, the chosen element of $Y \cap (p, q)$ is in $[a, b) \cap X$. On the other hand, if no such c exists, then, for rationals p, q such that $p < a < q < b$, the intersection $Y \cap (p, q)$ has a largest element, namely a , so $a \in [a, b) \cap X$. Thus, X is pseudo-dense in Y .

It follows that any symmetric subset A of the reals satisfies (b) and (c). Property (a) follows from (b) and (c) since \aleph_1 has no proper pseudo-dense subset. As we can obviously choose such an A of cardinality \aleph_1 , the lemma is proved. \square

The A provided by Lemma 2 will be the second of our three sets. To describe the third, we shall need some facts about trees.

A tree is a partially ordered set $(T, <)$ such that, for each $x \in T$, the set $\{y | y < x\}$ of strict predecessors of x is well-ordered. The height $h(x)$ of x is the order type of $\{y | y < x\}$, and the height $h(T)$ of T is the least ordinal not of the form $h(x)$ for any $x \in T$. If $\alpha \leq h(x)$, then $p_\alpha(x)$ is the (unique) predecessor of x of height α , the α th element of $\{y | y \leq x\}$. The α th level of T is $\{x | h(x) = \alpha\}$. A path through T is a linearly ordered subset of T which meets the α th level for all $\alpha < h(T)$.

Suppose $(T, <)$ is a tree and $<'$ is a linear ordering of T (which may be totally unrelated to the tree ordering $<$). If x and y are incomparable elements of T (with respect to $<$) and α is the smaller of their heights, then clearly $p_\alpha(x) \neq p_\alpha(y)$. Let β be the least ordinal such that $p_\beta(x) \neq p_\beta(y)$. We shall say that x is *left* of y (and y is *right* of x) if $p_\beta(x) < p_\beta(y)$. We define a relation $<$ on T by

$$x < y \quad \text{iff} \quad x < y \text{ or } x \text{ is left of } y.$$

The straightforward proof of the following lemma will be left to the reader.

LEMMA 3. *With the notations introduced above, $<$ linearly orders T .* \square

We shall need the following result of Aronszajn; see [4] or [7] for a proof.

LEMMA 4. *There is a tree of height \aleph_1 such that all its levels are countable and there is no path through it.* \square

Finally, we are ready to produce the third of our linearly ordered sets.

LEMMA 5. *There is a linearly ordered set B of cardinality \aleph_1 such that*
 (a) *There are no monotone sequences of length \aleph_1 in B .*
 (b) *No uncountable subset of B has a countable pseudo-dense subset.*

PROOF. Let $(T, <)$ be a tree with height \aleph_1 , countable levels, and no paths, as in Lemma 4. Note that then $|T| = \aleph_1$. Let $<'$ be any linear ordering of T , and let $<$ be as in the discussion preceding Lemma 3. We shall show that $(T, <)$ has both the properties required of B .

(a) Suppose $\{x_\alpha | \alpha < \aleph_1\}$ were a monotone sequence of length \aleph_1 in $(T, <)$. As each level of T is countable, we see that, for each $\gamma < \aleph_1$, the set $\bigcup_{h(y)=\gamma} \{z | z \geq y\}$ contains uncountably many x_α 's. As there are only

countably many terms in this union, there must be a y , of height γ , such that uncountably many x_α are $\geq y$.

I claim that, for each γ , there is only one such y . Suppose y' were another, and suppose, without loss of generality, that y' is left of y . Then we can successively choose $\xi, \eta, \zeta < \aleph_1$ such that

$$\begin{aligned} x_\xi &\geq y, \\ \xi < \eta \text{ and } x_\eta &\geq y', \\ \eta < \zeta \text{ and } x_\zeta &\geq y. \end{aligned}$$

It is easy to check that then x_η is left of both x_ξ and x_ζ , so $x_\eta < x_\xi, x_\zeta$ whereas $\xi < \eta < \zeta$. This contradicts the monotonicity of $\{x_\alpha \mid \alpha < \aleph_1\}$.

If y is \leq uncountably many x_α 's, then so are all its predecessors. Hence, these y 's form a path, contrary to the choice of T .

(b) Suppose $X \subseteq Y \subseteq T, |X| = \aleph_0, |Y| = \aleph_1$, and X is pseudo-dense in Y . The countable set $\{h(x) \mid x \in X\}$ is bounded above by an ordinal $\alpha < \aleph_1$. All but countably many elements of Y are in $\bigcup_{h(z)=\alpha} \{y \mid y \geq z\}$, so there is a z of height α such that two distinct elements of Y are $\geq z$. Let a and b be such elements, and let $a < b$. As X is pseudo-dense in Y , there is an $x \in X$ such that $a \leq x < b$. If $x < b$, then, as $z \leq b$ and z has greater height than x , we find $x < z \leq a$, so $x < a$, a contradiction. So x must be left of b ; that is, if β is the least ordinal ($\leq h(x)$) such that $p_\beta(x) \neq p_\beta(b)$, then $p_\beta(x) < p_\beta(b)$. But, as $\beta \leq h(x) < \alpha$, we see that $p_\beta(b) = p_\beta(z) = p_\beta(a)$, and β is also the least ordinal such that $p_\beta(x) \neq p_\beta(a)$. Thus, x is left of a , so $x < a$, a contradiction. \square

Let $(A, <_A)$ and $(B, <_B)$ be as in Lemmas 2 and 5, respectively, and let $<$ be the usual ordering of \aleph_1 . Let $f: \aleph_1 \rightarrow A$ and $g: \aleph_1 \rightarrow B$ be bijections, and let $h: A \rightarrow A$ be an anti-automorphism of $(A, <_A)$ by Lemma 2(c). Let $F: [\aleph_1]^2 \rightarrow 4$ be as follows. If $\alpha < \beta < \aleph_1$, then

$$\begin{aligned} F\{\alpha, \beta\} &= 0 \text{ if } f(\alpha) <_A f(\beta) \text{ and } g(\alpha) <_B g(\beta), \\ &= 1 \text{ if } f(\alpha) <_A f(\beta) \text{ and } g(\alpha) >_B g(\beta), \\ &= 2 \text{ if } f(\alpha) >_A f(\beta) \text{ and } g(\alpha) <_B g(\beta), \\ &= 3 \text{ if } f(\alpha) >_A f(\beta) \text{ and } g(\alpha) >_B g(\beta). \end{aligned}$$

If $P(\aleph_1, 3)$ were true, then, by Lemma 1, there would exist $C \subseteq \aleph_1$ such that $|C| = \aleph_1$ and $F(\{C\}^2)$ is included in a two element subset S of 4. If $S = \{0, 1\}$ or $S = \{2, 3\}$, then f maps C monotonically into A , contrary to Lemma 2(a). If $S = \{0, 2\}$ or $S = \{1, 3\}$, then g maps C monotonically into B , contrary to Lemma 5(a). If $S = \{0, 3\}$ then gf^{-1} maps the uncountable set $f(C) \subseteq A$ isomorphically to $g(C) \subseteq B$, while if $S = \{1, 2\}$, then $gf^{-1}h$ maps $h^{-1}f(C)$ isomorphically to $g(C)$. In either of the last two cases, an uncountable subset of A is isomorphic to an uncountable subset of B . A glance at Lemmas 2(b) and 5(b) shows that this is impossible. Each choice

of S has led to a contradiction, so $P(\aleph_1, 3)$ cannot hold. Theorem 1 is proved. \square

Theorem 1 can be generalized to apply to certain cardinals larger than \aleph_1 . An analogue of Lemma 2, with " κ " in place of " \aleph_1 " and "of cardinality $< \kappa$ " in place of "countable", can be proved for any κ provided there exists $\mu < cf(\kappa)$ such that $\kappa \leq 2^\mu$. The role of the real line in the proof of Lemma 2 is played by the lexicographic ordering of 2 for the least such μ , and the role of the rationals is played by the subset of 2 consisting of ultimately constant functions. A similar analogue of Lemma 5 can be proved provided κ is regular and the analogue of Lemma 4 holds. Thus, we can obtain the following result.

THEOREM 2. *If κ is regular and accessible, and if there is a tree of height κ with levels of cardinality $< \kappa$ and without paths, then not $P(\kappa, 3)$.*

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