Variable Lebesgue Spaces

David V. Cruz-Uribe, SFO

Trinity College

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The Maximal Operator on Variable Lebesgue Spaces

Outline

The maximal operator

For $f \in L^1_{\text{loc}}$, define

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy = \sup_{Q \ni x} \fint_Q |f(y)|\,dy$$

Can replace cubes by centered cubes or by (centered) balls

Motivation

Calderón-Zygmund philosophy:

To control a singular integral operator,

control the maximal operator

L^p estimates

- $p = \infty$: $\|Mf\|_\infty = \|f\|_\infty$
- $1 < p < \infty$:

$$\int_{\mathbb{R}^n} Mf(x)^p \, dx \leq C(p,n) \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

- $p = 1$: any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, dx$$

Calderón-Zygmund decomposition

Given $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $\lambda > 0$, there exist disjoint dyadic cubes $\{Q_j^\lambda\}$ such that

$$\lambda < \fint_{Q_j^\lambda} |f(y)|\,dy \leq 2^n \lambda$$

and $|f(x)| \leq \lambda$ a.e. on $\mathbb{R}^n \setminus \bigcup_j Q_j^\lambda$.

Further,

$$E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subset \bigcup_j 3Q_j^\lambda$$

Proof of L^p estimates

Step 1: weak $(1,1)$ via CZ-decomp:

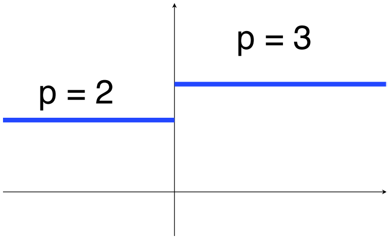
$$|E_\lambda| \leq \sum_j |3Q_j^\lambda| \leq \frac{3^n}{\lambda} \sum_j \int_{Q_j^\lambda} |f|\,dy \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f|\,dy.$$

Step 2: (p,p) inequalities follow by Marcinkiewicz interpolation

$$\int_{\mathbb{R}^n} Mf(x)^p\,dx = p\int_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda.$$

A simple example

$$p(x) = \begin{cases} 2 & -5 < x \leq 0 \\ 3 & 0 < x < 5. \end{cases}$$



A simple example (continued)

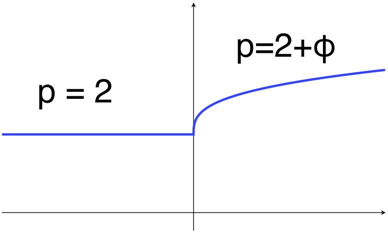
Let $f(x) = |x|^{-1/3}\chi_{(-5,0)}(x)$: $f \in L^{p(\cdot)}((-5,5))$

For $0 < x < 5$,

$$Mf(x) \geq \fint_{-x}^{x} |f(y)|\,dy = 3|x|^{-1/3} \notin L^{p(\cdot)}((-5,5))$$

First question

Do we need $p(\cdot)$ continuous? How much regularity?

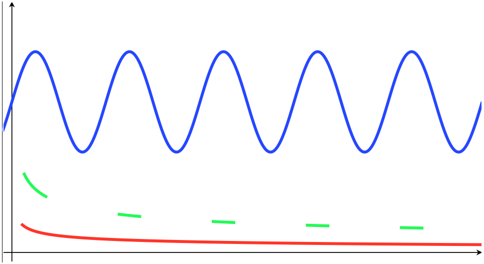


Behavior at infinity

Let $p(x) = 3 + sin(\pi x)$

Define $f(x) = |x|^{-1/3} \sum_k \chi_{[1/4+2k, 3/4+2k]} \in L^{p(\cdot)}(\mathbb{R})$

$Mf(x) \approx c|x|^{-1/3} \notin L^{p(\cdot)}(\mathbb{R})$



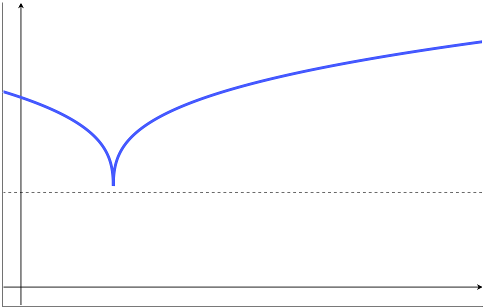
Second question

Do we need $p(\cdot)$ continuous at infinity? How much regularity?

Two more questions

Can $p(\cdot) = \infty$ on parts of the domain?

Can $p(\cdot)$ get arbitrarily close to 1?



Local log-Hölder continuity

Given $p(\cdot)$, we say $1/p(\cdot) \in LH_0$ if

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{C_0}{-\log(|x-y|)}, \qquad |x-y| < \frac{1}{2}$$

If $p_+ < \infty$, $p(\cdot) \in LH_0$ iff $1/p(\cdot) \in LH_0$.

$1/p(\cdot) \in LH_0$ iff $1/p'(\cdot) \in LH_0$

Log-Hölder continuity at infinity

Given $p(\cdot)$, we say $1/p(\cdot) \in LH_\infty$ if there exists p_∞ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{C_\infty}{\log(e + |x|)}.$$

If $p_+ < \infty$, $p(\cdot) \in LH_\infty$ iff $1/p(\cdot) \in LH_\infty$.

Hereafter, let $LH = LH_0 \cap LH_\infty$.

$L^{p(\cdot)}$ estimates for the maximal operator

Theorem

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1/p(\cdot) \in LH$. Then for all $\lambda > 0$,

$$\|\lambda\chi_{\{x:Mf(x)>\lambda\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

If in addition $p_- > 1$, then

$$\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Outline of proof: $p_+ < \infty$

- Assume f non-negative, bounded and compact support, $\|f\|_{p(\cdot)} = 1$
- Split f into $f \geq 1$ and $0 \leq f < 1$
- Apply Calderón-Zygmund decomposition
- Use LH_0 where f large; LH_∞ where f small
- Use classical result: M bounded on L^{p_-}/L^{p_∞}

First step: f small

Let $f_1 = f\chi_{\{x:f(x)\geq 1\}}$. Suffices to prove

$$\int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\,dx \leq C.$$

Apply Calderón-Zygmund decomposition:

$$\int_{\mathbb{R}^n} Mf_1(x)^{p(x)}\,dx \leq C\sum_{k,j}\int_{E_j^k}\left(\fint_{3Q_j^k} f_1(y)\,dy\right)^{p(x)}\,dx$$

where $\{Q_j^k\}$ are CZ cubes at height A^k and $E_j^k \subset 3Q_j^k$ pairwise disjoint.

Intuition: Hölder's inequality

$$
\begin{aligned}
\sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)\,dy \right)^{p(x)} dx
&\leq C \sum_{k,j} \int_{E_j^k} \left(\fint_{3Q_j^k} f_1(y)^{p(y)/p_-}\,dy \right)^{p_-} dx \\
&\leq C \sum_{k,j} \int_{E_j^k} M(f_1^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\leq C \int_{\mathbb{R}^n} M(f_1^{p(\cdot)/p_-})(x)^{p_-}\,dx \\
&\leq C \int_{\mathbb{R}^n} f_1(x)^{p(x)}\,dx \leq C.
\end{aligned}
$$

A lemma to make this work

Lemma

TFAE:

- $p(\cdot) \in LH_0$;
- *given any cube Q and $x \in Q$,*

$$|Q|^{p(x)-p_+(Q)} \leq C \quad \text{and} \quad |Q|^{p_-(Q)-p(x)} \leq C.$$

Second step: f large

Let $f_2 = f\chi_{\{x:0\leq f<1\}}$. Again suffices to prove

$$\int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\,dx \leq C.$$

Intuition: if function F "lives" far from origin, then

$$\int_{\mathbb{R}^n} F(x)^{p(x)}\,dx \approx \int_{\mathbb{R}^n} F(x)^{p_\infty}\,dx$$

Another key lemma

Lemma

If $p(\cdot) \in LH_\infty$*, and* $0 \leq F(x) \leq 1$*, then*

$$\int_{\mathbb{R}^n} F(x)^{p(x)}\,dx \leq C \int_{\mathbb{R}^n} F(x)^{p_\infty}\,dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx$$

and

$$\int_{\mathbb{R}^n} F(x)^{p_\infty}\,dx \leq C \int_{\mathbb{R}^n} F(x)^{p(x)}\,dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx$$

where $R(x) = (e + |x|)^{-n}$*.*

The final step

$$\begin{aligned}
\int_{\mathbb{R}^n} Mf_2(x)^{p(x)}\,dx \\
&\leq C\int_{\mathbb{R}^n} Mf_2(x)^{p_\infty}\,dx + C\int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx \\
&\leq C\int_{\mathbb{R}^n} f_2(x)^{p_\infty}\,dx + C \\
&\leq C\int_{\mathbb{R}^n} f_2(x)^{p(x)}\,dx + C\int_{\mathbb{R}^n} R(x)^{p_\infty}\,dx + C \\
&\leq C.
\end{aligned}$$

$p_- > 1$ necessary

If $p_- = 1$, then there exists a sequence of balls B_k and exponents $s_k \searrow 1$ such that

$$\frac{|E_k|}{|B_k|} = \frac{|\{x \in B_k : 1 < p(x) \leq s_k\}|}{|B_k|} \geq 1/2.$$

Let x_k be center of B_k. Define

$$f_k(x) = |x - x_k|^{-n+\frac{1}{k+1}} \chi_{E_k}(x).$$

Choose s_k so that $f_k \in L^{p(\cdot)}$ but

$$Mf_k(x) \geq c(n)(k+1)f_k(x).$$

LH_0 and LH_∞ pointwise sharp

Take any $\phi(\cdot) : [0,\infty) \to [0,1]$, increasing, smooth $\phi(x) = 0$ and

$$\lim_{x\to 0^+} \phi(x)\log(x) = -\infty.$$

Define

$$p(x) = \begin{cases} 2 + \phi(x) & x \geq 0 \\ 2 & x < 0 \end{cases}$$

Then $p(\cdot) \notin LH_0$ and M not bounded on $L^{p(\cdot)}(\mathbb{R})$.

Similar construction holds for LH_∞.

LH_0 and LH_∞ not necessary

Define

$$\frac{1}{p(x)} = \begin{cases} \frac{1}{2} + \log(|x|)^{-1/2} & |x| < e^{-9} \\ 5/6 & |x| \geq e^{-9}. \end{cases}$$

Then M bounded on $L^{p(\cdot)}(\mathbb{R})$ but $p(\cdot) \notin LH_0$.

Similar example holds at infinity.

Continuity not necessary

Define

$$p(x) = \begin{cases} 2 + \alpha \sin(\pi \log\log(1/|x|)) & |x| < e^{-e} \\ 2 & |x| \geq e^{-e} \end{cases}$$

Then M bounded on $L^{p(\cdot)}(\mathbb{R})$ but $p(\cdot)$ not continuous at 0.

Similar example holds at infinity.

Condition at infinity

$p(\cdot) \in N_\infty$ if

$$\int_{\{p(x)\neq p_\infty\}} \exp\left(-k\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|\right)\,dx < \infty.$$

If $p(\cdot) \in N_\infty$ and $1/p(\cdot) \in LH_0$, M bounded on $L^{p(\cdot)}$.

Use N_∞ condition to prove *Lemma at Infinity* above.

Local condition

$p(\cdot) \in K_0$ if $\sup_Q |Q|^{-1} \|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} < \infty$.

If $p(\cdot) \in K_0 \cap N_\infty$, M is bounded on $L^{p(\cdot)}$.

Use K_0 condition to prove the *Hölder inequality* argument above.

K_0 necessary and sufficient on bounded domains (only)!

$L^{p(\cdot)}$ version of Muckenhoupt A_p condition; proof uses theory of weights.

Averaging operators on cubes

Define

$$\mathcal{Y} = \big\{ \mathcal{Q} = \{Q\} : Q \text{ pairwise disjoint cubes} \big\}.$$

Given $\mathcal{Q} \in \mathcal{Y}$ define the averaging operator

$$A_{\mathcal{Q}} f(x) = \sum_{Q \in \mathcal{Q}} \fint_Q |f(y)|\, dy\, \chi_Q(x).$$

Equivalent conditions

Theorem

Given $p(\cdot)$ such that $1 < p_- \leq p_+ < \infty$, TFAE:

- *M bounded on $L^{p(\cdot)}$*
- *$A_{\mathcal{Q}}$ uniformly bounded for all $\mathcal{Q} \in \mathcal{Y}$*
- *M bounded on $L^{p'(\cdot)}$*
- *For some $s > 1$, $M(|\cdot|^s)^{1/s}$ bounded on $L^{p(\cdot)}$*
- $\lim_{r \to 1^-} (1-r)\|M(|\cdot|^r)^{1/r}\|_{p(\cdot)} = 0.$

Duality of M

Find a direct proof of the equivalence of

- M bounded on $L^{p(\cdot)}$
- M bounded on $L^{p'(\cdot)}$

Special case: Prove this for classical Lebesgue spaces.

Tail condition

Does there exist a “geometric” tail condition to replace LH_∞ that is necessary and sufficient?

Define

$$T_Q(x) = \frac{|Q|}{|Q| + |x - x_Q|^n} \approx M(\chi_Q)(x)$$

$$\sup_Q |Q|^{-1} \|T_Q\|_{p(\cdot)} \|T_Q\|_{p'(\cdot)} < \infty.$$

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