

## TWO $F_{\sigma\delta}$ IDEALS

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ABSTRACT. We find two  $F_{\sigma\delta}$  ideals on  $\mathbb{N}$  neither of which is  $F_\sigma$  whose quotient Boolean algebras are homogeneous but nonisomorphic. This solves a problem of Just and Krawczyk (1984).

We consider Boolean algebras of the form  $\mathcal{P}(\mathbb{N})/\mathcal{I}$ , where  $\mathcal{I}$  is an ideal on  $\mathbb{N}$  containing the ideal  $\text{Fin}$  of finite sets. In [3] Just and Krawczyk formulated several conditions on the ideals  $\mathcal{I}, \mathcal{J}$  that guarantee their quotients  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  and  $\mathcal{P}(\mathbb{N})/\mathcal{J}$  to be isomorphic. By identifying sets of integers with their characteristic functions, we equip  $\mathcal{P}(\mathbb{N})$  with the Cantor-space topology. We can therefore assign topological complexity to the ideals of sets of integers. In particular, we have  $F_\sigma$ ,  $F_{\sigma\delta}$ , Borel, and so on, ideals on  $\mathbb{N}$ .

Just and Krawczyk have proved that the Continuum Hypothesis implies that

- (1) all quotients over  $F_\sigma$  ideals are pairwise isomorphic, and
- (2) the quotient over the ideal of asymptotic density zero sets,  $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} |A \cap n|/n = 0\}$ , is isomorphic to the quotient over the ideal of logarithmic density zero sets,  $\mathcal{Z}_{\log} = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} (\sum_{i \in A \cap n} 1/i) / (\sum_{i < n} 1/i) = 0\}$ .

They have also introduced a class of *EU-ideals* that contains both  $\mathcal{Z}_0$  and  $\mathcal{Z}_{\log}$  and proved that under CH all quotients over these ideals are homogeneous and pairwise isomorphic. (A Boolean algebra  $\mathcal{B}$  is *homogeneous* if it is isomorphic to  $\mathcal{B}_A = \{B \in \mathcal{B} : B \leq A\}$ , for every  $A \in \mathcal{B} \setminus \{0_{\mathcal{B}}\}$ .) Motivated by this result, Just and Krawczyk posed the following problem.

**Problem 1** ([3, Problem C]). Is it true that if  $\mathcal{I}, \mathcal{J}$  are  $F_{\sigma\delta}$  and not  $F_\sigma$  and both  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  and  $\mathcal{P}(\mathbb{N})/\mathcal{J}$  are homogeneous, then  $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$ ?

We will prove that this problem has a negative answer. We will also prove that there is an  $F_{\sigma\delta}$  ideal whose quotient is not isomorphic to a quotient over any  $P$ -ideal. (Recall that  $\mathcal{I}$  is a *P-ideal* if for every sequence  $A_n$  ( $n \in \mathbb{N}$ ) in  $\mathcal{I}$  there is an  $A \in \mathcal{I}$  such that  $A_n \setminus A$  is finite for all  $n$ .)

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**Sequential topology.** If  $\mathcal{B}$  is a  $\sigma$ -complete Boolean algebra, one can define a topology on  $\mathcal{B}$  as follows. A sequence  $A_n$  ( $n \in \mathbb{N}$ ) *algebraically converges* to  $A$  if:

$$\bigvee_{m=1}^{\infty} \bigwedge_{n=m}^{\infty} A_n = \bigwedge_{m=1}^{\infty} \bigvee_{n=m}^{\infty} A_n.$$

A subset of  $\mathcal{B}$  is closed if it is closed under taking algebraic limits of sequences in it. Open sets are complements of closed sets. See [4] and [1] for more on this topology on complete Boolean algebras.

It is known that quotients over analytic ideals (or more generally, over ideals that have the property of Baire) are never  $\sigma$ -complete (see [3]). If  $\mathcal{B}$  is a (not necessarily  $\sigma$ -complete) Boolean algebra, define a topology  $\tau$  on  $\mathcal{B}$  as follows. A sequence  $A_n$  ( $n \in \mathbb{N}$ ) *algebraically converges* to  $A$  if:

- (1) For all  $m$ ,  $B_m = \bigwedge_{n=m}^{\infty} A_n$  exists.
- (2) For all  $m$ ,  $C_m = \bigvee_{n=m}^{\infty} A_n$  exists.
- (3) Both  $\bigwedge_{m=1}^{\infty} C_m$  and  $\bigvee_{m=1}^{\infty} B_m$  exist and are equal to  $A$ .

A subset of  $\mathcal{B}$  is  $\tau$ -closed if it is closed under taking algebraic limits of sequences in it.  $\tau$ -open sets are the complements of  $\tau$ -closed sets.

**Proposition 2.** *If  $\mathcal{I}$  is an analytic  $P$ -ideal, then there is a complete metric on  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  that induces  $\tau$ .*

*Proof.* Let  $\phi$  be a lower semicontinuous submeasure such that  $\mathcal{I} = \{A \subseteq \mathbb{N} : \limsup_{n \rightarrow \infty} \phi(A \setminus n) = 0\}$ , as guaranteed by [5]. Define a metric  $d_\phi$  on  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  by

$$d_\phi([A]_{\mathcal{I}}, [B]_{\mathcal{I}}) = \limsup_n \phi((A \Delta B) \setminus n).$$

This metric is complete (see [2, Lemma 1.3.3]). It is easily checked that a sequence is  $d_\phi$ -convergent if and only if it is  $\tau$ -convergent. □

**Theorem 3.** *There are two ideals  $\mathcal{I}$  and  $\mathcal{J}$  such that*

- (1) *both  $\mathcal{I}$  and  $\mathcal{J}$  are  $F_{\sigma\delta}$  and neither  $\mathcal{I}$  nor  $\mathcal{J}$  is  $F_\sigma$ ,*
- (2) *both quotient algebras over  $\mathcal{I}$  and  $\mathcal{J}$  are homogeneous,*
- (3) *these quotient algebras are not isomorphic.*

*Proof.* We will take  $\mathcal{I}$  and  $\mathcal{J}$  to be the following ideals on  $\mathbb{Q} \cap [0, 1]$ :

$$\begin{aligned} \text{NWD}(\mathbb{Q}) &= \{A \subseteq \mathbb{Q} \cap [0, 1] : A \text{ is nowhere dense}\}, \\ \text{NULL}(\mathbb{Q}) &= \{A \subseteq \mathbb{Q} \cap [0, 1] : \overline{A} \text{ is of Lebesgue measure } 0\}. \end{aligned}$$

(The closure  $\overline{A}$  is taken in  $\mathbb{R}$ .) To see that  $\text{NWD}(\mathbb{Q})$  is  $F_{\sigma\delta}$ , enumerate the basis of  $\mathbb{Q}$  as  $\{U_n\}$ ,  $\mathbb{Q}$  as  $\{q_n\}$ , and the basis of  $\mathbb{Q} \cap U_m$  as  $\{V_{mn}\}$ . The set

$$K_m = \{A \subseteq \mathbb{Q} : (\exists n) A \cap V_{mn} \subseteq \{q_1, \dots, q_n\}\}$$

is hereditary and  $F_\sigma$ , and  $A \in K_m$  if and only if  $A \cap U_m$  is nowhere dense. Therefore,  $\text{NWD}(\mathbb{Q}) = \bigcap_m K_m$ .

To see that  $\text{NULL}(\mathbb{Q})$  is  $F_{\sigma\delta}$ , for each  $n$  enumerate all finite unions of rational intervals of measure  $\leq 1/n$  and proceed as above, using the compactness of  $[0, 1]$ .

Neither of these ideals is  $F_\sigma$ .

Ideals  $\mathcal{I}$  and  $\mathcal{J}$  are *Rudin–Keisler isomorphic* if there are  $A \in \mathcal{I}$ ,  $B \in \mathcal{J}$ , and a bijection  $h$  between  $\mathbb{N} \setminus B$  and  $\mathbb{N} \setminus A$  such that for all  $X \subseteq \mathbb{N} \setminus A$  we have

$$X \in \mathcal{I} \iff h^{-1}(X) \in \mathcal{J}.$$

**Claim 1.** *The quotient  $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$  is homogeneous.*

*Proof.* Let  $A$  be a positive set. Then the interior  $B$  of  $\overline{A}$  is nonempty, hence  $B \cap A$  is dense in itself. Thus  $B \cap A$  is homeomorphic to  $\mathbb{Q}$ , and  $A \setminus B$  is nowhere dense. The homeomorphism is a Rudin–Keisler isomorphism between  $\text{NWD}(\mathbb{Q})$  and  $\text{NWD}(\mathbb{Q}) \upharpoonright A$ , and it induces an isomorphism between  $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$  and  $\mathcal{P}(A)/\text{NWD}(\mathbb{Q}) \upharpoonright A$ .  $\square$

**Claim 2.** *The quotient  $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$  is homogeneous.*

*Proof.* As in the proof of Claim 1, we need to prove that for every positive  $A$  the ideals  $\text{NULL}(\mathbb{Q})$  and  $\text{NULL}(\mathbb{Q}) \upharpoonright A$  are Rudin–Keisler isomorphic. We shall prove this in two steps.

If  $A, B$  are two subsets of  $\mathbb{Q} \cap [0, 1]$  with the same closure, there is a bijection  $f: A \rightarrow B$  such that  $\lambda(\overline{X}) = \lambda(\overline{f''X})$  for all  $X \subseteq A$ . Let  $A = \{a_i : i \in \mathbb{N}\}$  and  $B = \{b_i : i \in \mathbb{N}\}$  be one-to-one enumerations. Find a bijection  $f$  so that  $\lim_i d(a_i, f(a_i)) = 0$ , making sure that every isolated point is fixed by  $f$ . Such an  $f$  satisfies the requirements because  $\overline{f''X} \Delta \overline{X}$  is countable for every  $X$ .

In the second step we prove that for every  $K \subseteq [0, 1]$  of positive measure there is  $g: K \rightarrow [0, 1]$  such that  $\lambda(X) = 0$  if and only if  $\lambda(g''X) = 0$  for every closed  $X \subseteq K$ . The function defined by

$$g(a) = \frac{\lambda([0, a] \cap K)}{\lambda(K)}$$

has the property that  $\lambda(g''U) = \lambda(U)/\lambda(K)$  for every interval  $U$ . Therefore this equality holds for all Lebesgue-measurable sets, and  $g$  is as required.

To conclude the proof, let  $A \subseteq \mathbb{Q}$  be positive. By the above, we can find maps  $g: A \rightarrow [0, 1]$  and  $f: (g''A) \rightarrow \mathbb{Q}$  such that  $f \circ g$  is a Rudin–Keisler isomorphism.  $\square$

The proof of the following result is very similar to some arguments of [1].

**Lemma 4.** *The sequential topology on  $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$  is not Hausdorff.*

*Proof.* In this proof, by open we mean relatively open in  $\mathbb{Q}$  unless otherwise stated. Let us write  $\mathcal{I}$  for  $\text{NWD}(\mathbb{Q})$ . We claim that each open in  $\mathcal{P}(\mathbb{Q})/\mathcal{I}$  set containing  $[\emptyset]_{\mathcal{I}}$  contains  $[\mathbb{Q}]_{\mathcal{I}}$  in its closure. Let  $\mathcal{D}$  be an open neighborhood of  $[\emptyset]_{\mathcal{I}}$  in  $\mathcal{P}(\mathbb{Q})/\mathcal{I}$ .

It is straightforward to verify the following two facts about convergence in  $\mathcal{P}(\mathbb{Q})/\mathcal{I}$ . (The second of these facts is of a rather general nature while the first one is characteristic to  $\text{NWD}(\mathbb{Q})$ .)

- (1) If  $(U_n)$  is an increasing sequence of open sets, then  $[U_n]_{\mathcal{I}} \rightarrow [\bigcup_n U_n]_{\mathcal{I}}$ .
- (2) Let  $U$  be open (perhaps empty),  $q \in \mathbb{Q}$ , and let  $V_n$  be an open ball around  $q$  of radius  $1/n$ . Then  $[U \cup V_n]_{\mathcal{I}} \rightarrow [U]_{\mathcal{I}}$ .

List elements of  $\mathbb{Q}$ :  $q_0, q_1, q_2, \dots$ . By induction, using (2), we construct a sequence of open sets  $(U_n)$  with  $[U_n]_{\mathcal{I}} \in \mathcal{D}$  and with  $U_{n+1}$  being the union of  $U_n$  and an open ball around  $q_{n+1}$ . Then by (1),  $[U_n]_{\mathcal{I}} \rightarrow [\bigcup_n U_n]_{\mathcal{I}} = [\mathbb{Q}]_{\mathcal{I}}$ .  $\square$

**Lemma 5.** *The sequential topology on  $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$  is Hausdorff.*

*Proof.* Let  $\lambda(A)$  be the Lebesgue measure of  $A$ . Let us write  $\mathcal{J}$  for  $\text{NULL}(\mathbb{Q})$ , and let  $X = [x]_{\mathcal{J}}$ ,  $Y = [y]_{\mathcal{J}}$ , etc. We claim that whenever  $\lim_i X_i = Y$  in  $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$ , we have  $\lim_i \lambda(\overline{x_i \Delta y}) = 0$ . Assume the contrary, and fix a sequence  $X_i$  converging to  $Y$  such that  $\liminf_i \lambda(\overline{x_i \Delta y}) = \delta > 0$ . Let  $B_n = \bigvee_{i \geq n} X_i$

and  $C_n = \bigwedge_{i \geq n} X_i$ . Since  $B_n \geq X_n \geq C_n$  and  $B_n \geq Y \geq C_n$  for all  $n$ , for every  $n$  we have either  $\lambda(\overline{b_n \setminus y}) \geq \delta/2$  or  $\lambda(\overline{y \setminus c_n}) \geq \delta/2$ .

Let us assume that  $\lambda(\overline{y \setminus c_n}) \geq \delta/2$  for infinitely many  $n$ .

By making small changes to these sets, we may assume  $c_1 \subseteq c_2 \subseteq c_3 \subseteq \dots \subseteq y$ . Therefore, we have  $\lambda(\overline{y \setminus c_n}) \geq \delta/2$  for all  $n$ . The set  $F = \bigcap_{n=1}^\infty \overline{y \setminus c_n}$  has measure at least  $\delta/2$ , since  $\lambda(\overline{y \setminus c_n}) \geq \delta/2$  for all  $n$  and this is a decreasing sequence of closed subsets of  $[0, 1]$ . For each  $n$  find  $s_n \in y \setminus (c_1 \cup \dots \cup c_n)$  such that  $\inf_{a \in F} d(s_n, a) \leq 1/n$ , assuring that the closure of  $x = \{s_n : n \in \mathbb{N}\}$  includes  $F$ . Then  $x \cap c_n$  is finite for all  $n$ ; moreover,  $x \subseteq y$ , and  $[x]_{\mathcal{J}} \neq [\emptyset]_{\mathcal{J}}$ . Therefore, the sequence  $C_n$  does not converge to  $Y$ , contrary to our assumption.

Therefore, we have  $\lambda(\overline{b_n \setminus y}) \geq \delta/2$  for every  $n$ . The proof that this case leads to the contradiction is identical to the above.

An easy induction on the sequential rank shows that every  $\tau$ -closed set is closed in the metric topology induced by  $\lambda$ . Therefore, for  $y \subseteq \mathbb{Q}$  and  $\varepsilon > 0$  the set

$$\{[a]_{\mathcal{J}} : \lambda(\overline{a \cap y}) < \varepsilon\}$$

includes an open neighborhood of  $[y]_{\mathcal{J}}$ , in turn implying the space is Hausdorff.  $\square$

Since the sequential topology is defined in algebraic terms, an isomorphism between Boolean algebras is automatically a homeomorphism. Therefore, the two quotients are not isomorphic, and this concludes the proof.  $\square$

Note that Lemma 4 and Proposition 2 together imply

**Proposition 6.** *The quotient  $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$  is not isomorphic to  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  for any analytic  $P$ -ideal  $\mathcal{I}$ .  $\square$*

During the course of proving Lemma 5 we have proved that the sequential topology on  $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$  is stronger than a metric topology. It is not difficult to see that the two topologies differ, but even more is true. If  $I$  is an ideal on  $[0, 1]$  that contains all singletons, define the ideal  $I(\mathbb{Q})$  on  $\mathbb{Q} \cap [0, 1]$  by

$$I(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0, 1] : \overline{A} \in I\}.$$

(The closure  $\overline{A}$  is taken in  $\mathbb{R}$ .)

**Theorem 7.** *If  $I$  is a  $\sigma$ -ideal on  $\mathbb{Q} \cap [0, 1]$  containing all singletons, then the sequential topology on  $\mathcal{P}(\mathbb{Q})/I(\mathbb{Q})$  is not metric. Therefore, the quotient  $\mathcal{P}(\mathbb{Q})/I(\mathbb{Q})$  is not isomorphic to  $\mathcal{P}(\mathbb{N})/\mathcal{I}$  for any analytic  $P$ -ideal  $\mathcal{I}$ .*

*Proof.* Define a sequence  $a_n$  ( $n \in \mathbb{N}$ ) of subsets of  $\mathbb{Q} \cap [0, 1]$  by

$$a_{\frac{n(n+1)}{2} + i} = [i/n, (i + 1)/n],$$

if  $0 \leq i < n$ . Then  $\lim_{i \rightarrow \infty} \lambda(\overline{a_i}) = 0$ . However, the sequence  $A_i = [a_i]_{I(\mathbb{Q})}$  does not converge to  $[\emptyset]_{I(\mathbb{Q})}$  algebraically. This is because  $C_n = \bigwedge_{i \geq n} [a_n]_{I(\mathbb{Q})} = [\emptyset]_{I(\mathbb{Q})}$  and  $B_n = \bigvee_{i \geq n} [a_n]_{I(\mathbb{Q})} = [\mathbb{Q}]_{I(\mathbb{Q})}$  for all  $n$ .

We claim that every subsequence of  $\{a_n\}$  has a further subsequence that converges to  $[\emptyset]_{I(\mathbb{Q})}$ . Once proved, this will imply that the topology is not metric.

For  $i \in \mathbb{N}$  let  $x_i$  and  $y_i$  be the left and right endpoints of the interval  $a_i$ . For a subsequence  $a_{n_i}$  ( $i \in \mathbb{N}$ ) we can find a subsequence  $a_{m_i}$  such that  $\lambda(a_{m_i}) < 2^{-i}$

and  $\lim_i x_{m_i} = x$  and  $\lim_i y_{m_i} = y$  for some  $x, y$ . We necessarily have  $x = y$ , and therefore for  $k \in \mathbb{N}$  we have

$$b_k = \overline{\bigcup_{i \geq k} a_{m_i}} = \bigcup_{i \geq k} a_{m_i} \cup \{x\}.$$

Therefore  $\lambda(\overline{b_k}) < 2^{-k+1}$ . Since  $\overline{b_k} \supseteq \overline{b_{k+1}}$  for all  $k$  and  $I$  is a  $\sigma$ -ideal, the sequence  $B_k = \bigvee_{i \geq k} [a_{m_i}]_{I(\mathbb{Q})}$  converges to  $[\emptyset]_{I(\mathbb{Q})}$ . This proves our claim and concludes the proof.  $\square$

There are analytic ideals that are not P-ideals whose quotients are metrizable. For example, all  $F_{\sigma}$  ideals are of this form, because their quotients are discrete in the sequential topology (this follows from [3]).

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