Topological Relationships Between Complex Spatial Objects

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For a long time topological relationships between spatial objects have been a focus of research in a number of disciplines like artificial intelligence, cognitive science, linguistics, robotics, and spatial reasoning. Especially as predicates they support the design of suitable query languages for spatial data retrieval and analysis in spatial databases and geographical information systems (GIS). Unfortunately, they have so far only been defined for and applicable to simplified abstractions of spatial objects like single points, continuous lines, and simple regions. With the introduction of *complex spatial data types* an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. This article closes this gap and first introduces definitions of general and versatile spatial data types for *complex points*, *complex lines*, and *complex regions*. Based on the well known 9-intersection model, it then determines the complete sets of mutually exclusive topological relationships for all type combinations. Completeness and mutual exclusion are shown by a proof technique called *proof-by-constraint-and-drawing*. Due to the resulting large numbers of predicates and the difficulty of handling them, the user is provided with the concepts of *topological cluster predicates* and *topological predicate groups*, which permits one to reduce the number of predicates to be dealt with in a user-defined and/or application-specific manner.

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1. INTRODUCTION

In recent years, the exploration of topological relationships between objects in space has turned out to be a multidisciplinary research issue involving disciplines like artificial intelligence, CAD/CAM systems, cognitive science, computer vision, geographical information science, image databases, linguistics, psychology, robotics, spatial analysis, spatial database systems, and spatial reasoning. From a database and GIS perspective, their development has been motivated by the necessity of formally defined topological predicates as filter conditions for spatial selections and spatial joins in spatial query languages and as a support for spatial data retrieval and analysis tasks, both at the user definition level for reasons of conceptual clarity and at the query processing level for reasons of efficiency.

Topological relationships like *overlap*, *inside*, *disjoint*, or *meet* describe purely qualitative properties that characterize the relative positions of spatial objects toward each other and that are preserved under certain continuous transformations including all affine transformations. They enable the user, for example, to ask for land parcels that are *adjacent* to a hazardous waste site, to explore the *overlapping* of a river floodplain with a proposed highway network, or to determine for a collection of cities and states which city is located *within* which state. Assuming a relation *cities* with attributes *cname* of type *string* and *loc* of type *point* as well as a relation *states* with attributes *sname* of type *string* and *territory* of type *region*, we can express the last query example, which represents a spatial join, in an SQL-like style as follows:

```
SELECT cname, sname FROM cities, states WHERE  loc inside territory
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Further examples of topological queries can be found in [Shekar and Chawla 2003]. Topological relationships deliberately exclude any consideration of quantitative, metric measures like distance or direction measures and are associated with notions like adjacency, coincidence, connectivity, inclusion, and continuity.

Two important, fundamental approaches for the definition of topological relationships are the 9-*intersection model* [Egenhofer and Herring 1990b], which rests on point set theory and point set topology, as well as the *RCC model* [Cui et al. 1993], which employs spatial logic. Both models are essentially tailored to the treatment of simplified abstractions of spatial objects like simple lines and simple regions. Simple lines are one-dimensional, continuous features embedded in the plane with two end points, and simple regions are two-dimensional point sets topologically equivalent to a closed disc. Single points have not found much interest, since their interrelations are trivial. Although based on completely different foundations, both models yield rather similar results. The foundation of this work is the 9-intersection model, since we have an object-based view on spatial objects and intend to leverage the properties and relations of object components like boundaries, interiors, and exteriors for the definition of topological relationships.

Unfortunately, the described simple geometric structures are inadequate abstractions for real spatial applications since they are insufficient to cope with the variety and complexity of geographic reality. For example, the SQL query above assumes that an object of type *point* models a single point only and that an object of type *region* models a single-component area without holes. Hence, states consisting of several parts are not possible. Consequently, universal and versatile type specifications are needed for (more) complex spatial objects that are usable in many different applications. With regard to *complex points*, we allow finite collections of single points as point objects (e.g., to gather the positions of all lighthouses in the U.S.). With regard to *complex lines*, we permit arbitrary, finite collections of one-dimensional curves, that is, spatially embedded networks possibly consisting of several disjoint connected components, as line objects (e.g., to model the ramifications of the Nile Delta). With regard to *complex regions*, the two main extensions relate to separations of the exterior (holes) and to separations of the interior (multiple components). For example, countries (like Italy) can be made up of multiple components (like the mainland and the offshore islands) and can have holes (like the Vatican). From a formal point of view, spatial data types should be closed under the geometric set operations *union*, *intersection*, and *difference*. This is not the case for the simple types and means that such an operation applied to two-point, two-line, and two-region objects, respectively, must yield a well-defined object and may not leave the corresponding type definition. For the user this means obtaining precisely defined and reliable operation results. Hence, a first goal of this article is to introduce and formalize spatial data types for complex points, complex lines, and complex regions. The formal basis is point set theory and point set topology.

With the increasing integration of *complex spatial data types* into geographical information systems (GIS) and into spatial extension packages of commercial database systems, an issue arises regarding the design, definition, and number of topological relationships operating on these complex types. For example, the predicate *inside* in the SQL query above is limited to simple spatial objects; it must be defined and made applicable to complex spatial objects. Hence, a second goal of this article is to explore and derive the possible topological relationships between all combinations of complex spatial data types on the basis of the well-known 9-intersection model. For this purpose, we draw up collections of constraints specifying conditions for valid topological relationships and satisfying the properties of *completeness* and *exclusiveness*. The property of completeness ensures a full coverage of all topological situations. This implies that we are able to express any topological relationship between two spatial objects. The property of exclusiveness ensures that two different relationships cannot hold for the same two spatial objects. This ensures the uniqueness of each topological relationship and avoids ambiguity and confusion. Topological relationships between complex spatial objects are also interesting from a simply theoretical point of view for the aforementioned disciplines and are important for an extension of spatiotemporal predicates [Erwig and Schneider 2002] from simple to complex spatiotemporal objects (moving objects) [Güting et al. 2000].

The satisfaction of the first two goals leads us to our third goal. With the increasing complexity of spatial data types and topological predicates operating on them, we also increase their expressiveness. As we will later see in this article, we will obtain a larger and more expressive variety of topological predicates for each combination of spatial data types. The currently known topological predicates between simple spatial objects are all contained in our predicate collections. The way of formulating topological SQL queries, however, remains unchanged, which is advantageous for the user.

The resulting large numbers of predicates for each type combination make them difficult to handle for the user. Hence, a fourth goal of this article is to provide the user with the concepts of *topological cluster predicates* and *topological predicate groups*. These permit the user to reduce the number of predicates to be dealt with in a user-defined and/or application-specific manner.

The remainder of the article is organized as follows: Section 2 discusses related work regarding spatial objects and topological relationships. Section 3 formalizes the spatial data model for which topological relationships will be investigated. Section 4 explains the general strategy for deriving topological relationships from the 9-intersection model. In Section 5 all topological relationships between spatial objects of the same type are analyzed. Section 6 deals with topological relationships between spatial objects of different types. Section 7 introduces the concepts of topological cluster predicates and topological predicate groups and shows their possible specification by means of the DDL of a query language. Finally, Section 8 draws some conclusions and discusses future work.

2. RELATED WORK

In this section we discuss related work about spatial objects as the operands of topological relationships (predicates) (Section 2.1) and about topological relationships themselves (Section 2.2).

2.1 Spatial Objects

In the past, numerous data models and query languages for spatial data have been proposed with the aim of formulating and processing spatial queries in databases and GIS (e.g., Egenhofer [1994], Güting [1988], Güting and Schneider [1995], Orenstein and Manola [1988], Roussopoulos et al. [1988], and Schneider [1997]). *Spatial data types* (see [Schneider 1997] for a survey) like *point*, *line*, or *region* are the central concept of these approaches. They provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. Topological predicates operate on instances of these data types, called *spatial objects*. So far, only simple object structures like single points, continuous lines, and simple regions have been specified and used as arguments of topological predicates.

In the Introduction we have motivated the need for complex spatial objects. So far only a few models [Clementini and Di Felice 1996; ESRI 1995; Güting and Schneider 1993, 1995; OGC 1999, 2001; Worboys and Bofakos 1993] have been developed for complex spatial objects. The methods in Clementini and Di Felice [1996], Güting and Schneider [1993, 1995], and Worboys and Bofakos [1993] are the only formal approaches; they all share a number of structural features with our type specifications. The approach in Worboys and Bofakos [1993] provides the concept of "generic area" but does not offer concepts for more complex lines and points. Generic areas are structurally similar to our complex regions. But their definition was different and based on a labeled tree representation. The root node represents the region object. All nonroot nodes are labeled by names of *atoms*, which are subsets of $\mathbb{R}^2$ and topologically equivalent to closed discs. For each node of the tree, its child nodes form a *base area*. A base area is a set of atoms so that each pair of atoms has a finite, possibly empty intersection and the composite object has no holes. Further, each nonroot node contains all its child nodes spatially, and the intersection of a child node with its father node is finite. This model is capable of representing areal objects to any finite level where each level is described by a set of base areas. We can describe this also as a "horizontal" object view. Our region model, however, is based on the concept of "face," which offers a "vertical" view on region components. A face represents a closed disc that can contain one or more closed discs as holes. Thus, it extends only over two levels. All faces of a region object are conceptually at the same level. This holds also for a face located within a hole of another face.

The approach in Clementini and Di Felice [1996] allows self-intersecting and self-touching lines, and the object representations resulting from its definitions are not necessarily unique. We disallow self-intersecting and self-touching spatial objects in general since besides ambiguous object representations they do not arise in spatial applications. In our opinion, these topological constraints should be part of the type definitions, and it should not be the user's burden to check them. Further, we offer a "structured" and an "unstructured" view on spatial objects (see Section 3).

The definitions of complex spatial data types in [Güting and Schneider 1993, 1995] are based on a finite geometric domain (so-called *realm*) whereas we define our data types in the infinite Euclidean plane. From a structural perspective, they are quite similar to ours so that they can be used as an implementation of our model. In particular, they also forbid self-intersections of spatial objects.

The OpenGIS Consortium (OGC) has proposed geometric structures, called *simple features*, in their OGC Abstract Specification [OGC 1999] and in their Geography Markup Language (GML) [OGC 2001], which is an XML encoding for the transport and storage of geographic information. These geometric structures are only described informally and called *MultiPoint*, *MultiLineString*, and *MultiPolygon*. Spatial objects may have several components, and region objects may have holes. Different components of the same spatial object may overlap. This specification is going to become the industry standard for spatial data types.

Another widely accepted spatial data type specification is provided by ESRI's Spatial Database Engine (SDE) [ESRI 1995]. The SDE also permits multicomponent spatial objects and regions with holes. But a formal type definition is missing. Several database vendors have more or less integrated SDE functionality into their spatial extension packages through extensibility mechanisms. Examples are the Informix Geodetic DataBlade [Informix 1997], the Oracle Spatial Cartridge [Oracle 1997], and DB2's Spatial Extender [Davis 1998].

$$\begin{pmatrix} A^\circ \cap B^\circ \neq \varnothing & A^\circ \cap \partial B \neq \varnothing & A^\circ \cap B^- \neq \varnothing \\ \partial A \cap B^\circ \neq \varnothing & \partial A \cap \partial B \neq \varnothing & \partial A \cap B^- \neq \varnothing \\ A^- \cap B^\circ \neq \varnothing & A^- \cap \partial B \neq \varnothing & A^- \cap B^- \neq \varnothing \end{pmatrix}$$

Fig. 1. The nine-intersection matrix.

2.2 Topological Relationships

Topological predicates between spatial objects in two-dimensional space belong to the most investigated topics of spatial data modeling and reasoning. Their importance and necessity has already been recognized for a long time [Abler 1987; Freeman 1975]. An important approach for characterizing them rests on the so-called 9-*intersection model* [Egenhofer and Herring 1990a] as an extension and generalization of the original 4-*intersection model* [Egenhofer 1989; Egenhofer and Herring 1990b]. Both models use point sets and point set topology as their formal framework. Based on the 9-intersection model, a complete collection of mutually exclusive topological relationships can be determined for each combination of simple spatial data types. The model is based on the nine possible intersections of boundary (∂A), interior (A°), and exterior (A^-) of a spatial object A with the corresponding components of another object B.[1] Each intersection is tested with regard to the topologically invariant criteria of emptiness and nonemptiness. The topological relationship between two spatial objects A and B can be expressed by evaluating the matrix in Figure 1.

For this matrix, $2^9 = 512$ different configurations are possible, from which only a certain subset makes sense depending on the *definition* and *combination* of spatial objects just considered. For each combination of spatial types, this means that each of its predicates is associated with a unique intersection matrix so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and nonemptiness. Topological relationships that have been investigated so far are restricted in the sense that their argument objects are not allowed to have a very general structure. It is just the objective of this article to give very general and versatile definitions of spatial objects and to identify the topological relationships between them.

Topological relationships have been first investigated for simple regions [Clementini et al. 1993; Egenhofer 1989; Egenhofer and Franzosa 1991; Egenhofer and Herring 1990b]. For two simple regions, eight meaningful (out of 512 possible) configurations have been identified which lead to the well known eight predicates called *disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, and *coveredBy* (Figure 2).

The 4-intersection/9-intersection model on simple regions has been extended and refined in a number of ways especially with further topological invariants (like the dimension of the intersection components, their types (touching, crossing), the number of components) to discover more details about topological relationships (e.g., Clementini et al. [1993]; Egenhofer and Franzosa [1995]). A generalization of the aforementioned eight topological predicates to complex regions can be found in [Schneider 2001]; their implementation is described in

[1]The 4-intersection model only considers the interior and the boundary of each spatial object.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

disjoint *meet* *overlap* *equal*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

inside *contains* *covers* *coveredBy*

Fig. 2. The eight topological predicates for two simple regions.

Table I. Numbers of Topological Predicates Between Two Simple Spatial Objects

	Simple Point	Simple Line	Simple Region
Simple point	2	3	3
Simple line	3	33	19
Simple region	3	19	8

[Schneider 2002]. In this article, we are interested in exploring the complete collection of purely topological predicates on complex regions.

For two simple lines, 33 topological relationships [Clementini and Di Felice 1998; Egenhofer 1993; Egenhofer and Herring 1990a] can be found. We aim at identifying them for complex lines. Topological predicates between simple points are trivial: either two simple points are *disjoint* or they are *equal*. We will consider predicates between complex points as finite collections of single points. A simple point can be located *on* one of the endpoints of a simple line, *in* the interior of a simple line, or it can be *disjoint* from a simple line. We are interested in the relationships between complex points and complex lines. For a simple point and a simple region, we obtain the three predicates *disjoint*, *meet*, and *inside*. We will identify the relationships between complex points and complex regions. For a simple line and a simple region, 19 topological relationships [Egenhofer and Mark 1995] can be distinguished. Here we are interested in the nature and number of topological relationships between a complex line and a complex region. Table I summarizes the results obtained so far.

It is surprising that topological predicates on complex spatial objects have so far not been defined. But the definition of these predicates is particularly important for spatial query languages that aim at integrating complex spatial objects like regions having holes and separations, or lines being multicomponent networks. Behr and Schneider [2001], Clementini et al. [1995], and Egenhofer et al. [1994] have so far contributed to a definition of topological relationships for more complex regions. In Clementini et al. [1995]; the so-called TRCR (Topological Relationships for Composite Regions) model only allows sets of disjoint simple regions without holes. Topological relationships between these composite regions are defined in an ad hoc manner and are not systematically derived from the underlying model. Further, the model is only related to but not directly based on the 9-intersection model. In Egenhofer et al. [1994], topological relationships of simple regions with holes were considered. Unfortunately, multipart regions are not permitted. While the authors took into account the number of components (areas without holes, holes) of two regions and the large number of topological relationships between all component pairs of both regions, we

(a) (b) (c)

Fig. 3. Examples of a complex point object (a), a complex line object (b), and a complex region object (c).

pursue a global approach that is independent of the number of components. Hence, a further goal of this article is to provide an integrated treatment of holes and separations for regions and to define topological predicates on complex regions in a systematic way. Our work in [Behr and Schneider 2001] already gave the rough idea of our mechanism for deriving topological relationships for complex regions.

Besides the 33 topological relationships between two simple lines, an additional 24 relationships exist for more complex lines which are allowed to be a connected graph without loops and without self-intersections [Egenhofer and Herring 1990a]. We generalize this approach in the sense that we also permit complex lines to have loops and to consist of several graph components.

Table I also indicates that the numbers of topological predicates can already become quite large for topological predicates between simple spatial objects so that the predicates are difficult to handle by the user. A further increase of the number of predicates has even to be expected for complex spatial objects. Hence, a grouping or clustering of these predicates can be meaningful and support user-friendliness. The calculus-based method in [Clementini et al. 1993] is an example of an approach that groups all its possible cases into a few meaningful topological relationships. The grouping is performed on the basis of the emptiness and nonemptiness of component intersections, inclusion and noninclusion of one object in another object, and the dimension of component intersections. In contrast to this, our concept of *predicate clustering* will exclusively rest on the emptiness and nonemptiness of component intersections.

3. COMPLEX SPATIAL OBJECTS

This section defines the underlying spatial data model for our topological predicates. We strive for a very general, abstract definition of complex spatial objects (see Figure 3) in the Euclidean plane $\mathbb{R}^2$. The task is to determine those point sets that are admissible for complex-point (Section 3.1), complex-line (Section 3.2), and complex-region (Section 3.3) objects. For complex lines and complex regions, we give both an "unstructured" and a "structured" definition; for complex points both definitions coincide. The unstructured definition purely determines the point set of a line or region. The structured definition gives a unique representation and emphasizes the component view of a spatial object. A complex point may include several points, a complex line may be a spatially embedded network possibly consisting of several components, and a complex region may be a multipart region possibly with holes. The formal framework purely employs point set theory and point set topology [Gaal 1964] since we disregard shape aspects and metric properties and focus on the study of topological relationships. For each spatial data type, we specify the topological notions of *boundary*, *interior*, *exterior*, and *closure* since these notions are later needed for the specification of topological relationships.

3.1 Complex Points

A value of type *point* is defined as a finite set of isolated points in the plane. Finiteness of the number of components is required in all parts of the spatial data model since we are only able to handle finite collections of spatial objects in geographic applications.

Definition 3.1. The spatial data type *point* is defined as

$$point = \{P \subset \mathbb{R}^2 \mid P \text{ is finite}\}.$$

We call a value of this type *complex point*. If $P \in point$ is a singleton set, that is, $|P| = 1$, P is denoted as a *simple point*.

In other words, type *point* (Figure 3(a)) contains all finite sets of the power set of $\mathbb{R}^2$. In particular, the empty set, which is the identity of geometric union, is admitted since it can be the result of a geometric operation, for example, if a point object has nothing in common with another point object in a geometric intersection operation.

Since we intend to later apply the 9-intersection model to complex points, we have to give a definition for the topological notions of boundary, interior, and exterior of a complex point. For a simple point p, we specify $\partial p = \varnothing$ and $p^\circ = p$, which is the commonly accepted definition. For a complex point $P = \{p_1, \ldots, p_n\}$, we then obtain $\partial P = \varnothing$, $P^\circ = \bigcup_{i=1}^n p_i^\circ$, and $P^- = \mathbb{R}^2 - (\partial P \cup P^\circ) = \mathbb{R}^2 - P^\circ$. The closure $\overline{P}$ is given as $\overline{P} = \partial P \cup P^\circ = P^\circ$.

3.2 Complex Lines

Before we start with a definition for complex lines (Figure 3(b)), we need a few definitions of some well-known and needed topological concepts. We assume the existence of the Euclidean distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $d(p, q) = d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. With the notion of distance, we can now proceed to define what is meant by a *neighborhood* of a point in $\mathbb{R}^2$.

Definition 3.2. Let $q \in \mathbb{R}^2$ and $\epsilon \in \mathbb{R}^+$. The set $N_\epsilon(q) = \{p \in \mathbb{R}^2 \mid d(p, q) < \epsilon\}$ is called the *open neighborhood of radius ϵ and center q*. The set $\overline{N}_\epsilon(q) = \{p \in \mathbb{R}^2 \mid d(p, q) \leq \epsilon\}$ is called the *closed neighborhood of radius ϵ and center q*. Any open (closed) neighborhood with center q is denoted by $N(q)$ ($\overline{N}(q)$). Any neighborhood (closed or open) with center q is denoted by $N^*(q)$.

We are now able to define the notion of a *continuous mapping* which preserves neighborhood relations between mapped points in two spaces of the plane. Hence, the property of continuity of this mapping ensures the maintenance of the closure and connectivity of the mapping domain for its image. These mappings are also called *topological transformations* and include translation, rotation, and scaling.

Definition 3.3. Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}^2$. Then f is said to be *continuous at a point* $x_0 \in X$ if, given an arbitrary number $\epsilon > 0$, there exists a number

$\delta > 0$ (usually depending on ϵ) such that for every $x \in N_\delta(x_0) \cap X$ we obtain that $f(x) \in N_\epsilon(f(x_0))$. The mapping f is said to be *continuous on* X if it is continuous at every point of X.

For a function $f : X \to Y$ and a set $A \subseteq X$, we introduce the notation $f(A) = \{f(x) \,|\, x \in A\}$. Definition 3.3 already enables us to give an unstructured definition for complex lines as the union of the images of a finite number of continuous mappings. Note that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 3.4. The spatial data type *line* is defined as

$$
\begin{aligned}
line = \{L \subset \mathbb{R}^2 \,|\, & \text{(i)} \quad L = \textstyle\bigcup_{i=1}^{n} f_i([0,1]) \text{ with } n \in \mathbb{N}_0 \\
& \text{(ii)} \;\; \forall\, 1 \le i \le n : f_i : [0,1] \to \mathbb{R}^2 \text{ is a continuous mapping} \\
& \text{(iii)} \; \forall\, 1 \le i \le n : |f_i([0,1])| > 1\}.
\end{aligned}
$$

We call a value of this type *complex line*.

The first condition also allows a line object to be the empty set ($n = 0$ in Definition 3.4). The third condition avoids degenerate line objects consisting only of a single point.

For a structured definition, we separate the point set of a complex line into appropriate components. We first consider a *single-component line* as the image of a single continuous mapping and distinguish several special cases.

Definition 3.5. Let $l \subset \mathbb{R}^2, l \neq \emptyset$.

(i) l is a *single-component line* $\Leftrightarrow$ $l = f([0,1])$ where $f : [0,1] \to \mathbb{R}^2$ is a continuous mapping and $|f([0,1])| > 1$;

(ii) l is a *simple line* $\Leftrightarrow$ l is a single-component line $\wedge$ $\forall a, b \in [0,1], a \neq b : f(a) \neq f(b)$;

(iii) l is a *self-touching line* $\Leftrightarrow$ l is a single-component line $\wedge$ $\exists a \in \{0,1\}\ \exists b \in\ ]0,1[\ : f(a) = f(b)$;

(iv) l is a *closed line* $\Leftrightarrow$ l is a single-component line $\wedge$ $f(0) = f(1)$;

(v) l is a *self-intersecting line* $\Leftrightarrow$ l is a single-component line $\wedge$ $\exists a, b \in\ ]0,1[, a \neq b : f(a) = f(b)$;

(vi) l is a *non-self-intersecting line* $\Leftrightarrow$ l is a single-component line $\wedge$ $\forall a, b \in\ ]0,1[, a \neq b : f(a) \neq f(b) \wedge$ $\forall a \in \{0,1\}\ \forall b \in\ ]0,1[\ : f(a) \neq f(b)$.

The values $f(0)$ and $f(1)$ are called the *end points* of l.

Intuitively, a single-component line models all curves that can be drawn on a sheet of paper from a starting point to an end point without lifting the pen. Definitions 3.5 (ii) to 3.5 (vi) describe five special cases. We obtain a simple line if during the drawing process the pen does not meet an already occupied point of the line again. A non-self-intersecting line is similar to a simple line but it allows that the endpoints coincide and thus the line forms a loop. In a self-touching line, one of its endpoints touches its interior. In a closed line, its two endpoints coincide. In a self-intersecting line, the pen crosses an already

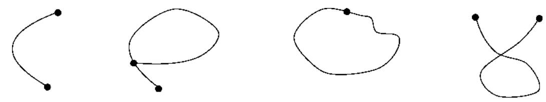


Fig. 4. A single-component line as a simple line (a), as a self-touching line (b), as a closed line (c), and as a self-intersecting line (d). The thick points indicate the end points of the different kinds of lines.

occupied interior point of the line but does not stop there. Note that the five special cases are not mutually exclusive. For instance, a single-component line can be self-intersecting and closed. Figure 4 shows some examples.

For a unique definition of complex lines, only non-self-intersecting lines are adequate since the other line types in Definition 3.5 lead to modeling or uniqueness problems. Obviously, self-touching and closed lines are too specialized. Simple lines are also too limited because they do not allow closed lines. Self-intersecting and single-component lines are too general because they allow multiple representations. The self-intersecting line in Figure 4(d) can, for example, also be represented by three continuous mappings, namely, one mapping for the closed part and one mapping for each remaining part. Non-self-intersecting lines, also denoted as *curves*, cover simple and closed lines and can serve as a "canonical" representation.

Definition 3.6. Let S be the set of all non-self-intersecting lines over $\mathbb{R}^2$. Two lines $l_1, l_2 \in S$ with corresponding continuous mappings f_1 and f_2 are called *quasidisjoint* if, and only if, $(\forall a, b \in]0,1[\, : f_1(a) \neq f_2(b)) \wedge \neg((f_1(0) = f_2(0) \wedge f_1(1) = f_2(1)) \vee (f_1(0) = f_2(1) \wedge f_1(1) = f_2(0)))$. They *meet* in an end point p if, and only if, they are quasi-disjoint and $\exists\, a, b \in \{0, 1\} : f_1(a) = p = f_2(b)$.

Note that due to the uniqueness constraint the definition of *quasidisjoint* forbids that two non-self-intersecting lines form a loop.

Next, we introduce the concept of a *block*, which is used in Definition 3.8 to specify a connected component of a complex line. Such a connected component is represented as a spatially embedded planar graph, in which, again due to the uniqueness constraint, an end point is either the end point of only a single non-self-intersecting line or shared by more than two non-self-intersecting lines. Intuitively, this definition means that in Figure 3(b) we have to draw an explicit end point at each intersection point.

Definition 3.7. The set B of *blocks* over S is defined as

$$
\begin{aligned}
B = \{\textstyle\bigcup_{i=1}^{m} l_i \mid\ & \text{(i) } m \in \mathbb{N}, \forall\, 1 \leq i \leq m : l_i \in S \\
& \text{(ii) } \forall\, 1 \leq i < j \leq m : l_i \text{ and } l_j \text{ are quasidisjoint} \\
& \text{(iii) } m > 1 \Rightarrow \forall\, 1 \leq i \leq m\ \exists\, 1 \leq j \leq m,\ i \neq j : l_i \text{ and } l_j \text{ meet} \\
& \text{(iv) } \forall\, p \in \textstyle\bigcup_{i=1}^{m} \{f_i(0), f_i(1)\} : card(\{f_i \mid 1 \leq i \leq m \wedge \\
& \quad f_i(0) = p \wedge f_i(1) = p\}) \neq 0 \\
& \quad \vee\, card(\{f_i \mid 1 \leq i \leq m \wedge (f_i(0) = p \wedge f_i(1) \neq p) \vee \\
& \qquad (f_i(0) \neq p \wedge f_i(1) = p)\}) \neq 2\}.
\end{aligned}
$$

Two blocks $b_1, b_2 \in B$ are *disjoint* if, and only if, $b_1 \cap b_2 = \varnothing$.

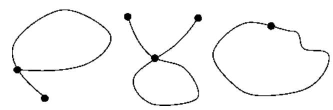


Fig. 5. A complex line. The thick points indicate end points of non-self-intersecting lines.

Condition (iv) ensures uniqueness of representation and disallows end points which have exactly two different emanating segments. We are now able to give the structured definition for complex lines.

Definition 3.8. The spatial data type *line* is defined as

$$line = \{\bigcup_{i=1}^{n} b_i \mid \text{(i) } n \in \mathbb{N}_0, \forall\, 1 \leq i \leq n : b_i \in B \\ \text{(ii) } \forall\, 1 \leq i < j \leq n : b_i \text{ and } b_j \text{ are disjoint}\}.$$

We call a value of this type *complex line*.

A line object is the empty object (empty set) if $n = 0$ in Definition 3.8. Figure 5 shows a complex line consisting of six continuous mappings according to the structured view. The boundary of a complex line L is the set of its end points minus those end points that are shared by several non-self-intersecting lines. The shared points belong to the interior of a complex line. Based on Definition 3.4, let $E(L) = \bigcup_{i=1}^{n}\{f_i(0), f_i(1)\}$ be the set of end points of all non-self-intersecting lines. We obtain

$$\partial L = E(L) - \{p \in E(L) \mid card(\{f_i \mid 1 \leq i \leq m \ \wedge\ f_i(0) = p\}) + \\ card(\{f_i \mid 1 \leq i \leq m \ \wedge\ f_i(1) = p\}) \neq 1\}.$$

Let $L \neq \varnothing$. It is possible that ∂L is empty. The closure $\overline{L}$ of L is the set of all points of L including the end points. Therefore $\overline{L} = L$ holds. For the interior of L we obtain $L^{\circ} = \overline{L} - \partial L = L - \partial L \neq \varnothing$, and for the exterior we get $L^{-} = \mathbb{R}^2 - L$, since $\mathbb{R}^2$ is the embedding space.

3.3 Complex Regions

Our definition of complex regions is based on point set theory and point set topology [Gaal 1964]. Regions are embedded into the two-dimensional Euclidean space $\mathbb{R}^2$ and modeled as special infinite point sets. We briefly introduce some needed concepts from point set topology in $\mathbb{R}^2$.

Definition 3.9. Let $X \subseteq \mathbb{R}^2$ and $q \in \mathbb{R}^2$. q is an *interior point* of X if there exists a neighborhood N^* such that $N^*(q) \subseteq X$. q is an *exterior point* of X if there exists a neighborhood N^* such that $N^*(q) \cap X = \varnothing$. q is a *boundary point* of X if q is neither an interior nor exterior point of X. q is a *closure point* of X if q is either an interior or boundary point of X.

The set of all interior points of X is called the *interior* of X and is denoted by X°. The set of all exterior points of X is called the *exterior* of X and is denoted by X^{-}. The set of all boundary points of X is called the *boundary* of X and is denoted by ∂X. The set of all closure points of X is called the *closure* of X and is denoted by $\overline{X}$.

Fig. 6. Examples of possible geometric anomalies of a region object.

A point q is a *limit point* of X if for every neighborhood $N^*(q)$ it holds that $(N^* - \{q\}) \cap X \neq \varnothing$. X is called an *open set* in $\mathbb{R}^2$ if $X = X^\circ$. X is called a *closed set* in $\mathbb{R}^2$ if every limit point of X is a point of X.

It follows from the definition that every interior point of X is a limit point of X. Thus, limit points need not be boundary points. The converse is also true. A boundary point of X need not be a limit point; it is then called an *isolated* point of X. For the closure of X, we obtain that $\overline{X} = \partial X \cup X^\circ$.

It is obvious that arbitrary point sets do not necessarily form a region. But open and closed point sets in $\mathbb{R}^2$ are also inadequate models for complex regions since they can suffer from undesired geometric anomalies (Figure 6). A complex region defined as an open point set runs into the problem that it may have missing lines and points in the form of cuts and punctures. At any rate, its boundary is missing. A complex region defined as a closed point set admits isolated or dangling point and line features. *Regular closed* point sets [Tilove 1980] avoid these anomalies.

Definition 3.10. Let $X \subseteq \mathbb{R}^2$. X is called *regular closed* if, and only if, $X = \overline{X^\circ}$.

The effect of the *interior* operation is to eliminate dangling points, dangling lines, and boundary parts. The effect of the *closure* operation is to eliminate cuts and punctures by appropriately supplementing points and to add the boundary. Closed neighborhoods (Definition 3.2), for example, are regular closed sets.

For the specification of the *region* data type, definitions are needed for bounded and connected sets.

Definition 3.11. (i) Two sets $X, Y \subseteq \mathbb{R}^2$ are said to be *separated* if, and only if, $X \cap \overline{Y} = \varnothing = \overline{X} \cap Y$. A set $X \subseteq \mathbb{R}^2$ is *connected* if, and only if, it is not the union of two nonempty separated sets. (ii) Let $q = (x, y) \in \mathbb{R}^2$. Then the *length* or *norm* of q is defined as $||q|| = \sqrt{x^2 + y^2}$. (iii) A set $X \subseteq \mathbb{R}^2$ is said to be *bounded* if there exists a number $r \in \mathbb{R}^+$ such that $||q|| < r$ for every $q \in X$.

We are now able to give an unstructured type definition for complex regions:

Definition 3.12. The spatial data type *region* is defined as

$$\textit{region} = \{R \subset \mathbb{R}^2 \mid \begin{array}{l} \text{(i) } R \text{ is regular closed} \\ \text{(ii) } R \text{ is bounded} \\ \text{(iii) The number of connected sets of } R \text{ is finite}\}. \end{array}$$

We call a value of this type *complex region*.

A region object can also be the empty object (empty set). In fact, this very "unstructured" definition models complex regions possibly consisting of several components and possibly having holes. But since the topological predicates of the 9-intersection model only work on simpler regions, we have to take a more fine-grained and structured view of regions. The structured definition of type *region* distinguishes simple regions, simple regions with holes, and complex regions.

Definition 3.13. A *simple region* is a bounded, regular closed set homeomorphic (i.e., topologically equivalent) to a closed neighborhood in $\mathbb{R}^2$.

This, in particular, means that a simple region has a connected interior, a connected boundary, and a single connected exterior. Hence, it does not consist of several components, and it does not have holes.

The concept of a hole is topologically not directly inferable since point set topology does not distinguish between "outer" exterior and "inner" exterior of a set. This requires an explicit and constructive definition of a region containing holes and a use of the topological predicates *meet*, *covers*, *coveredBy*, *contains*, and *disjoint* for simple regions, as they are defined by the 9-intersection model in Figure 2.

Definition 3.14. Let $\{F_0, \dots, F_n\}$ be a set of simple regions, and let $\pi : \{1, \dots, k\} \to \{1, \dots, n\}, k, n \in \mathbb{N}, k \le n$, be a total, injective mapping. The regular set $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ is called a *simple region with holes* or a *face*, and $F_1, \dots, F_n$ are called *holes* if, and only if,

(i) $\forall\, 1 \le i \le n : contains(F_0, F_i) \vee (covers(F_0, F_i) \wedge |F_0 \cap F_i| = 1)$;
(ii) $\forall\, 1 \le i < j \le n : disjoint(F_i, F_j) \vee (meet(F_i, F_j) \wedge |F_i \cap F_j| = 1)$;
(iii) $\nexists\, \{\pi(1), \dots, \pi(k)\} \subseteq \{1, \dots, n\} : covers(F_0, F_{\pi(1)}) \wedge$
$meet(F_{\pi(1)}, F_{\pi(2)}) \wedge \cdots \wedge meet(F_{\pi(k-1)}, F_{\pi(k)}) \wedge coveredBy(F_{\pi(k)}, F_0)$;
(iv) $\nexists\, \{\pi(1), \dots, \pi(k)\} \subseteq \{1, \dots, n\} : meet(F_{\pi(1)}, F_{\pi(2)}) \wedge$
$meet(F_{\pi(2)}, F_{\pi(3)}) \wedge \cdots \wedge meet(F_{\pi(k-1)}, F_{\pi(k)}) \wedge meet(F_{\pi(k)}, F_{\pi(1)})$.

Let *srh* be the set of all simple regions with holes.

Figure 7(c) gives an example of a face. Figure 7(d) shows its boundary, and Figure 7(e) its interior. The first two conditions allow a hole within a face to touch the boundary of F_0 or of another hole in at most a single point. This is necessary in order to achieve closure under the geometric operations *union*, *intersection*, and *difference* (see also Güting and Schneider [1993] and Schneider [1997]). For example, a (regularized) subtraction of a face A from a face B may lead to such a hole in B. On the other hand, to allow two holes to have a partially common border makes no sense because then adjacent holes could be merged into a single hole by eliminating the common border (similarly for adjacency of a hole with the boundary of F_0). The third condition prevents the formation of "open hole chains" where any two subsequent holes meet and both the first and the last holes touch F_0. The fourth condition prevents the formation of "closed hole chains" within the face where any two subsequent holes meet and both the first and the last holes meet. All four conditions together ensure uniqueness of representation, that is, there are no two different interpretations of the point set describing a face. Hence, a face is atomic and cannot be decomposed into

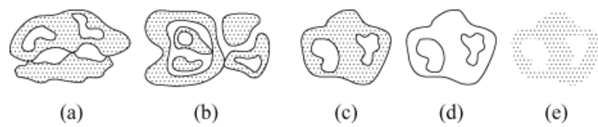


Fig. 7. Unique representation of a complex region with two faces where the upper face has two holes (a), a complex region with five faces (b), a simple region with two holes (c), its boundary (d), and its interior (e).

two or more faces. For example, the configuration shown in Figure 7(a) must be interpreted as two faces with two holes and not as a single face with four holes.

Let $F = F_0 - \bigcup_{i=1}^{n} F_i^\circ$ be a simple region with holes $F_1, \ldots, F_n$. Then the boundary of F is given as $\partial F = \bigcup_{i=0}^{n} \partial F_i$, and the interior of F is defined as $F^\circ = F_0^\circ - \bigcup_{i=1}^{n} F_i$.

We are now able to give the structured definition for complex regions.

Definition 3.15. The spatial data type *region* is defined as

$$
\begin{aligned}
\mathit{region} = \{R \subseteq \mathbb{R}^2 \mid \; & \text{(i) } R = \textstyle\bigcup_{i=1}^{n} F_i \text{ with } n \in \mathbb{N}_0 \\
& \text{(ii) } \forall\, 1 \leq i \leq n : F_i \in \mathit{srh} \\
& \text{(iii) } \forall\, 1 \leq i < j \leq n : F_i^\circ \cap F_j^\circ = \varnothing \\
& \text{(iv) } \forall\, 1 \leq i < j \leq n : \partial F_i \cap \partial F_j = \varnothing \vee \\
& \qquad |\partial F_i \cap \partial F_j| \text{ is finite}\}.
\end{aligned}
$$

We call a value of this type *complex region*.

A region object is the empty object (empty set) if $n = 0$ in Definition 3.15. Figure 7(b) shows an example of a region with five faces. The definition requires of a face to be disjoint to another face, or to meet another face in one or several single boundary points, or to lie within a hole of another face and possibly share one or several single boundary points with the boundary of the hole. Faces having common connected boundary parts with other faces or holes are disallowed. The argumentation is similar to that for the face definition.

Let $F = \bigcup_{i=1}^{n} F_i$ be a nonempty region with faces $\{F_1, \ldots, F_n\}$. Then the boundary of F is given as $\partial F = \bigcup_{i=1}^{n} \partial F_i$ ($\neq \varnothing$), and the interior of F is given as $F^\circ = \bigcup_{i=1}^{n} F_i^\circ = F - \partial F$ ($\neq \varnothing$). Further, we obtain $\overline{F} = \partial F \cup F^\circ = F$ and $F^- = \mathbb{R}^2 - \overline{F} = \mathbb{R}^2 - F$ ($\neq \varnothing$).

4. DERIVING TOPOLOGICAL RELATIONSHIPS FROM THE 9-INTERSECTION MODEL

The preceding section has provided structured and unstructured views of three kinds of complex spatial objects. An apparently promising approach to deriving topological relationships from them is to leverage the structured view of a spatial object. But the following expositions reveal that considering components leads to rather complicated and impractical models.

We demonstrate this by first considering two simple regions A and B with n and m holes, respectively. If we take into account the regions A and B *without* holes and call them A^* and B^*, respectively, the total number of topological relationships that can be specified between A^* and its holes with B^* and its holes amounts to $(n + m + 2)^2$ [Egenhofer et al. 1994]. It has also been shown in Egenhofer et al. [1994] that this number can be reduced to $mn + m + n + 1$. The problems of this approach are the dependency on the number of holes and the resulting large number of topological relationships.

We are confronted with a similar problem if we take another strategy and have a look on the topological relationships between two complex regions A and B with n and m faces, respectively, possibly with holes. Each face of A is in relationship with any face of B. This gives a total of $8^{n \cdot m}$ possible topological configurations if we take the eight topological relationships between two simple regions with holes, as they are specified in [Schneider 2001], as the basis. As a result, the total number of relationships between the faces of two complex regions depends on the numbers of faces, is therefore not bounded by a constant, and increases in an exponential way. This approach is obviously not manageable and thus not acceptable.

Hence, the comparison of structural elements of the objects with respect to their topological relationships does not seem to be an adequate and efficient method. On the contrary, this detailed investigation is usually not desired and thus even unnecessary. For instance, if two regions intersect (according to some definition), the number of intersecting face pairs, as long as it is greater than zero, is irrelevant since it does not influence the fact of intersection. Consequently, the analysis of topological relationships between two complex spatial objects requires a more general strategy.

Our strategy for the analysis of topological relationships between two complex spatial objects is simple and yet very general and expressive. Instead of applying the 9-intersection model to point sets belonging to simple spatial objects, we extend it to point sets belonging to complex spatial objects. Due to the special features of the objects (point, linear, areal properties), the embedding space (here: $\mathbb{R}^2$), the relation between the objects and the embedding space (e.g., it makes a difference whether we consider a point in $\mathbb{R}$ or in $\mathbb{R}^2$), and the employed spatial data model (e.g., discrete, continuous), a number of topological configurations cannot exist and have to be excluded. For each pair of complex spatial data types, our goal is to determine topological constraints or conditions that have to be satisfied. These serve as criteria for excluding all impossible configurations. The approach taken employs a proof technique called *proof-by-constraint-and-drawing*. It starts with the 512 possible matrices and is a two-step process:

(i) For each type combination we give the formalization of a collection of topological *constraint* rules for existing relationships in terms of the nine intersections. For each constraint rule we give reasons for its validity, correctness, and meaningfulness. The evaluation of each constraint rule gradually reduces the set of the currently valid matrices by all those matrices not fulfilling the constraint rule under consideration.

(ii) The existence of topological relationships given by the remaining matrices is verified by realizing prototypical spatial configurations in $\mathbb{R}^2$, that is, these configurations can be *drawn* in the plane.

Still open issues relate to the evaluation order, completeness, and minimality of the collection of constraint rules. Each constraint rule is a predicate that is matched with all intersection matrices under consideration. All constraint rules must be satisfied together so that they represent a conjunction of predicates. In other words, constraint rules are all formulated in conjunctive normal form. Since the conjunction (logical *and*) operator is commutative and associative, the *evaluation order* of the constraint rules is irrelevant; the final result is always the same.

The completeness issue is the following: for each type combination, we give a list of constraint rules which eliminate invalid intersection matrices. But we cannot be sure that these rules remove *all* invalid matrices. There could still be invalid matrices which are not covered by any of the specified constraints. Hence, the question arises for the *completeness* of the given collection of constraint rules. In step (ii) of the proof technique, we try to draw for each remaining matrix a prototypical spatial configuration. If we are able to do this, the given collection of constraint rules is *complete*.

The aspect of *minimality* addresses the possible redundancy of constraint rules. Redundancy can arise for two reasons. First, several constraint rules may be correlated in the sense that one of them is more general than the others, that is, it eliminates at least the matrices excluded by all the other, covered constraints. This can be easily checked by analyzing the constraint rules themselves and searching for the most nonrestrictive and common constraint rule. Even then the same matrix can be excluded by several constraint rules simultaneously. Second, a constraint rule can be covered by some combination of other constraint rules. This can be checked by a comparison of the matrix collection fulfilling all n constraint rules with the matrix collection fulfilling $n - 1$ constraint rules. If both collections are equal, then the omitted constraint rule was implied by the combination of the other constraint rules and is therefore redundant. In this article, we are not so much interested in the aspect of minimality since our goal is to identify the topologically invalid intersection matrices. We are willing to accept a certain (but small) degree of redundancy.

5. TOPOLOGICAL RELATIONSHIPS BETWEEN TWO COMPLEX SPATIAL OBJECTS OF EQUAL TYPE

First we analyze the topological relationships between two complex spatial objects which have the same type and thus share the same properties and dimension. This leads to the three type combinations *point/point* (Section 5.1), *line/line* (Section 5.2), and *region/region* (Section 5.3).

For each of these three type combinations we identify pairs (p_1, p_2) of different topological predicates p_1 and p_2 that are *converse* to each other. That is, $p_1(A, B) = p_2(B, A)$ where $A, B \in \alpha$ with $\alpha \in \{point, line, region\}$. Such a pair of converse predicates has the square intersection matrices M_1 and M_2 such that $M_1 = M_2^T$, where the superscript T denotes the transpose of a matrix.

5.1 Topological Relationships Between Two Complex Points

Topological relationships on complex point objects have so far not been explored in the literature, since the assumption has always been that a point object only consists of a single point and since the topological relationships between two single points are rather trivial (either disjointedness or equality). In the Introduction and in Section 2.1 we have motivated why more complex spatial objects are needed in general. For example, consider an operation *crossing* : *line* × *line* → *point* taking two line objects as operands, computing all single intersection points (common linear parts are ignored here), and collecting them in a single *point* object. Another example is a relation "terrorists" storing for each terrorist in a complex point object all refuges as point locations that the terrorist has taken. Then we can ask, for example, which terrorists used common refuges (overlap, equal). We can also ask for those terrorists that used all refuges of other terrorists (contains). Finally, we can determine those terrorists who never used the refuges of other terrorists (disjoint). This shows the necessity to deal with topological predicates on *complex* point objects.

We now present the constraint rules for two complex point objects A and B defined according to Section 3.1. Each constraint rule is first formulated colloquially and afterwards formalized by employing the nine intersections. Then a rationale is given explaining why the constraint rule makes sense and is correct. We presuppose that A and B are not empty, because topological relationships for empty operands are not meaningful.

LEMMA 5.1.1. *All intersections comprising an operand with a boundary operator yield the empty set, that is,*

$$\forall\, C \in \{A^\circ, \partial A, A^-\} : C \cap \partial B = \varnothing \ \wedge\ \forall\, D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing.$$

PROOF. According to the definition of a complex point $\partial A = \partial B = \varnothing$ holds. Hence, the intersection of the empty set with any other component yields the empty set. □

LEMMA 5.1.2. *The exteriors of two complex point objects always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. We know that $A \cup A^- = \mathbb{R}^2$ and $B \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $A = \mathbb{R}^2$, or (ii) $B = \mathbb{R}^2$, or (iii) $A \cup B = \mathbb{R}^2$. All three situations are impossible, since A, B, and $A \cup B$ are finite sets and $\mathbb{R}^2$ is an infinite set. Thus $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$, and $A \cup B \subset \mathbb{R}^2$ hold, and $(\mathbb{R}^2 - A) \cap (\mathbb{R}^2 - B) \neq \varnothing$. □

LEMMA 5.1.3. *Each nonempty part of a complex point intersects at least one nonempty part of the other complex point, that is,*

$$(\forall\, C \in \{A^\circ, A^-\} :\ C \cap B^\circ \neq \varnothing \ \vee\ C \cap B^- \neq \varnothing) \wedge$$
$$(\forall\, D \in \{B^\circ, B^-\} :\ A^\circ \cap D \neq \varnothing \ \vee\ A^- \cap D \neq \varnothing).$$

PROOF. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup B^- = \mathbb{R}^2$. That is, the complex point A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Since only nonempty object parts are considered, the interior and the exterior of A, respectively B, must hence intersect at least either the interior or the exterior or both parts of B, respectively A. □

Lemma 5.1.1 means that the second row and the second column of an intersection matrix only yield empty intersections so that we do not have to consider

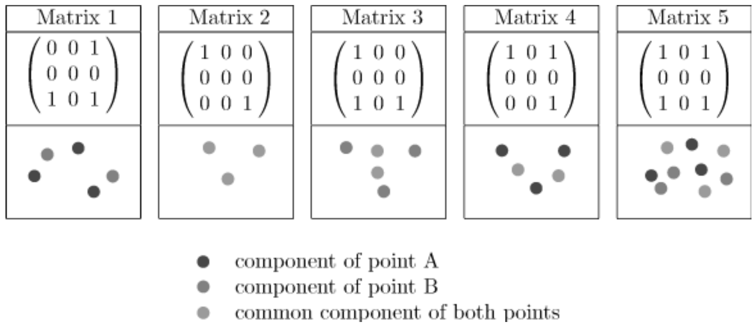


Fig. 8. The five topological relationships between two complex points.

them any further. Lemma 5.1.2 indicates that the exteriors of two point objects always intersect. This leads to a value 1 at position (3, 3) in each valid matrix. Lemma 5.1.3 says that in the first and third rows and in the first and third columns of a matrix at least one “corner” intersection must yield true so that we find the value 1 in the matrix there.

The application of the three constraint rules results in a reduction of the 512 possible intersection matrices to five remaining matrices, which describe the characteristic features of the existing topological relationships between complex points. The corresponding matrices and their geometric interpretations are given in Figure 8.

Due to the small number of topological relationships, with each matrix we can associate a name for the corresponding topological predicate. Matrix 1 describes the relationship *disjoint*, matrix 2 the relationship *equal*, matrix 3 the relationship *inside*, matrix 4 the relationship *contains*, and matrix 5 the relationship *overlap*. The relationships *inside* and *contains* are converse to each other. The two topological relationships *disjoint* and *equal* between two *simple* point objects are included in this collection of relationships and represented in matrix 1 and matrix 2, respectively.

Finally, we can summarize our result as follows:

THEOREM 5.1. *Based on the 9-intersection model, five different topological relationships can be identified between two complex* point *objects.*

PROOF. The argumentation is based on the proof-by-constraint-and-drawing method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.1.1, 5.1.2, and 5.1.3, reduce the number of the 512 possible intersection matrices to five matrices. The ability to draw prototypes of the corresponding five topological configurations (Figure 8) proves that the constraint rules are complete. □

5.2 Topological Relationships Between Two Complex Lines

Next, we discuss topological relationships between two nonempty, complex lines A and B defined according to Section 3.2. We pursue the same strategy as before and present constraint rules to filter out nonexistent topological configurations.

LEMMA 5.2.1. *The exteriors of two complex line objects always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. We know that $\overline{A} \cup A^- = A \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = B \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $A = \mathbb{R}^2$, or (ii) $B = \mathbb{R}^2$, or (iii) $A \cup B = \mathbb{R}^2$. These situations are all impossible, since A, B, and $A \cup B$ as bounded, one-dimensional shapes of finite length are unable to cover the unbounded, two-dimensional plane $\mathbb{R}^2$. □

Intuitively, the following constraint rule means that neither the first row nor the first column of a 3×3-intersection matrix may only contain zeros.

LEMMA 5.2.2. *The interior of a complex line object intersects either the interior, the boundary, or the exterior of the other line object, that is,*

$$\begin{aligned}&(A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap \partial B \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing) \wedge\\&(A^\circ \cap B^\circ \neq \varnothing \;\vee\; \partial A \cap B^\circ \neq \varnothing \;\vee\; A^- \cap B^\circ \neq \varnothing).\end{aligned}$$

PROOF. Assuming that the constraint rule is false. Then $(A^\circ \cap B^\circ = \varnothing \wedge A^\circ \cap \partial B = \varnothing \wedge A^\circ \cap B^- = \varnothing) \vee (A^\circ \cap B^\circ = \varnothing \wedge \partial A \cap B^\circ = \varnothing \wedge A^- \cap B^\circ = \varnothing)$. We show that the first argument of the disjunction (similar for the second argument) leads to a contradiction. It can be summarized as $A^\circ \cap (B^\circ \cup \partial B \cup B^-) = A^\circ \cap \mathbb{R}^2 = \varnothing$. This is a contradiction to the assumed nonemptiness of a *line* object requiring that $A^\circ \neq \varnothing$ and $A^\circ \cap \mathbb{R}^2 = A^\circ$, that is, $A^\circ \subset \mathbb{R}^2$. □

LEMMA 5.2.3. *If the boundary of a complex line object intersects the interior of another line object, its exterior also intersects the interior of the other line object, that is,*

$$\begin{aligned}&((\partial A \cap B^\circ \neq \varnothing \Rightarrow A^- \cap B^\circ \neq \varnothing) \;\wedge\; (A^\circ \cap \partial B \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing))\\\Leftrightarrow\;&((\partial A \cap B^\circ = \varnothing \;\vee\; A^- \cap B^\circ \neq \varnothing) \;\wedge\; (A^\circ \cap \partial B = \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing)).\end{aligned}$$

PROOF. Without loss of generality, let P be an endpoint of the boundary of A located in the interior of B. From P exactly one curve of A starts or ends. Either P divides a curve of B into two subcurves, or P is endpoint of more than one curve of B. Hence, in P at least two curves of B end. Since the curve of A can coincide with at most one of the curves of B, at least one of the curves of B must be situated in the exterior of A. □

LEMMA 5.2.4. *If the boundary of a complex line object intersects the exterior of another line object, its interior also intersects the exterior of the other line object, that is,*

$$\begin{aligned}&((\partial A \cap B^- \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing) \;\wedge\; (A^- \cap \partial B \neq \varnothing \Rightarrow A^- \cap B^\circ \neq \varnothing))\\\Leftrightarrow\;&((\partial A \cap B^- = \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing) \;\wedge\; (A^- \cap \partial B = \varnothing \;\vee\; A^- \cap B^\circ \neq \varnothing)).\end{aligned}$$

PROOF. For each point $p \in B^-$ we can find a neighborhood $N^*(p)$ such that $N^*(p) \subset B^-$. If $p \in \partial A$, in *each* neighborhood of p we must find points of A°, since a curve of A starts at p. Hence, interior points of A exist that intersect B^-. □

An evaluation of all 512 3 × 3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that 82 matrices satisfy these rules and thus represent the possible topological relationships between two complex lines. The matrices and their geometric interpretations are shown in Figure 9. The 33 topological relationships between simple lines [Clementini and Di Felice 1998; Egenhofer 1993; Egenhofer and Herring 1990a] are contained and correspond to the matrices with the numbers 4, 5, 8, 10, 12, 14, 19, 20, 23, 25–29, 36, 40, 42, 48, 50, 54, 56, 58, 60, 66–68, 72, 73, and 75–79. If we consider line objects as connected components, as it is also done in [Egenhofer and Herring 1990a], the additional 24 topological relationships are also contained and correspond to the intersection matrices with the numbers 6, 7, 13, 15, 16, 21, 22, 24, 30–32, 38, 51–53, 59, 61, 62, 70, 71, 74, and 80–82.

Among the 82 topological relationships, we find 32 pairs of converse relationships with the pairs (2, 3), (6, 7), (9, 17), (10, 19), (11, 18), (12, 20), (13, 21), (14, 23), (15, 22), (16, 24), (26, 27), (30, 31), (34, 43), (35, 46), (37, 49), (38, 52), (39, 63), (40, 66), (41, 69), (42, 72), (45, 47), (51, 53), (55, 64), (56, 67), (57, 65), (58, 68), (59, 70), (60, 73), (61, 71), (62, 74), (76, 77), (80, 81) of matrix numbers.

Finally, we can summarize our result as follows:

THEOREM 5.2. *Based on the 9-intersection model, 82 different topological relationships can be identified between two complex* line *objects.*

PROOF. The argumentation is based on the proof-by-constraint-and-drawing method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.2.1 to 5.2.4, reduce the number of the 512 possible intersection matrices to 82 matrices. The ability to draw prototypes of the corresponding 82 topological configurations (Figure 9) proves that the constraint rules are complete. □

5.3 Topological Relationships Between Two Complex Regions

In this section we identify those topological relationships that can be realized between two nonempty, complex regions A and B defined according to Section 3.3. We present constraint rules that exclude nonexistent topological configurations. Note that a *part* of a complex region denotes either its boundary, interior, or exterior and that all parts are nonempty (see Section 3.3).

LEMMA 5.3.1. *Each part of a complex region intersects at least one part of the other complex region, that is,*

$$(\forall C \in \{A^\circ, \partial A, A^-\} : C \cap B^\circ \neq \varnothing \ \vee \ C \cap \partial B \neq \varnothing \ \vee \ C \cap B^- \neq \varnothing) \wedge$$
$$(\forall D \in \{B^\circ, \partial B, B^-\} : A^\circ \cap D \neq \varnothing \ \vee \ \partial A \cap D \neq \varnothing \ \vee \ A^- \cap D \neq \varnothing).$$

PROOF. We know that $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$ and that $B^\circ \cup \partial B \cup B^- = \mathbb{R}^2$. That is, the complex region A, respectively B, together with its exterior forms a complete partition of the Euclidean plane. Hence, each part of A, respectively B, must intersect at least one part of B, respectively A. □

Since a row in the matrix represents the possible intersections of a part of A with all parts of B and since a column represents the possible

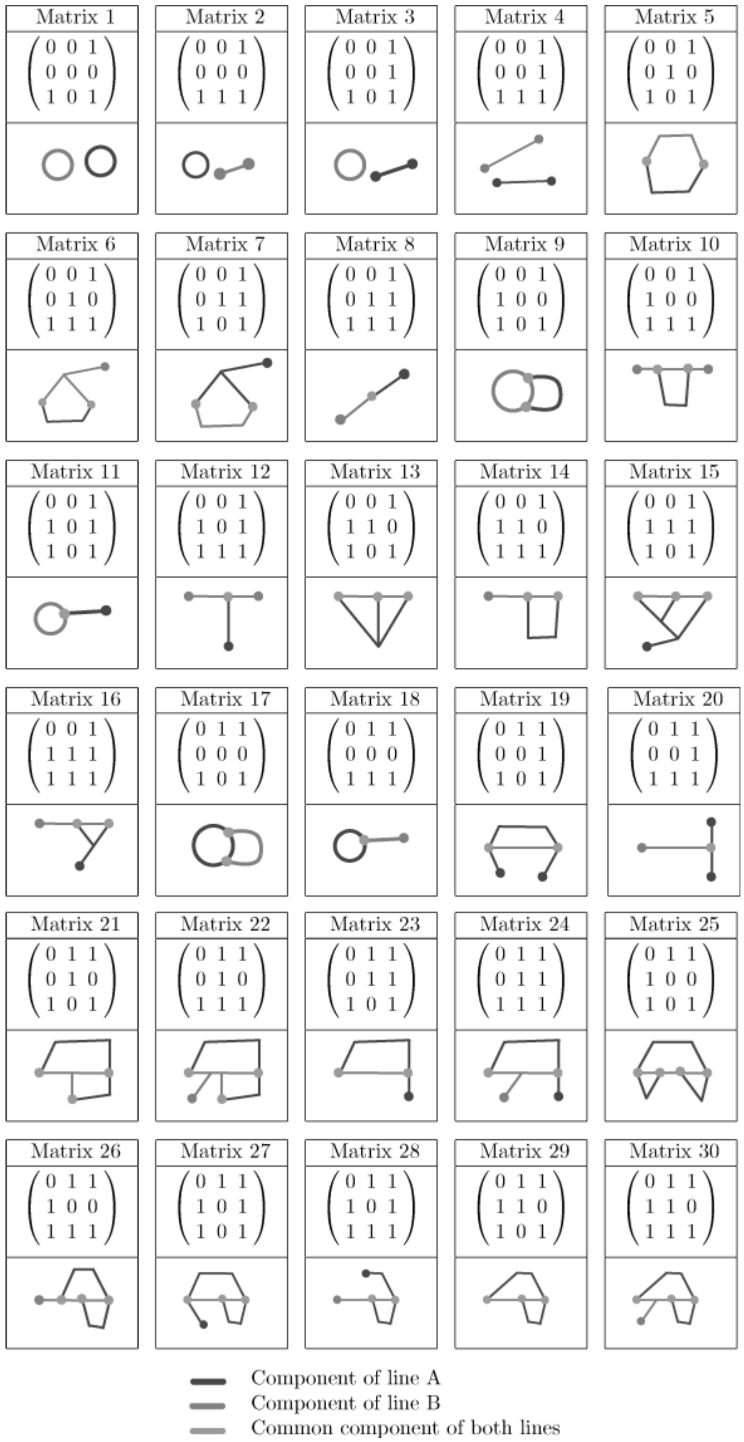


Fig. 9. The 82 topological relationships between two complex lines *(continues)*.

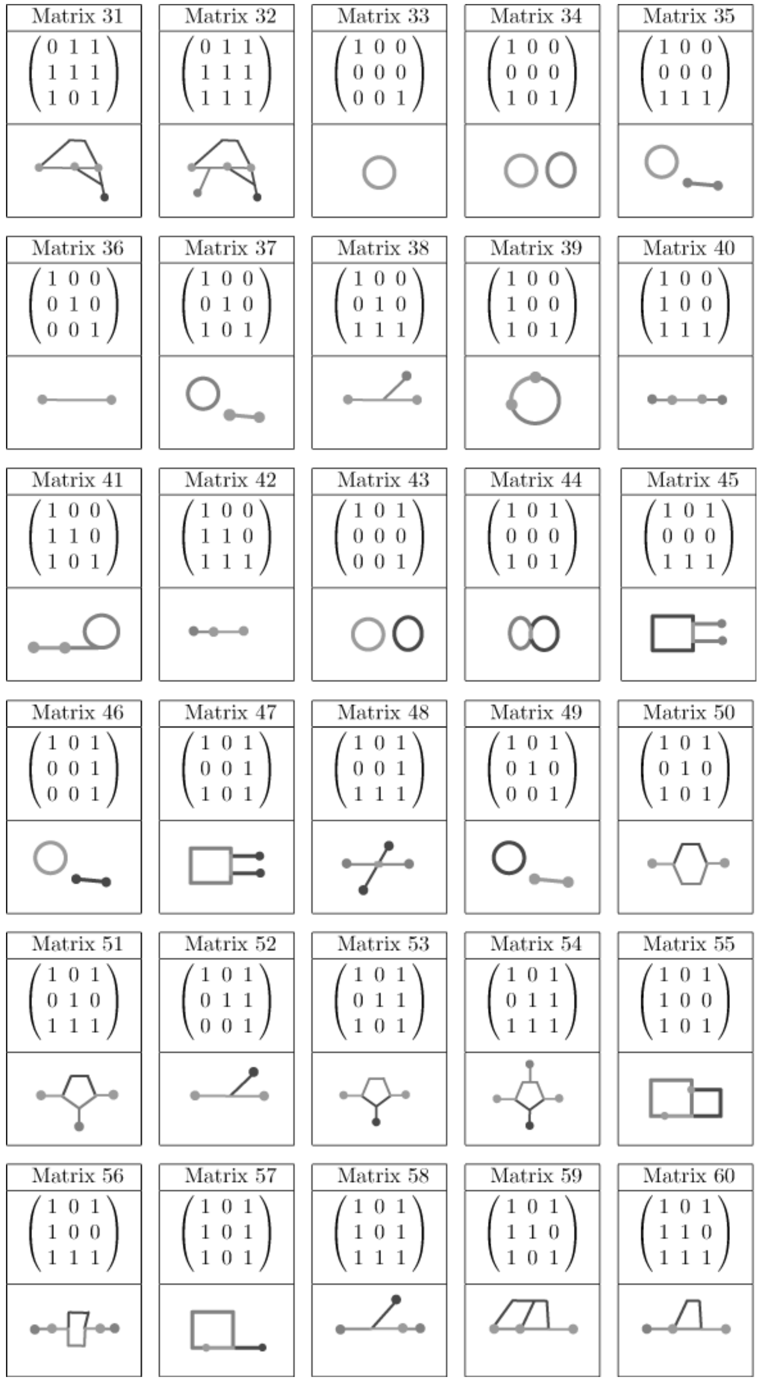


Fig. 9. *(Continued).*

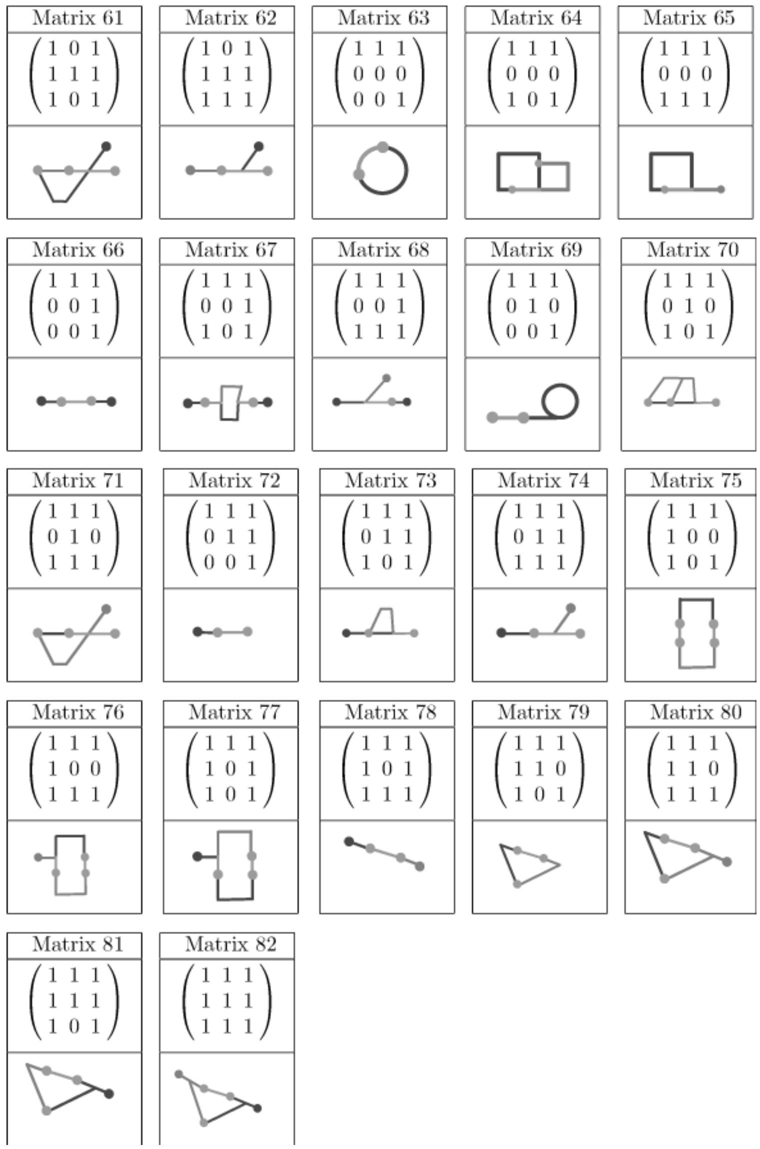


Fig. 9. *(Continued)*.

intersections of a part of B with all parts of A, in each row and in each column at least one intersection must yield true so that we find the value 1 in the matrix there.

In the following lemma we formulate a constraint rule on the basis of subsets. Since the 9-intersection model rests on the equality or inequality of the intersection of sets, we express the subset relationships in terms of the nine intersections and show the equivalence.

LEMMA 5.3.2. *Neither the interior nor the exterior of a complex region can be completely contained in or equal to the boundary of the other complex region, that is,*

$$
\begin{aligned}
& A^\circ \not\subseteq \partial B \;\wedge\; A^- \not\subseteq \partial B \;\wedge\; B^\circ \not\subseteq \partial A \;\wedge\; B^- \not\subseteq \partial A \\
\Leftrightarrow\; & (A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing) \;\wedge\; (A^- \cap B^\circ \neq \varnothing \;\vee\; A^- \cap B^- \neq \varnothing) \;\wedge \\
& (A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^- \cap B^\circ \neq \varnothing) \;\wedge\; (A^\circ \cap B^- \neq \varnothing \;\vee\; A^- \cap B^- \neq \varnothing).
\end{aligned}
$$

PROOF. This lemma follows from the fact that the dimension of a boundary with its linear structure is less than the dimensions of the interior and the exterior with their areal structures.

We show the equivalence considering the subexpression $A^\circ \not\subseteq \partial B \Leftrightarrow (A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing)$. For the other subexpressions, the argumentation is similar. We first show the "$\Rightarrow$" direction. If $A^\circ \not\subseteq \partial B$, then $A^\circ \cap (\mathbb{R}^2 - \partial B) = A^\circ \cap (B^\circ \cup B^-) \neq \varnothing$. Due to distributivity, $A^\circ \cap B^\circ \neq \varnothing \;\vee\; A^\circ \cap B^- \neq \varnothing$ follows. For the "$\Leftarrow$" direction, we argue like before but in the opposite direction. □

LEMMA 5.3.3. *The exteriors of two complex region objects always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. Let $r_1, r_2 \in \mathbb{R}^+$ be numbers according to Definition 3.11 (iii) such that the norms of A and B, respectively, are less than r_1 and r_2, respectively. Let $r_m = \max(r_1, r_2)$. Since we can find a point $p = (x, y) \in \mathbb{R}^2$ with $\sqrt{x^2 + y^2} > r_m$, this point is in the exterior of A and in the exterior of B. Hence, we obtain that $p \in A^- \cap B^-$, which proves the lemma. □

LEMMA 5.3.4. *The boundaries of two complex regions are equal if, and only if, the interiors and the exteriors, respectively, of both regions are equal, that is,*

$$
\begin{aligned}
& (\partial A = \partial B \Leftrightarrow A^\circ = B^\circ \;\wedge\; A^- = B^-) \\
\Leftrightarrow\; & (c \Leftrightarrow d) \Leftrightarrow ((c \;\wedge\; d) \;\vee\; (\neg c \;\wedge\; \neg d)) \text{ where} \\
& c = A^\circ \cap \partial B = \varnothing \;\wedge\; \partial A \cap B^\circ = \varnothing \;\wedge\; \partial A \cap \partial B \neq \varnothing \;\wedge \\
& \quad \partial A \cap B^- = \varnothing \;\wedge\; A^- \cap \partial B = \varnothing \text{ and} \\
& d = A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap \partial B = \varnothing \;\wedge\; A^\circ \cap B^- = \varnothing \;\wedge \\
& \quad \partial A \cap B^\circ = \varnothing \;\wedge\; A^- \cap B^\circ = \varnothing \;\wedge\; \partial A \cap B^- = \varnothing \;\wedge \\
& \quad A^- \cap \partial B = \varnothing \;\wedge\; A^- \cap B^- \neq \varnothing.
\end{aligned}
$$

PROOF. This very special constraint rule expresses that complex regions are uniquely characterized by their boundaries. This is ensured by the Jordan Curve Theorem [Gaal 1964]. □

LEMMA 5.3.5. *If the boundary of a complex region intersects the interior of the other complex region, both its interior and its exterior intersect the interior of the other region, that is,*

$$
\begin{aligned}
& ((\partial A \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^- \cap B^\circ \neq \varnothing)) \;\wedge \\
& \;(A^\circ \cap \partial B \neq \varnothing \Rightarrow (A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- \neq \varnothing))) \\
\Leftrightarrow\; & ((\partial A \cap B^\circ = \varnothing \;\vee\; (A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^- \cap B^\circ \neq \varnothing)) \;\wedge \\
& \;(A^\circ \cap \partial B = \varnothing \;\vee\; (A^\circ \cap B^\circ \neq \varnothing \;\wedge\; A^\circ \cap B^- \neq \varnothing))).
\end{aligned}
$$

PROOF. Let $p \in \partial A \cap B^\circ$. Since $p \in \partial A$, according to the boundary definition in Definition 3.9, for each neighborhood $N^*(p)$ it holds that $N^*(p) \cap A^\circ \neq \varnothing$ and $N^*(p) \cap A^- \neq \varnothing$. Since $p \in B^\circ$, there is a neighborhood $M^*(p)$ which is fully contained in B°. The nonempty intersection of one of the $N^*(p)$ and $M^*(p)$ proves the lemma. □

LEMMA 5.3.6. *If the boundary of a complex region intersects the exterior of the other complex region, both its interior and its exterior intersect the exterior of the other region, that is,*

$$
\begin{aligned}
&((\partial A \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \wedge \\
&\ (A^- \cap \partial B \neq \varnothing \Rightarrow (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing))) \\
\Leftrightarrow \ &((\partial A \cap B^- = \varnothing \ \vee \ (A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing)) \wedge \\
&\ (A^- \cap \partial B = \varnothing \ \vee \ (A^- \cap B^\circ \neq \varnothing \ \wedge \ A^- \cap B^- \neq \varnothing))).
\end{aligned}
$$

PROOF. The argumentation is similar as for the previous constraint. But we can also argue by employing the Jordan Curve Theorem [Gaal 1964]. Due to this theorem, on each side of the boundary of a region there is either the region's interior or exterior. On both sides of a line intersecting the exterior of this region, we find the exterior of the region. If the line is part of the boundary of another region, we obtain the intersection of both regions' exteriors and the intersection between the interior of the first region and the exterior of the other region. □

LEMMA 5.3.7. *Either the boundaries of two complex regions intersect, or the boundary of one region intersects the exterior of the other region, that is,*

$$\partial A \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing.$$

PROOF. Assume that the constraint rule is false. Then $\partial A \cap \partial B = \varnothing \ \wedge \ \partial A \cap B^- = \varnothing \ \wedge \ A^- \cap \partial B = \varnothing$. With Lemma 5.3.1, $\partial A \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap \partial B \neq \varnothing$ holds. Without loss of generality, let us consider a point $p \in A^\circ \cap \partial B$ and an infinite ray s emanating from p in an arbitrary direction. Since the component (face) of A containing p is bounded, s encounters the boundary of A in a point, say, q. This boundary could potentially intersect the exterior, the boundary, or the interior of B. But according to our assumption, the first two cases cannot hold so that q must lie inside the interior of B. We obtain a similar situation as before, except for the fact that now A and B change their roles. We continue to observe the course of s: the ray over and over again alternately encounters a point of $A^\circ \cap \partial B$ and then a point of $\partial A \cap B^\circ$. Since the ray can be prolonged arbitrarily, A and B must be unbounded or consist of infinitely many components. But this is a contradiction to the definition of the *region* data type. □

LEMMA 5.3.8. *If the interiors of two complex regions intersect, the interior of one region also intersects the boundary of the other region, or the regions' boundaries intersect, that is,*

$$
\begin{aligned}
&(A^\circ \cap B^\circ \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \\
\Leftrightarrow \ &(A^\circ \cap B^\circ = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing).
\end{aligned}
$$

PROOF. Without loss of generality, let us consider a component of the first region and a component of the second region with intersecting interiors. We have to distinguish three situations. First, if the interiors of both components are equal, also their boundaries are equal and hence intersect. Consequently, also the regions' boundaries intersect. Second, if the interiors of both components but not their boundaries intersect, one component is contained in the other. Since this is a proper containment (otherwise the boundaries would intersect), the boundary of one component must be inside the interior of the other component. Consequently, the interior of one region intersects the boundary of the other region. Third, if the interiors as well as the boundaries of the two components intersect, the remaining two conclusions of the constraint rule hold. □

LEMMA 5.3.9. *If the interior of a complex region intersects the exterior of the other region, either the interior of the first region intersects the boundary of the second region, or the boundary of the first region intersects the exterior of the second region, or both regions' boundaries intersect, that is,*

$$\begin{aligned} & ((A^\circ \cap B^- \neq \varnothing \Rightarrow (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)) \wedge \\ & (A^- \cap B^\circ \neq \varnothing \Rightarrow (\partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing))) \\ \Leftrightarrow \ & ((A^\circ \cap B^- = \varnothing \ \vee \ A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^- \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing) \wedge \\ & (A^- \cap B^\circ = \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing)). \end{aligned}$$

PROOF. If there is an intersection between the interior of a complex region and the exterior of the other complex region, a few different situations for each component causing the intersection can be distinguished. The first situation is that a component partially intersects the interior and the exterior of the other region. Then the boundary of the other region intersects the interior of the first region. The second situation is that the interior of a component lies completely inside the exterior of the other region. Several cases can now be distinguished. The first case is that also the boundary (and thus the entire component) lies inside and consequently intersects the exterior of the other region. The second case is that the boundary of a component lies only partially inside the exterior of the other region. Again we obtain an intersection between boundary and exterior. The third case is that the boundary of a component intersects the boundary of the other region. Note that the boundary of the component cannot cross the interior of the other region, since then the interior of the component would not be entirely within the exterior of the other region. □

An evaluation of all 512 3×3-intersection matrices against these nine constraint rules with the aid of a simple test program reveals that 33 matrices satisfy these rules and thus represent the possible topological relationships between two complex regions. The matrices and their geometric interpretations are shown in Figure 10. The eight topological relationships between simple regions [Egenhofer 1989; Egenhofer and Franzosa 1991; Egenhofer and Herring 1990b] are contained and correspond to the intersection matrices with the numbers 1, 4, 5, 7, 9, 19, 24, and 33.

Among the 33 topological relationships, we find 13 pairs of converse relationships with the pairs (2, 3), (6, 11), (7, 19), (8, 21), (9, 24), (10, 12), (14, 20), (15, 22), (16, 25), (17, 23), (18, 26), (27, 28), (31, 32) of matrix numbers.

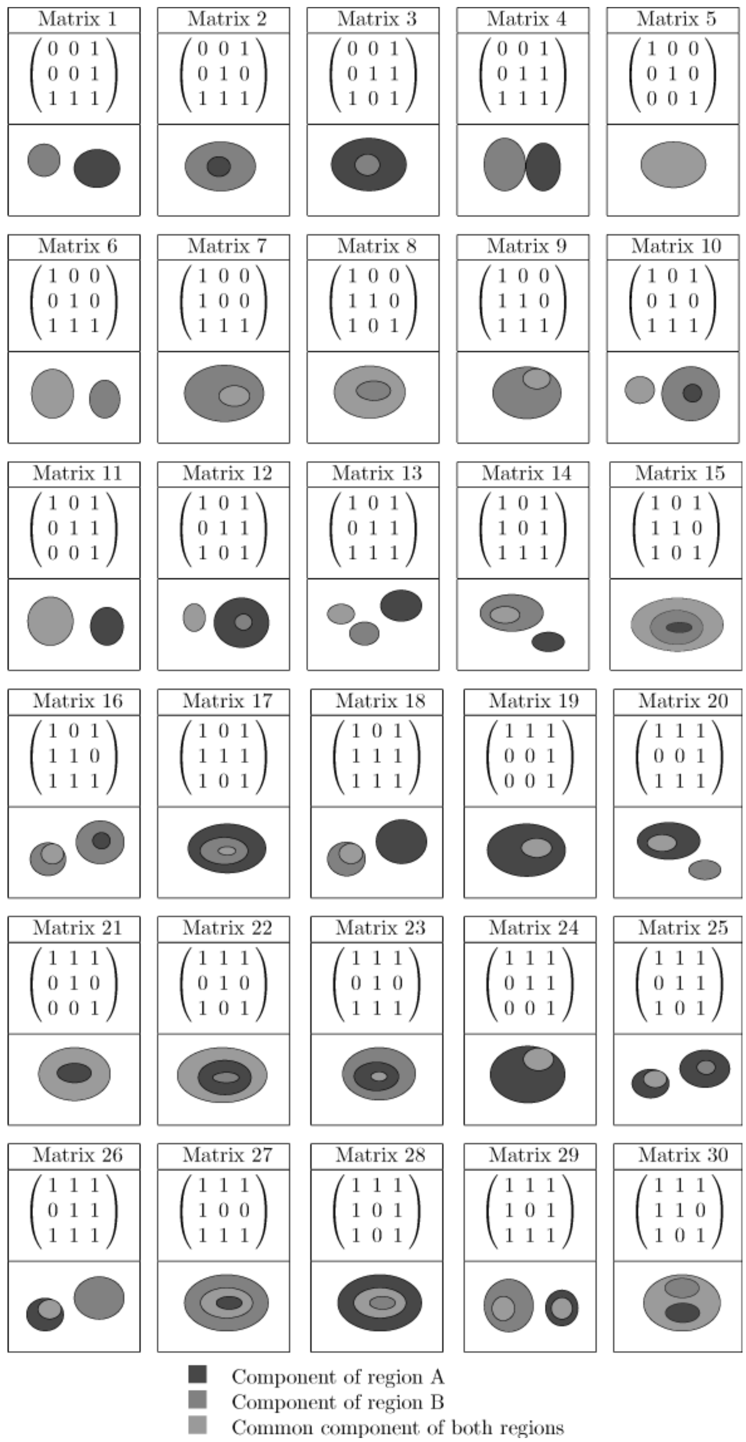


Fig. 10. The 33 topological relationships between two complex regions *(Continues)*.

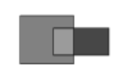
Matrix 31	Matrix 32	Matrix 33
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
		

Fig. 10. *(Continued)*.

Finally, we can summarize our result as follows:

THEOREM 5.3. *Based on the 9-intersection model, 33 different topological relationships can be identified between two complex* region *objects.*

PROOF. The argumentation is based on the *proof-by-constraint-and-drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 5.3.1 to 5.3.9, reduce the number of the 512 possible intersection matrices to 33 matrices. The ability to draw prototypes of the corresponding 33 topological configurations (Figure 10) proves that the constraint rules are complete. □

6. TOPOLOGICAL RELATIONSHIPS BETWEEN TWO COMPLEX SPATIAL OBJECTS OF DISTINCT TYPE

Next we analyze the topological relationships between two nonempty complex spatial objects that have *different* types and thus different properties and dimensions. This leads to the six type combinations *point/line* (Section 6.1), *point/region* (Section 6.2), *line/region* (Section 6.3), *line/point*, *region/point*, and *region/line*. The latter three type combinations are symmetric counterparts of the first three type combinations so that they do not have to be treated separately. It is obvious that a number of topological relationships cannot exist. For example, higher-dimensional objects can never be located inside lower-dimensional objects.

6.1 Topological Relationships Between a Complex Point and a Complex Line

Constraint rules for points and lines can also be based on the 9-intersection matrix. In the following, we assume that A is a nonempty object of type *point* and B is a nonempty object of type *line*.

LEMMA 6.1.1. *All intersections comprising an operand with the boundary operator of the complex point object yield the empty set, that is,*

$$\forall D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing.$$

PROOF. According to the definition of a complex point, $\partial A = \varnothing$ holds. Hence, the intersection of the empty set with any component of B yields the empty set. □

LEMMA 6.1.2. *The intersection of the interior of the complex line object and the exterior of the complex point object cannot be empty, that is,*

$$A^- \cap B^\circ \neq \varnothing.$$

PROOF. Assume that the constraint rule is wrong. Then $A^- \cap B^\circ = \varnothing$. Since we know that $\partial A = \varnothing$, we can conclude that $A^\circ = B^\circ$. This leads to a contradiction since the finite set representing the point object $A = A^\circ$ cannot cover the infinite set representing B°. □

LEMMA 6.1.3. *The exteriors of the complex point and the complex line always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. We know that $\overline{A} \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = \mathbb{R}^2$. Hence, $A^- \cap B^-$ is only empty if either (i) $\overline{A} = \mathbb{R}^2$, or (ii) $\overline{B} = \mathbb{R}^2$, or (iii) $\overline{A} \cup \overline{B} = \mathbb{R}^2$. The situations are all impossible, since A, B, and hence $A \cup B$ are bounded, but $\mathbb{R}^2$ is unbounded. □

LEMMA 6.1.4. *The interior of the complex point intersects at least one part of the complex line, that is,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee \ A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing.$$

PROOF. We know that $A^\circ \cup A^- = \mathbb{R}^2$ and that $\partial B \cup B^\circ \cup B^- = \mathbb{R}^2$. Since only nonempty object parts of both objects are considered, we obtain $A^\circ \cap \mathbb{R}^2 = A^\circ \cap (\partial B \cup B^\circ \cup B^-) \neq \varnothing$. This statement is equivalent to the constraint rule. □

An evaluation of all 512 3 × 3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that 14 matrices satisfy these rules and thus represent the possible topological relationships between a complex point and a complex line. The matrices and their geometric interpretations are shown in Figure 11. Between a *simple* point and a *simple* line we can distinguish three topological relationships. Either a simple point and a simple line are disjoint, or the simple point is located in one of the end points of the simple line, or the simple point is situated in the interior of the simple line. These topological predicates are contained in the 14 general ones and correspond to the matrices 2, 4, and 8, respectively.

Finally, we can summarize our result as follows:

THEOREM 6.1. *Based on the 9-intersection model, 14 different topological relationships can be identified between a complex* point *object and a complex* line *object.*

PROOF. The argumentation is based on the *proof-by-constraint-and-drawing* method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.1.1 to 6.1.4, reduce the number of the 512 possible intersection matrices to 14 matrices. The ability to draw prototypes of the corresponding 14 topological configurations (Figure 11) proves that the constraint rules are complete. □

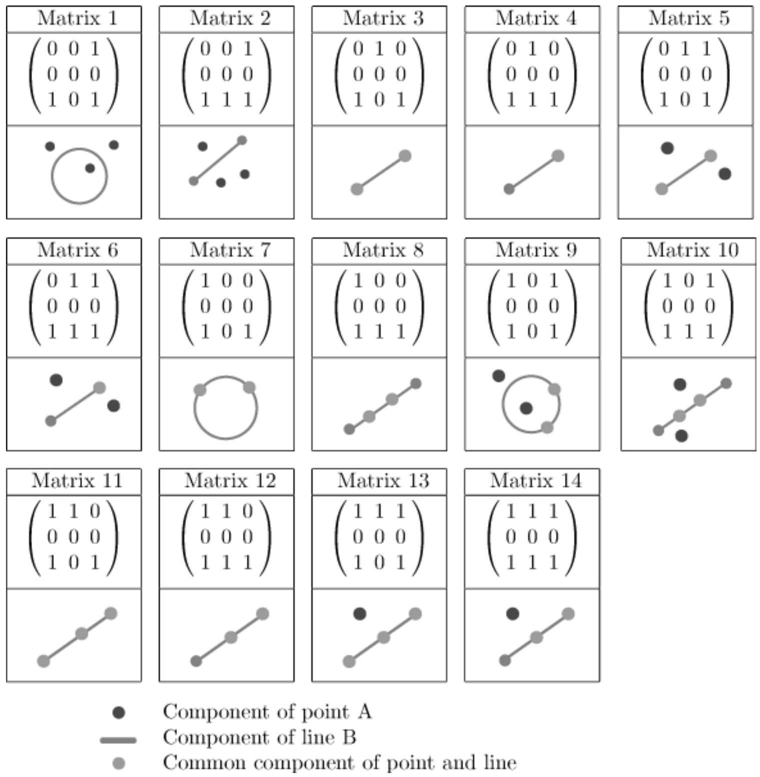


Fig. 11. The 14 topological relationships between two complex points and lines.

6.2 Topological Relationships Between a Complex Point and a Complex Region

The point-in-polygon test is probably the most classical representative of a topological predicate between a point and a polygon. In the following, we assume that A is a nonempty object of type *point* and B is a nonempty object of type *region*. The following constraint rules lead us to all topological predicates between a complex point object and a complex region object.

LEMMA 6.2.1. *The exteriors of the complex point and the complex region always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. The argumentation is the same as for Lemma 6.1.3. □

LEMMA 6.2.2. *All intersections comprising an operand with the boundary operator of the complex point object yield the empty set, that is,*

$$\forall\, D \in \{B^\circ, \partial B, B^-\} : \partial A \cap D = \varnothing.$$

PROOF. The argumentation is the same as for Lemma 6.1.1. □

LEMMA 6.2.3. *The interior and the boundary of the complex region object intersect the exterior of the point object, that is,*

$$A^- \cap B^\circ \neq \varnothing \;\wedge\; A^- \cap \partial B \neq \varnothing.$$

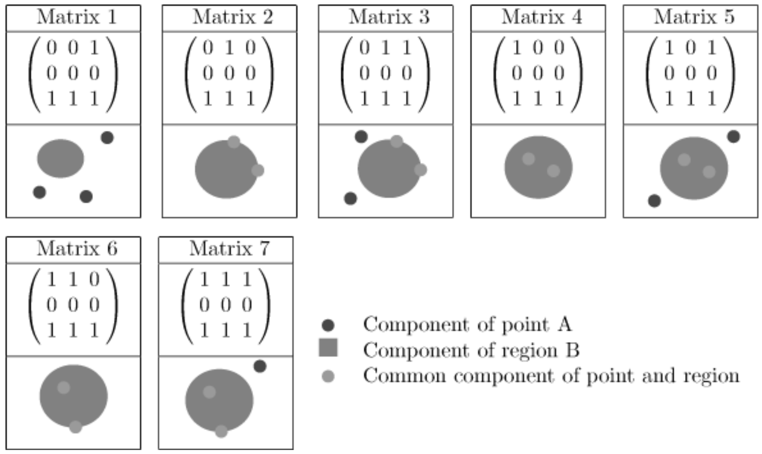


Fig. 12. The seven topological relationships between a complex point and a region.

PROOF. Both the interior and the boundary of a complex region object are infinite point sets whereas the complex point object is a finite point set. Hence, the complex point object can cover neither the interior nor the boundary of the complex region object so that its exterior must intersect these region parts. □

LEMMA 6.2.4. *The interior of the complex point intersects at least one part of the complex region, that is,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee \ A^\circ \cap B^\circ \neq \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing.$$

PROOF. The argumentation is the same as for Lemma 6.1.4. □

An evaluation of all 512 3×3-intersection matrices against these four constraint rules with the aid of a simple test program reveals that seven matrices satisfy these rules and thus represent the possible topological relationships between a complex point and a complex region. The matrices and their geometric interpretations are shown in Figure 12. Between a *simple* point and a *simple* region we can distinguish three topological relationships. Either a simple point and a simple region are disjoint (we also say the point is *outside* the region), or the simple point is located *on* the boundary of the simple region, or the simple point is *inside* the simple region. These topological predicates are contained in the seven general ones and correspond to the matrices 1, 2, and 4, respectively.

Finally, we can summarize our result as follows:

THEOREM 6.2. *Based on the 9-intersection model, seven different topological relationships can be identified between a complex* point *object and a complex* region *object.*

PROOF. The argumentation is based on the proof-by-constraint-and-drawing method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.2.1 to 6.2.4, reduce the number of the 512 possible intersection matrices to 7 matrices. The ability to draw prototypes of the corresponding 7 topological configurations (Figure 12) proves that the constraint rules are complete. □

6.3 Topological Relationships Between a Complex Line and a Complex Region

In the following last case, we assume that A is a nonempty object of type *line* and B is a nonempty object of type *region*. The following constraint rules identify all topological predicates between a complex line object and a complex region object.

LEMMA 6.3.1. *The exteriors of the complex line and the complex region always intersect with each other, that is,*

$$A^- \cap B^- \neq \varnothing.$$

PROOF. The argumentation is the same as for Lemma 6.1.3. □

LEMMA 6.3.2. *The interior of the complex region always intersects the exterior of the complex line, that is,*

$$A^- \cap B^\circ \neq \varnothing.$$

PROOF. Assume that this constraint rule is wrong. Then $A^- \cap B^\circ = \varnothing$, and we can conclude that $A \supseteq B^\circ$. From this we obtain that $\forall\, p \in B^\circ\ \exists \epsilon \in \mathbb{R}^+ : N_\epsilon(p) \subseteq B^\circ \Rightarrow N_\epsilon(p) \subseteq A$. This leads to a contradiction since $\forall\, p \in B^\circ\ \forall \epsilon \in \mathbb{R}^+ : N_\epsilon(p) \not\subseteq A$. □

Intuitively, a line object as a one-dimensional, linear entity cannot cover a region object, which is a two-dimensional, areal entity.

LEMMA 6.3.3. *The interior or the exterior of the complex line intersects the boundary of the complex region, that is,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing.$$

PROOF. We know that $\partial B \neq \varnothing$ and that hence $\mathbb{R}^2 \cap \partial B \neq \varnothing$. Since $A^\circ \cup \partial A \cup A^- = \mathbb{R}^2$, we obtain that $(A^\circ \cup \partial A \cup A^-) \cap \partial B \neq \varnothing$. This leads to $A^\circ \cap \partial B \neq \varnothing \ \vee\ \partial A \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing$. Since ∂A is a finite point set and ∂B is an infinite point set, either $\partial A \subset \partial B$ or $\partial A \cap \partial B = \varnothing$. This means that the constraint rule $A^\circ \cap \partial B \neq \varnothing \ \vee\ A^- \cap \partial B \neq \varnothing$ must hold. □

LEMMA 6.3.4. *The interior of the complex line intersects at least one part of the complex region, that is,*

$$A^\circ \cap \partial B \neq \varnothing \ \vee\ A^\circ \cap B^\circ \neq \varnothing \ \vee\ A^\circ \cap B^- \neq \varnothing.$$

PROOF. The argumentation is the same as for Lemma 6.1.4. □

LEMMA 6.3.5. *If the boundary of the complex line intersects the interior of the complex region, its interior also intersects the interior of the complex region, that is,*

$$\begin{aligned} &(\partial A \cap B^\circ \neq \varnothing \Rightarrow A^\circ \cap B^\circ \neq \varnothing) \\ \Leftrightarrow\ &(\partial A \cap B^\circ = \varnothing \ \vee\ A^\circ \cap B^\circ \neq \varnothing). \end{aligned}$$

PROOF. Without loss of generality, let $p \in \partial A \cap B^\circ$. Since $p \in B^\circ$, an $\varepsilon \in \mathbb{R}^+$ exists such that $N_\varepsilon(p) \subset B^\circ$. Fixing such an ε, and because a continuous curve with an infinite number of points starts in p, we obtain that $N_\varepsilon(p) \cap A^\circ \neq \varnothing$. This leads to the conclusion that $A^\circ \cap B^\circ \neq \varnothing$. □

LEMMA 6.3.6. *If the boundary of the complex line intersects the exterior of the complex region, its interior also intersects the exterior of the complex region, that is,*

$$\begin{aligned} & (\partial A \cap B^- \neq \varnothing \Rightarrow A^\circ \cap B^- \neq \varnothing) \\ \Leftrightarrow\ & (\partial A \cap B^- = \varnothing \ \vee \ A^\circ \cap B^- \neq \varnothing). \end{aligned}$$

PROOF. The argumentation is analogous to the argumentation for the constraint rule in Lemma 6.3.5. □

LEMMA 6.3.7. *If the boundary of the complex line intersects the boundary of the complex region, its exterior also intersects the boundary of the complex region, that is,*

$$\begin{aligned} & (\partial A \cap \partial B \neq \varnothing \Rightarrow A^- \cap \partial B \neq \varnothing) \\ \Leftrightarrow\ & (\partial A \cap \partial B = \varnothing \ \vee \ A^- \cap \partial B \neq \varnothing). \end{aligned}$$

PROOF. The boundary of a region is a line object whose components are all closed curves. Hence, this line object only consists of interior points. This leads to the case we have discussed in Lemma 5.2.3. □

An evaluation of all 512 3×3-intersection matrices against these seven constraint rules with the aid of a simple test program reveals that 43 matrices satisfy these rules and thus represent the possible topological relationships between a complex line and a complex region. The matrices and their geometric interpretations are shown in Figure 13. Between a *simple* line and a *simple* region we can distinguish 19 topological relationships [Egenhofer and Herring 1990a]. These topological predicates are contained in the 43 general ones and correspond to the matrices 2–4, 7, 11–13, 15–17, 28, 30, 31, 35–37, 39, 41, and 42, respectively.

Finally, we can summarize our result as follows:

THEOREM 6.3. *Based on the 9-intersection model, 43 different topological relationships can be identified between a complex* line *object and a complex* region *object.*

PROOF. The argumentation is based on the proof-by-constraint-and-drawing method described in Section 4. The constraint rules, whose correctness has been shown in Lemmas 6.3.1 to 6.3.7, reduce the number of the 512 possible intersection matrices to 43 matrices. The ability to draw prototypes of the corresponding 43 topological configurations (Figure 13) proves that the constraint rules are complete. □

7. CLUSTERING OF TOPOLOGICAL PREDICATES

Based on the 9-intersection model and the predicate derivation mechanism described in Section 4, in Sections 5 and 6, we have systematically identified the topological relationships between any two spatial objects of the data types *point*, *line*, and *region* defined in Section 3. Objects of these data types have a much

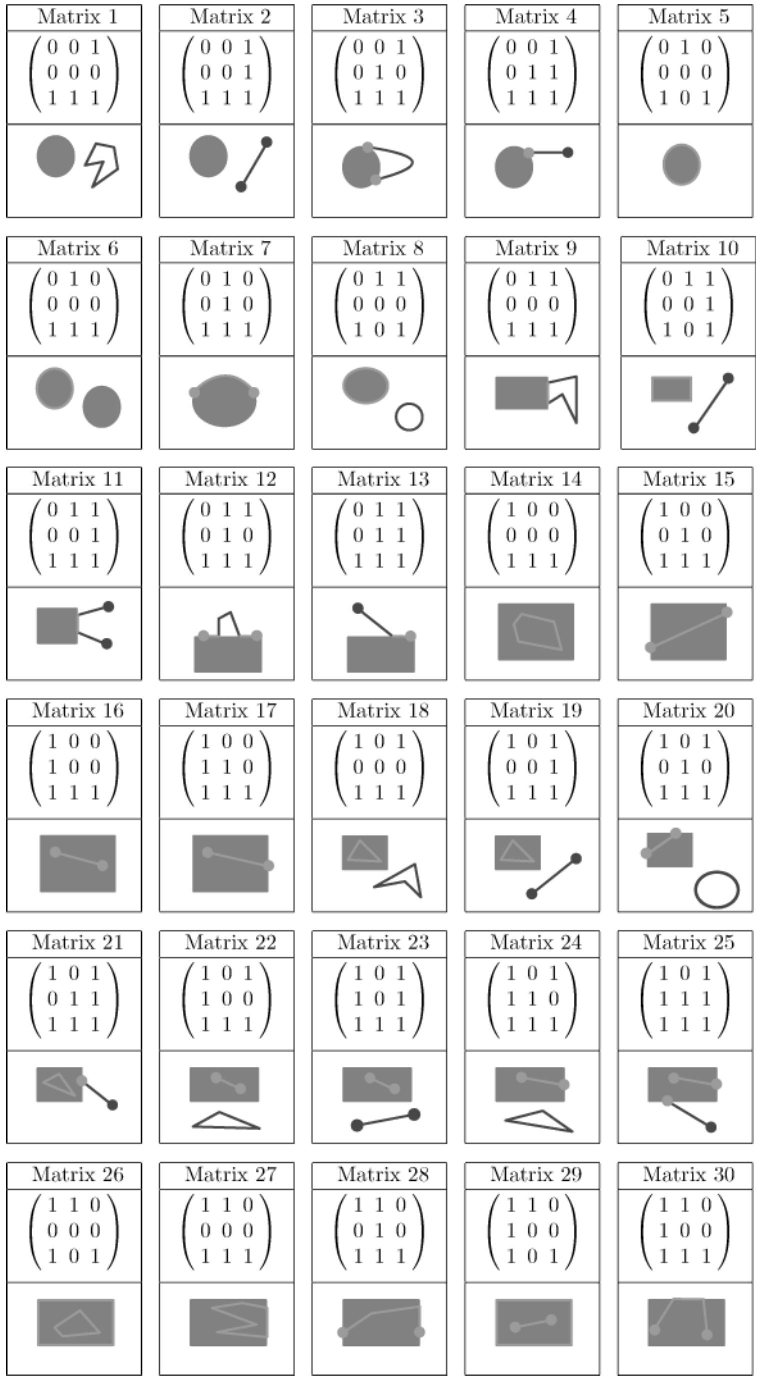


Fig. 13. The 43 topological relationships between a complex line and a complex region *(Continues)*.

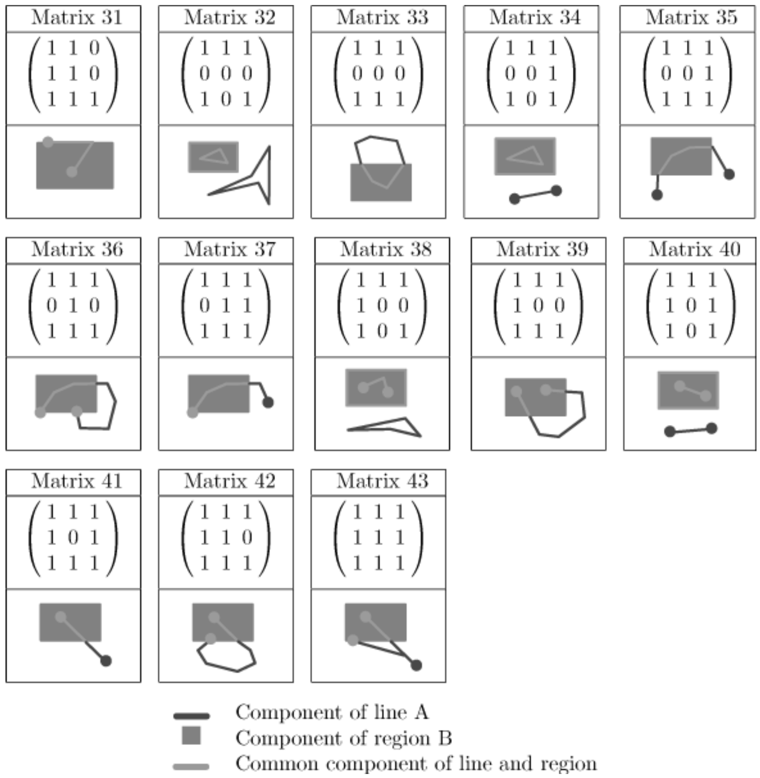


Fig. 13. *(Continued)*.

Table II. Number of Topological Predicates Between Two Complex Spatial Objects

	Complex Point	Complex Line	Complex Region
Complex point	5	14	7
Complex line	14	82	43
Complex region	7	43	33

more complex internal structure than simple objects (Section 2.1). We have seen that topological predicates operating on complex spatial objects comprise, generalize, and extend the predicates found so far for simple object structures (see Section 2.2 and Table I). Table II summarizes the number of predicates obtained for each type combination.

In the following, Section 7.1 addresses the problem of the large numbers and the manageability of the predicates in Table II. The next two subsections present two possible solutions based on clustering rules (Section 7.2) and topological cluster predicates (Section 7.3) to cope with this quantity problem.

7.1 The Quantity Problem

Compared to Table I, unsurprisingly the number of topological predicates has increased for each combination of complex data types. Already for topological predicates on simple spatial objects, the large numbers of predicates have been considered a problem since they make it difficult for users to distinguish, remember, and handle them [Clementini et al. 1993]. Consequently, this is, in particular, the case for the topological predicates on complex spatial objects. For example, whereas the five relationships between complex points are manageable and distinguishable by the user, this is certainly not the case for the 82 relationships between complex lines.

Frequently the user will not be interested in such a large, overwhelming collection of detailed predicates for a particular type combination and will prefer a reduced and manageable set instead. We call such a reduced set of predicates a *group* or *cluster*. Such a cluster should be user-defined and/or application-specific. It should be user-defined and thus flexible since the user should be able to select which predicates she wants to merge to more general predicates for her purposes. It should be application-specific since different applications may have a different understanding and thus definition of topological predicates carrying the same name. For example, one application could employ the *inside* predicate according to its original meaning, whereas another application could perhaps wish not to distinguish between *inside* and *coveredBy* but merge them and call the result predicate *inside* too.

Two possible solutions consist in (i) the design of *clustering rules* for topological predicates and (ii) in the explicit construction of user-defined and/or application-specific *topological cluster predicates* and *topological predicate groups*. We will see that the first solution is more a designer approach than a user approach like the second solution and that its outcome implicitly leads to cluster predicates.

7.2 Implicit Definition of Topological Cluster Predicates Through Clustering Rules

Clustering rules are relaxed constraint rules which do not take into account all nine intersections of the 9-intersection matrix. Only a predicate designer but not a user can usually specify them. They must be defined in a way so that they are mutually exclusive and cover all basic topological predicates identified in Sections 5 or 6 for the respective type combination. As an example, we will now give a specification of eight constraint rules that is generic in the sense that it is valid for all type combinations considered. Let $A \in \alpha$, $A \neq \varnothing$, and $B \in \beta$, $B \neq \varnothing$, for $\alpha, \beta \in \{point, line, region\}$. In the following, the notation p_c indicates that a predicate p is clustered. We define:

Clustering Rule 1. Two spatial objects are *disjoint* if the parts of one object intersect at most with the exterior of the other object, that is,

$$disjoint_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ = \varnothing \ \wedge \ A^\circ \cap \partial B = \varnothing \ \wedge \ \partial A \cap B^\circ = \varnothing \ \wedge \\ \partial A \cap \partial B = \varnothing.$$

Clustering Rule 2. Two spatial objects *meet* if both interiors do not intersec, but the interior or the boundary of one object intersects the boundary of the other object, that is,

$$meet_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ = \varnothing \wedge (A^\circ \cap \partial B \neq \varnothing \ \vee \ \partial A \cap B^\circ \neq \varnothing \ \vee \ \partial A \cap \partial B \neq \varnothing).$$

Clustering Rule 3. A spatial object is located *inside* another object if (i) their interiors intersect, (ii) the inner object does not share anything with the exterior of the other object, (iii) the interior of the containing object is partially located in the exterior of the inner object, and (iv) the boundaries of both objects do not intersect, that is,

$$inside_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing \ \wedge \ \partial A \cap \partial B = \varnothing.$$

Clustering Rule 4. The relationships that are symmetric to $inside_c$ describe the predicate $contains_c$, that is,

$$contains_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing \ \wedge \ \partial A \cap \partial B = \varnothing.$$

Clustering Rule 5. A spatial object is *covered by* another object if (i) their interiors intersect, (ii) the inner object does not share anything with the exterior of the other object, (iii) the interior of the containing object is partially located in the exterior of the inner object, and (iv) the boundaries of both objects intersect, that is,

$$coveredBy_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing \ \wedge \ \partial A \cap \partial B \neq \varnothing.$$

Clustering Rule 6. The relationships that are symmetric to $coveredBy_c$ describe the predicate $covers_c$, that is,

$$covers_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing \ \wedge \ \partial A \cap \partial B \neq \varnothing.$$

Clustering Rule 7. Two spatial objects are *equal* if at most corresponding parts intersect, that is,

$$equal_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap \partial B = \varnothing \ \wedge \ A^\circ \cap B^- = \varnothing \ \wedge \ \partial A \cap B^\circ = \varnothing \ \wedge \ \partial A \cap B^- = \varnothing \ \wedge \ A^- \cap B^\circ = \varnothing \ \wedge \ A^- \cap \partial B = \varnothing.$$

Clustering Rule 8. Two spatial objects *overlap* if the interior of each object intersects both the interior and the exterior of the other object, that is,

$$overlap_c(A, B) \stackrel{\text{def}}{=} A^\circ \cap B^\circ \neq \varnothing \ \wedge \ A^\circ \cap B^- \neq \varnothing \ \wedge \ A^- \cap B^\circ \neq \varnothing.$$

In order to prove the mutual exclusion of the obtained topological cluster predicates as well as the complete coverage of all basic topological predicates by them, we do *not* argue on the basis of our definitions of clustering rules. Instead, the strategy is to match each clustering rule against all basic topological predicates, which are then associated with possible cluster predicates. Table III shows the result.

Table III. Generic Topological Cluster Predicates and Assigned Basic Topological Predicates Depending on the Type Combination and Indicated by Their Matrix Numbers

Cluster predicate	Point/ Point	Line/ Line	Region/ Region	Point/ Line	Point/ Region	Line/ Region
$disjoint_c$	1	1–4	1	1, 2	1	1, 2
$meet_c$	–	5–32	2–4	3–6	2, 3	3–13
$covers_c$	–	49, 52, 69, 72	11, 21, 24	–	–	–
$coveredBy_c$	–	37, 38, 41, 42	6, 8, 9	–	–	15, 17, 28, 31
$inside_c$	3	34, 35, 39, 40	7	7, 8, 11, 12	4, 6	14, 16, 26, 27, 29, 30
$contains_c$	4	43, 46, 63, 66	19	–	–	–
$overlap_c$	5	44, 45, 47, 48, 50, 51, 53–62, 64, 65, 67, 68, 70, 71, 73–82	10, 12–18, 20 22, 23, 25–33	9, 10, 13, 14	5, 7	18–25, 32–43
$equal_c$	2	33, 36	5	–	–	–

For each type combination, it is easy to verify that each basic topological predicate is assigned to at most one topological cluster predicate and that there is no basic topological predicate that is not assigned to a cluster predicate. Hence, each basic predicate is assigned to exactly one cluster predicate, and all basic predicates are covered.

7.3 Explicit Definition of Topological Cluster Predicates Through Disjunctions

The lists of matrix numbers in Table III express that each clustered predicate is equal to a disjunction of basic predicates. Hence, from a user perspective, a cluster predicate summarizes basic topological predicates under a common name. In a database context, a user should be able to use disjunctions of basic topological predicates for explicitly constructing cluster predicates.

A first problem is how to address basic topological predicates, since they are nameless because of their large amount. For this purpose, we introduce the six type-combination specific predicates *tp_pp*, *tp_pl*, *tp_pr*, *tp_ll*, *tp_lr*, and *tp_rr* (the prefix *tp_* stands for *topological predicate*; the last two letters denote the type combination) and parameterize them by their matrix number. Since we do not have predicate names and since the user can identify topological relationships only by looking at the example scenarios, we regard this as a practical solution. For example, the two basic predicates *tp_pr*(5) and *tp_pr*(7) denote the two possible overlap situations between a point object and a region object.

A second problem is how user-defined topological cluster predicates can be specified in a database context. Here we propose an extension of the data description language (DDL) of SQL. For example, a DDL command

```
create tpclpred lr_inside(tp_lr(14), tp_lr(16), tp_lr(26),
                          tp_lr(27), tp_lr(29), tp_lr(30));
```

could specify a cluster predicate *lr_inside* which for two objects $A \in line$ and $B \in region$ computes the logical expression

$$lr\_inside(A, B) \stackrel{\text{def}}{=} tp\_lr(14)(A, B) \lor tp\_lr(16)(A, B) \lor tp\_lr(26)(A, B) \lor tp\_lr(27)(A, B) \lor tp\_lr(29)(A, B) \lor tp\_lr(30)(A, B).$$

The predicate *lr_inside* can now be employed in a query. For example, assuming the two relations `rivers(rname:string, route:line)` and `states(sname:string, area:region)`, we can pose the query *Determine river names and state names where the river is located within the state* as follows:

```
SELECT rname, sname FROM   rivers, states WHERE  route lr_inside area
```

A user can, of course, define arbitrary cluster predicates. It is especially possible to define different cluster predicates which do not completely exclude each other since they contain common basic topological predicates. To define a collection of mutually exclusive cluster predicates which covers all basic topological predicates, we allow the specification of *topological predicate groups*. For example, each column of Table III represents such a predicate group. An extension of the DDL of SQL could formulate the predicate group for the point/line case in Table III as follows:

```
create tppredgroup pl_cluster
  (pl_disjoint(tp_pl(1), tp_pl(2)),
   pl_meet(tp_pl(3), tp_pl(4), tp_pl(5), tp_pl(6)),
   pl_inside(tp_pl(7), tp_pl(8), tp_pl(11), tp_pl(12))
   pl_overlap(tp_pl(9), tp_pl(10), tp_pl(13), tp_pl(14)))
```

Since the same cluster predicate can be defined differently in different predicate groups, a user must be able to indicate which group she would like to use. A group can be selected by the DDL command

```
use tppredgroup pl_cluster
```

In the same way as a query language should be prepared to incorporate basic topological predicates, cluster predicates, and predicate groups, these concepts should also be integrated into the application programming interface of a spatial database or GIS.

8. CONCLUSIONS AND FUTURE WORK

From a formal and an application point of view, spatial applications require by far more complex geometric structures than the usual simple points, lines, and regions that can be currently found in spatial database systems, spatial query languages, and GIS. In the meantime, some GIS and database vendors have recognized this shortcoming and begun to incorporate more complex spatial data types into their systems. A first contribution of this article is that we have defined very general and versatile spatial data types for complex points, complex lines, and complex regions in the two-dimensional Euclidean space on the basis of point set theory and point set topology. Complex points may be composed of a finite set of isolated points, complex lines may represent spatially embedded graphs possibly consisting of several connected components, and complex regions may consist of several components where each component possibly contains holes.

The introduction of complex spatial data types leads to a larger variety of topological relationships. The investigation and formalization of complete collections of mutually exclusive topological relationships between all combinations of complex spatial data types has been the second main contribution of this article. It has been done on the basis of the well-known 9-intersection model.

Due to the large amount of predicates for each type combination and the user's difficulty in handling them, our third contribution consists in the introduction of topological cluster predicates and topological predicate groups. These two concepts allow the user to group basic topological predicates under a common name and thus to reduce the number of predicates.

A main topic of future work consists in the efficient implementation of all basic topological predicates as well as cluster predicates. Literature on the implementation of topological predicates is rare (see Section 2). We plan to apply techniques from computational geometry [de Berg et al. 2000]. But the implementation of a single algorithm for each topological predicate of each type combination can become rather troublesome due to the large number of predicates. We plan to pursue the idea of designing a single evaluation algorithm for each type combination that manages *without* an exhaustive case analysis. The task of such an algorithm would be efficiently determine the topological relationship for a given scenario of two spatial objects. The predicate determined would then be efficiently matched against the query predicate. The implementation will be part of SPAL2D, which is a sophisticated *spatial algebra* under development for two-dimensional applications.

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