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THE DIRICHLET PROBLEM WITHOUT THE MAXIMUM PRINCIPLE

by Wolfgang ARENDT & A. F. M. TER ELST (*)

Abstract. — Consider the Dirichlet problem with respect to an elliptic operator

$$A = -\sum_{k,l=1}^{d} \partial_k \, a_{kl} \, \partial_l - \sum_{k=1}^{d} \partial_k \, b_k + \sum_{k=1}^{d} c_k \, \partial_k + c_0$$

on a bounded Wiener regular open set $\Omega \subset \mathbb{R}^d$, where $a_{kl}, c_k \in L_\infty(\Omega, \mathbb{R})$ and $b_k, c_0 \in L_\infty(\Omega, \mathbb{C})$. Suppose that the associated operator on $L_2(\Omega)$ with Dirichlet boundary conditions is invertible. Then we show that for all $\varphi \in C(\partial\Omega)$ there exists a unique $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ such that $u|_{\partial\Omega} = \varphi$ and $Au = 0$.

In the case when Ω has a Lipschitz boundary and $\varphi \in C(\overline{\Omega}) \cap H^{1/2}(\overline{\Omega})$, then we show that u coincides with the variational solution in $H^1(\Omega)$.

Résumé. — Considérons le problème de Dirichlet par rapport à un opérateur elliptique

$$A = -\sum_{k,l=1}^{d} \partial_k \, a_{kl} \, \partial_l - \sum_{k=1}^{d} \partial_k \, b_k + \sum_{k=1}^{d} c_k \, \partial_k + c_0$$

sur un ensemble ouvert régulier de Wiener borné $\Omega \subset \mathbb{R}^d$, où $a_{kl}, c_k \in L_\infty(\Omega, \mathbb{R})$ et $b_k, c_0 \in L_\infty(\Omega, \mathbb{C})$. Supposons que 0 n'est pas une valeur propre de A avec conditions aux limites Dirichlet. Alors nous montrons que pour tout $\varphi \in C(\partial\Omega)$ il existe un unique $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ tel que $u|_{\partial\Omega} = \varphi$ et $Au = 0$.

Dans le cas où Ω a une frontière Lipschitz et $\varphi \in C(\overline{\Omega}) \cap H^{1/2}(\overline{\Omega})$, nous montrons que u coïncide avec la solution variationnelle dans $H^1(\Omega)$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with boundary Γ. Throughout this paper we assume that $d \geqslant 2$. The classical Dirichlet problem is to find for each $\varphi \in C(\Gamma)$ a function $u \in C(\overline{\Omega})$ such that $u|_\Gamma = \varphi$ and $\Delta u = 0$ as distribution on Ω. The set Ω is called *Wiener regular* if for every $\varphi \in C(\Gamma)$ there exists a unique $u \in C(\overline{\Omega})$ such that $u|_\Gamma = \varphi$ and $\Delta u = 0$ as distribution on Ω.

The Dirichlet problem has been extended naturally to more general second-order operators. For all $k, l \in \{1, \ldots, d\}$ let $a_{kl} \colon \Omega \to \mathbb{R}$ be a bounded measurable function and suppose that there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} a_{kl}(x)\, \xi_k\, \overline{\xi_l} \geqslant \mu\, |\xi|^2 \tag{1.1}$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Further, for all $k \in \{1, \ldots, d\}$ let $b_k, c_k, c_0 \colon \Omega \to \mathbb{C}$ be bounded and measurable. Define the map $\mathcal{A} \colon H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \mathcal{A}u, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \sum_{k,l=1}^{d} \int_\Omega a_{kl}\, (\partial_k u)\, \overline{\partial_l v} + \sum_{k=1}^{d} \int_\Omega b_k\, u\, \overline{\partial_k v} + \sum_{k=1}^{d} \int_\Omega c_k\, (\partial_k u)\, \overline{v} + \int_\Omega c_0\, u\, \overline{v}$$

for all $u \in H^1_{\mathrm{loc}}(\Omega)$ and $v \in C_c^\infty(\Omega)$. Given $\varphi \in C(\Gamma)$, by a *classical solution* of the Dirichlet problem we understand a function $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ satisfying $\mathcal{A}u = 0$ and $u|_\Gamma = \varphi$. For the pure second-order case (that is $b_k = c_k = c_0 = 0$) Littman–Stampacchia–Weinberger [11] proved that for all $\varphi \in C(\Gamma)$ there exists a unique classical solution u. Then Stampacchia [13, Théorème 10.2] added real valued lower order terms, under the condition (see [13], (9.2')) that there exists a $\mu' > 0$ such that

$$\int_\Omega c_0\, v + \sum_{k=1}^{d} \int_\Omega b_k\, \partial_k v \geqslant \mu' \int_\Omega v \tag{1.2}$$

for all $v \in C_c^\infty(\Omega)^+$. Gilbarg–Trudinger [10, Theorem 8.31] merely assume that

$$\int_\Omega c_0\, v + \sum_{k=1}^{d} \int_\Omega b_k\, \partial_k v \geqslant 0 \tag{1.3}$$

for all $v \in C_c^\infty(\Omega)^+$ in order to obtain the same conclusion. A consequence of these assumptions is a weak maximum principle, which implies that

$\|u\|_{C(\overline{\Omega})} \leqslant \|\varphi\|_{C(\Gamma)}$ for all $u \in H^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfying $\mathcal{A}u = 0$ and $u|_\Gamma = \varphi$. We may consider (1.3) as a kind of submarkov condition since it is equivalent to $-\mathcal{A}\mathbb{1}_\Omega \leqslant 0$ in $\mathcal{D}'(\Omega)$.

The aim of this paper is to show that the positivity condition (1.3) and the maximum principle are not needed for the well-posedness of the Dirichlet problem. In addition we allow the b_k and c_0 to be complex valued. In order to state the main results of this paper in a more precise way we need a few definitions. Define the form $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^{d} \int_\Omega a_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \sum_{k=1}^{d} \int_\Omega b_k\, u\, \overline{\partial_k v} + \sum_{k=1}^{d} \int_\Omega c_k\,(\partial_k u)\,\overline{v} + \int_\Omega c_0\, u\,\overline{v}. \tag{1.4}$$

Let A^D be the operator in $L_2(\Omega)$ associated with the form $\mathfrak{a}|_{H^1_0(\Omega)\times H^1_0(\Omega)}$. In other words, A^D is the realisation of the elliptic operator $\mathcal{A}$ in $L_2(\Omega)$ with Dirichlet boundary conditions. This operator has a compact resolvent. Moreover, if (1.3) is valid, then $\ker A^D = \{0\}$ by [10, Corollary 8.2]. Instead of (1.3) we assume the condition $\ker A^D = \{0\}$, which is equivalent to the uniqueness of the Dirichlet problem (cf. Proposition 2.3 below).

The main result of this paper is the following well-posedness result for the Dirichlet problem.

THEOREM 1.1. — *Let $\Omega \subset \mathbb{R}^d$ be an open bounded Wiener regular set with $d \geqslant 2$. For all $k,l \in \{1,\dots,d\}$ let $a_{kl}\colon \Omega \to \mathbb{R}$ be a bounded measurable function and suppose that there exists a $\mu > 0$ such that*

$$\operatorname{Re} \sum_{k,l=1}^{d} a_{kl}(x)\,\xi_k\,\overline{\xi_l} \geqslant \mu\,|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Further, for all $k \in \{1,\dots,d\}$ let $b_k, c_0\colon \Omega \to \mathbb{C}$ and $c_k\colon \Omega \to \mathbb{R}$ be bounded and measurable. Let A^D be as above. Suppose $0 \notin \sigma(A^D)$. Then for all $\varphi \in C(\Gamma)$ there exists a unique $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ such that $u|_\Gamma = \varphi$ and $\mathcal{A}u = 0$.

Moreover, there exists a constant $c > 0$ such that

$$\|u\|_{C(\overline{\Omega})} \leqslant c\,\|\varphi\|_{C(\Gamma)}$$

for all $\varphi \in C(\Gamma)$, where $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ is such that $u|_\Gamma = \varphi$ and $\mathcal{A}u = 0$.

Instead of the homogeneous equation $\mathcal{A}u = 0$ one can also consider the inhomogeneous equation $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$. We shall do that in Theorem 2.13.

Adopt the notation and assumptions of Theorem 1.1. Define $P\colon C(\Gamma) \to C(\overline{\Omega})$ by $P\varphi = u$, where $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ is such that $u|_\Gamma = \varphi$ and $\mathcal{A}u = 0$. Note that $P\varphi$ is the *classical solution* of the Dirichlet problem.

If Ω has even a Lipschitz boundary (which implies Wiener regularity), then there is also a variational solution of the Dirichlet problem that we describe next. Denote by $\mathrm{Tr}\colon H^1(\Omega) \to L_2(\Gamma)$ the trace operator. Again let $a_{kl}, b_k, c_k, c_0 \in L_\infty(\Omega)$ and suppose that the ellipticity condition (1.1) is satisfied. Further suppose that $0 \notin \sigma(A^D)$. Then for each $\varphi \in \mathrm{Tr}\, H^1(\Omega)$ there exists a unique $u \in H^1(\Omega)$, called the *variational solution*, such that $\mathcal{A}u = 0$ and $\mathrm{Tr}\, u = \varphi$ (cf. Lemma 2.1). Define $\gamma\colon\ \mathrm{Tr}\, H^1(\Omega) \to H^1(\Omega)$ by setting $\gamma\varphi = u$.

The second result of this paper says that the variational solution and the classical solution coincide, if both are defined.

Theorem 1.2. — *Adopt the notation and assumptions of Theorem 1.1. Suppose that Ω has a Lipschitz boundary. Let $\varphi \in C(\Gamma) \cap \mathrm{Tr}\, H^1(\Omega)$. Then $P\varphi = \gamma\varphi$ almost everywhere on Ω.*

The last main result of this paper concerns a parabolic equation. Let A_c denote the part of the operator A^D in $C_0(\Omega)$. So

$$D(A_c) = \{u \in D(A^D) \cap C_0(\Omega) : A^D u \in C_0(\Omega)\}$$

and $A_c = A^D|_{D(A_c)}$.

Theorem 1.3. — *Adopt the notation and assumptions of Theorem 1.1. Then $-A_c$ generates a holomorphic C_0-semigroup on $C_0(\Omega)$. Moreover, $\mathrm{e}^{-tA_c} u = \mathrm{e}^{-tA^D} u$ for all $u \in C_0(\Omega)$ and $t > 0$.*

In Section 2 we prove Theorem 1.1 via an iteration argument. Section 3 is devoted to the comparison of the classical and the variational solutions of the Dirichlet problem. Theorem 1.2 is proved there with the help of a deep result of Dahlberg [7]. We consider the semigroup on $C_0(\Omega)$ in Section 4 and prove Theorem 1.3.

2. The Dirichlet problem

In this section we prove Theorem 1.1 on the well-posedness of the Dirichlet problem. The technique is a reduction to the Stampacchia result mentioned in the introduction. For this reason we introduce the following two forms and operators.

Adopt the notation and assumptions of Theorem 1.1. For all $\lambda \in \mathbb{R}$ define the forms $\mathfrak{a}_\lambda, \mathfrak{b}_\lambda \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_\lambda(u,v) = \mathfrak{a}(u,v) + \lambda\,(u,v)_{L_2(\Omega)}$$

$$\text{and} \quad \mathfrak{b}_\lambda(u,v) = \sum_{k,l=1}^{d} \int_\Omega a_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \sum_{k=1}^{d} \int_\Omega c_k\,(\partial_k u)\,\overline{v} + \lambda \int_\Omega u\,\overline{v},$$

where $\mathfrak{a}$ is as in (1.4). Define similarly $\mathcal{A}_\lambda, \mathcal{B}_\lambda \colon H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$ and let B^D be the operator associated with the sesquilinear form $\mathfrak{b}_0|_{H^1_0(\Omega)\times H^1_0(\Omega)}$. It follows from ellipticity that there exists a $\lambda_0 > 0$ such that

$$\frac{\mu}{2}\,\|v\|^2_{H^1(\Omega)} \leqslant \operatorname{Re}\mathfrak{a}_{\lambda_0}(v) \quad \text{and} \quad \frac{\mu}{2}\,\|v\|^2_{H^1(\Omega)} \leqslant \operatorname{Re}\mathfrak{b}_{\lambda_0}(v)$$

for all $v \in H^1(\Omega)$. Note that $\mathcal{B}_\lambda$ satisfies the submarkovian condition $-\mathcal{B}_\lambda \mathbb{1}_\Omega \leqslant 0$, that is (1.3), and even Stampacchia's condition (1.2) for all $\lambda > 0$. So we can and will apply Stampacchia's result (in the proof of Lemma 2.8).

We first investigate the operator A^D in $L_2(\Omega)$. Note that $f_0 + \sum_{k=1}^d \partial_k f_k \in \mathcal{D}'(\Omega)$ for all $f_0, f_1, \ldots, f_d \in L_1(\Omega)$. The next lemma is also valid if the a_{kl} and c_k are complex valued.

Lemma 2.1. — *Let $f_1, \ldots, f_d \in L_2(\Omega)$. Let $\tilde{p} \in (1,\infty)$ be such that $\tilde{p} \geqslant \frac{2d}{d+2}$. Further let $f_0 \in L_{\tilde{p}}(\Omega)$. Then there exists a unique $u \in H^1_0(\Omega)$ such that $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$.*

Proof. — There exists a unique $T \in \mathcal{L}(H^1_0(\Omega))$ such that $(Tu,v)_{H^1_0(\Omega)} = \mathfrak{a}(u,v)$ for all $u,v \in H^1_0(\Omega)$. Then T is injective because $\ker A^D = \{0\}$. Moreover, the inclusion $H^1_0(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Hence the operator T is invertible by the Fredholm–Lax–Milgram lemma, [5, Lemma 4.1]. Clearly $v \mapsto \sum_{k=1}^d (f_k, \partial_k v)_{L_2(\Omega)}$ is continuous from $H^1_0(\Omega)$ into $\mathbb{C}$. Define $F \colon C_c^\infty(\Omega) \to \mathbb{C}$ by $F(v) = \langle f_0, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$. We claim that F extends to a continuous function from $H^1_0(\Omega)$ into $\mathbb{C}$. If $d \geqslant 3$, then $H^1_0(\Omega) \subset L_r(\Omega)$, where $r = \frac{2d}{d-2}$. So $H^1_0(\Omega) \subset L_q(\Omega)$, where q is the dual exponent of $\tilde{p}$. The last inclusion is also valid if $d = 2$. So in any case the map F extends to a continuous function from $H^1_0(\Omega)$ into $\mathbb{C}$. Then the lemma follows. □

The next lemma is valid for a general bounded open set Ω and does not use the condition $0 \notin \sigma(A^D)$. It is an extension of [1, Lemma 4.2].

Lemma 2.2. — *Let $u \in C_0(\Omega) \cap H^1_{\mathrm{loc}}(\Omega)$ and $f_1, \ldots, f_d \in L_2(\Omega)$. Let $\tilde{p} \in (1,\infty)$ be such that $\tilde{p} \geqslant \frac{2d}{d+2}$. Further let $f_0 \in L_{\tilde{p}}(\Omega)$. Suppose that $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$. Then $u \in H^1_0(\Omega)$.*

Proof. — As at the end of the previous proof there exists an $M_0 > 0$ such that $| \int_\Omega f_0 \, \overline{v}| \leqslant M_0 \, \|v\|_{H^1(\Omega)}$ for all $v \in H^1_0(\Omega)$. Set $M = M_0 + \sum_{k=1}^d \|f_k\|_2$.

Let $\varepsilon > 0$. Set $v_\varepsilon = (\operatorname{Re} u - \varepsilon)^+$. Then $\operatorname{supp} v_\varepsilon \subset \Omega$ is compact. Hence there exists an open $\Omega_1 \subset \mathbb{R}^d$ such that $\operatorname{supp} v_\varepsilon \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega$. Then $v_\varepsilon \in H^1_0(\Omega_1)$. Moreover,

$$\sum_{k,l=1}^d \int_{\Omega_1} a_{kl} \, (\partial_k u) \, \overline{\partial_l v} + \sum_{k=1}^d \int_{\Omega_1} b_k \, u \, \overline{\partial_k v} + \sum_{k=1}^d \int_{\Omega_1} c_k \, (\partial_k u) \, \overline{v} + \int_{\Omega_1} c_0 \, u \, \overline{v}$$
$$= \int_{\Omega_1} f_0 \, \overline{v} + \sum_{k=1}^d \int_{\Omega_1} f_k \, \overline{\partial_k v} \tag{2.1}$$

for all $v \in C_c^\infty(\Omega_1)$. Since $u|_{\Omega_1} \in H^1(\Omega_1)$ it follows that (2.1) is valid for all $v \in H^1_0(\Omega_1)$. Choosing $v = v_\varepsilon$ gives

$$\left| \sum_{k,l=1}^d \int_\Omega a_{kl} \, (\partial_k u) \, \partial_l v_\varepsilon + \sum_{k=1}^d \int_\Omega b_k \, u \, \partial_k v_\varepsilon + \sum_{k=1}^d \int_\Omega c_k \, (\partial_k u) \, v_\varepsilon + \int_\Omega c_0 \, u \, v_\varepsilon \right|$$
$$\leqslant M_0 \, \|v_\varepsilon\|_{H^1(\Omega)} + \sum_{k=1}^d \|f_k\|_2 \, \|\partial_k v_\varepsilon\|_2 \leqslant M \, \|v_\varepsilon\|_{H^1(\Omega)}.$$

On the other hand, $\partial_k v_\varepsilon = \partial_k((\operatorname{Re} u - \varepsilon)^+) = \mathbb{1}_{[\operatorname{Re} u > \varepsilon]} \, \partial_k \operatorname{Re} u$ for all $k \in \{1, \ldots, d\}$ by [10, Lemma 7.6]. Therefore

$$\begin{aligned}
&\operatorname{Re} \sum_{k,l=1}^d \int_\Omega a_{kl} \, (\partial_k u) \, \partial_l v_\varepsilon + \operatorname{Re} \sum_{k=1}^d \int_\Omega b_k \, u \, \partial_k v_\varepsilon \\
&\qquad + \operatorname{Re} \sum_{k=1}^d \int_\Omega c_k \, (\partial_k u) \, v_\varepsilon + \operatorname{Re} \int_\Omega c_0 \, u \, v_\varepsilon \\
&= \sum_{k,l=1}^d \int_\Omega a_{kl} \, (\partial_k v_\varepsilon) \, \partial_l v_\varepsilon + \operatorname{Re} \sum_{k=1}^d \int_\Omega b_k \, u \, \partial_k v_\varepsilon \\
&\qquad + \sum_{k=1}^d \int_\Omega c_k \, (\partial_k \operatorname{Re} u) \, v_\varepsilon + \operatorname{Re} \int_\Omega c_0 \, u \, v_\varepsilon \\
&= \operatorname{Re} \mathfrak{a}(v_\varepsilon) + \varepsilon \sum_{k=1}^d \int_\Omega (\operatorname{Re} b_k) \, \partial_k v_\varepsilon - \sum_{k=1}^d \int_\Omega (\operatorname{Im} b_k) \, (\operatorname{Im} u) \, \partial_k v_\varepsilon \\
&\qquad + \varepsilon \int_\Omega (\operatorname{Re} c_0) \, v_\varepsilon - \int_\Omega (\operatorname{Im} c_0) \, (\operatorname{Im} u) \, v_\varepsilon \\
&\geqslant \frac{\mu}{2} \, \|v_\varepsilon\|^2_{H^1(\Omega)} - \lambda_0 \, \|v_\varepsilon\|_2^2 - \varepsilon \, M' \, |\Omega|^{1/2} \, \|v_\varepsilon\|_{H^1(\Omega)} - M' \, \|u\|_2 \, \|v_\varepsilon\|_{H^1(\Omega)},
\end{aligned}$$

where $M' = \|c_0\|_\infty + \sum_{k=1}^d \|b_k\|_\infty$. Since $\|v_\varepsilon\|_2 = \|(\operatorname{Re} u - \varepsilon)^+\|_2 \leqslant \|u\|_2 \leqslant |\Omega|^{1/2} \|u\|_{C_0(\Omega)}$, it follows that

$$\frac{\mu}{2} \|(\operatorname{Re} u - \varepsilon)^+\|_{H^1(\Omega)}^2 \leqslant M'' \|(\operatorname{Re} u - \varepsilon)^+\|_{H^1(\Omega)} + \lambda_0 |\Omega| \|u\|_{C_0(\Omega)}^2$$

for all $\varepsilon \in (0,1]$, where $M'' = M + M' |\Omega|^{1/2} (\|u\|_{C_0(\Omega)} + 1)$.

Therefore the sequence $((\operatorname{Re} u - 2^{-n})^+)_{n \in \mathbb{N}_0}$ is bounded in $H_0^1(\Omega)$. Passing to a subsequence if necessary, we may assume without loss of generality that there exists a $w \in H_0^1(\Omega)$ such that $\lim (\operatorname{Re} u - 2^{-n})^+ = w$ weakly in $H_0^1(\Omega)$. Then $\lim (\operatorname{Re} u - 2^{-n})^+ = w$ in $L_2(\Omega)$. But $\lim (\operatorname{Re} u - 2^{-n})^+ = (\operatorname{Re} u)^+$ in $L_2(\Omega)$. So $(\operatorname{Re} u)^+ = w \in H_0^1(\Omega)$. Similarly one proves that $(\operatorname{Re} u)^-, (\operatorname{Im} u)^+, (\operatorname{Im} u)^- \in H_0^1(\Omega)$. So $u \in H_0^1(\Omega)$. $\square$

Lemma 2.2 together with the condition $0 \notin \sigma(A^D)$ gives the uniqueness in Theorem 1.1.

Proposition 2.3. — *For all $\varphi \in C(\Gamma)$ there exists at most one $u \in C(\overline{\Omega}) \cap H_{\mathrm{loc}}^1(\Omega)$ such that $u|_\Gamma = \varphi$ and $\mathcal{A} u = 0$.*

Proof. — Let $u \in C(\overline{\Omega}) \cap H_{\mathrm{loc}}^1(\Omega)$ and suppose that $u|_\Gamma = 0$ and $\mathcal{A} u = 0$. Then $u \in C_0(\Omega)$. Hence $u \in H_0^1(\Omega)$ by Lemma 2.2. Also $\mathcal{A} u = 0$. Therefore $u \in D(A^D)$ and $A^D u = 0$. But $0 \notin \sigma(A^D)$. So $u = 0$. $\square$

In the next proposition we use that Ω is Wiener regular.

Proposition 2.4. — *Let $\lambda > \lambda_0$ and $p \in (d, \infty]$. Let $f_0 \in L_{p/2}(\Omega)$ and $f_1, \ldots, f_d \in L_p(\Omega)$. Then there exists a unique $u \in H_0^1(\Omega) \cap C_0(\Omega)$ such that $\mathcal{B}_\lambda u = f_0 + \sum_{k=1}^d \partial_k f_k$.*

Proof. — Since a_{kl} and c_k are real valued for all $k, l \in \{1, \ldots, d\}$ we may assume that $f_0, \ldots, f_d$ are real valued. By [10, Theorem 8.31] there exists a unique $u \in C(\overline{\Omega}) \cap H_{\mathrm{loc}}^1(\Omega)$ such that $\mathcal{B}_\lambda u = f_0 + \sum_{k=1}^d \partial_k f_k$ and $u|_\Gamma = 0$. Then $u \in C_0(\Omega)$ and the existence follows from Lemma 2.2. The uniqueness follows from Proposition 2.3. $\square$

Corollary 2.5. — *Let $\lambda > \lambda_0$ and $p \in (d, \infty]$. Let $f_0 \in L_{p/2}(\Omega)$ and $f_1, \ldots, f_d \in L_p(\Omega)$. Let $u \in H_0^1(\Omega)$ and suppose that $\mathcal{B}_\lambda u = f_0 + \sum_{k=1}^d \partial_k f_k$. Then $u \in C_0(\Omega)$.*

Proof. — By Proposition 2.4 there exists a $\tilde{u} \in H_0^1(\Omega) \cap C_0(\Omega)$ such that $\mathcal{B}_\lambda \tilde{u} = f_0 + \sum_{k=1}^d \partial_k f_k$. Then $\mathcal{B}_\lambda (u - \tilde{u}) = 0$. So $\mathfrak{b}_\lambda (u - \tilde{u}, v) = 0$ first for all $v \in C_c^\infty(\Omega)$ and then by density for all $v \in H_0^1(\Omega)$. Choose $v = u - \tilde{u}$. Then $\frac{\mu}{2} \|u - \tilde{u}\|_{H^1(\Omega)}^2 \leqslant \operatorname{Re} \mathfrak{b}_\lambda (u - \tilde{u}) = 0$. So $u = \tilde{u} \in C_0(\Omega)$. $\square$

We next wish to add the other lower order terms.

Proposition 2.6. — *There exists a $c > 0$ such that for all $\Phi \in C^1(\mathbb{R}^d)$ there exists a unique $u \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $u|_\Gamma = \Phi|_\Gamma$ and $\mathcal{A}u = 0$. Moreover,*

$$\|u\|_{C(\overline{\Omega})} \leqslant c \, \|\Phi|_\Gamma\|_{C(\Gamma)}.$$

For the proof we need some lemmas. In the next lemma we introduce a parameter δ in order to avoid duplication of the proof.

Lemma 2.7. — *Fix $\delta \in [0, \lambda_0 + 1]$.*

(1) *For all $f \in L_2(\Omega)$ and $\lambda > \lambda_0$ there exists a unique $u \in H_0^1(\Omega)$ such that*

$$\mathfrak{b}_\lambda(u, v) = \sum_{k=1}^d (b_k \, f, \partial_k v)_{L_2(\Omega)} + ((c_0 - \delta \, \mathbb{1}_\Omega) \, f, v)_{L_2(\Omega)} \tag{2.2}$$

for all $v \in H_0^1(\Omega)$.

For all $\lambda > \lambda_0$ define $R_\lambda \colon L_2(\Omega) \to L_2(\Omega)$ by $R_\lambda f = u$, where $u \in H_0^1(\Omega)$ is as in (2.2).

(2) *There exists a $c_1 > 0$ such that*

$$\|R_\lambda f\|_{L_q(\Omega)} \leqslant c_1 \, (\lambda - \lambda_0)^{-1/4} \, \|f\|_{L_2(\Omega)}$$

for all $\lambda > \lambda_0$ and $f \in L_2(\Omega)$, where $\frac{1}{q} = \frac{1}{2} - \frac{1}{4d}$.

(3) *There exists a $c_2 \geqslant 1$ such that*

$$\|R_\lambda f\|_{L_q(\Omega)} \leqslant c_2 \, \|f\|_{L_p(\Omega)}$$

for all $\lambda \in [\lambda_0 + 1, \infty)$, $p, q \in [2, \infty]$ and $f \in L_p(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{4d}$.

(4) *If $\lambda > \lambda_0$, $p \in (d, \infty]$ and $f \in L_p(\Omega)$, then $R_\lambda f \in C_0(\Omega)$.*

Proof.

(1). This follows from the Lax–Milgram theorem.

(2). Define $M = \|c_0 - \delta \, \mathbb{1}_\Omega\|_{L_\infty(\Omega)} + \sum_{k=1}^d \|b_k\|_{L_\infty(\Omega)}$. Let $\lambda > \lambda_0$, $f \in L_2(\Omega)$ and set $u = R_\lambda f$. Then

$$\begin{aligned}
\frac{\mu}{2} \, \|u\|_{H^1(\Omega)}^2 + (\lambda - \lambda_0) \|u\|_{L_2(\Omega)}^2 \\
&\leqslant \operatorname{Re} \mathfrak{b}_{\lambda_0}(u) + (\lambda - \lambda_0) \|u\|_{L_2(\Omega)}^2 \\
&= \operatorname{Re} \mathfrak{b}_\lambda(u) \\
&= \operatorname{Re} \sum_{k=1}^d (b_k \, f, \partial_k u)_{L_2(\Omega)} + \operatorname{Re}((c_0 - \delta \, \mathbb{1}_\Omega) \, f, u)_{L_2(\Omega)} \\
&\leqslant M \, \|f\|_{L_2(\Omega)} \, \|u\|_{H^1(\Omega)}.
\end{aligned}$$

So $\|u\|_{H^1(\Omega)} \leqslant 2\mu^{-1} M \|f\|_{L_2(\Omega)}$ and

$$\|R_\lambda f\|_{L_2(\Omega)} = \|u\|_{L_2(\Omega)} \leqslant \sqrt{\frac{2}{\mu(\lambda - \lambda_0)}}\, M \|f\|_{L_2(\Omega)}.$$

The Sobolev embedding theorem implies that there exists a $c_1 > 0$ such that $\|v\|_{L_{q_1}(\Omega)} \leqslant c_1 \|v\|_{H^1(\Omega)}$ for all $v \in H^1_0(\Omega)$, where $\frac{1}{q_1} = \frac{1}{2} - \frac{1}{2d}$. (The extra factor 2 is to avoid a separate case for $d = 2$.) Then $\|R_\lambda f\|_{L_{q_1}(\Omega)} \leqslant 2\mu^{-1} c_1 M \|f\|_{L_2(\Omega)}$. Hence

$$\|R_\lambda f\|_{L_q(\Omega)} \leqslant \|R_\lambda f\|^{1/2}_{L_2(\Omega)} \|R_\lambda f\|^{1/2}_{L_{q_1}(\Omega)} \leqslant c_2 (\lambda - \lambda_0)^{-1/4} \|f\|_{L_2(\Omega)},$$

where $c_2 = (2/\mu)^{3/4} c_1^{1/2} M$.

(3). Apply Corollary 2.5 with $p = 4d$ and $\lambda = \lambda_0 + 1$. It follows that $R_{\lambda_0+1} f \in C_0(\Omega)$ for all $f \in L_p(\Omega)$. Clearly the map $R_{\lambda_0+1}|_{L_p(\Omega)} \colon L_p(\Omega) \to C_0(\Omega)$ has a closed graph. Hence it is continuous. In particular, there exists a $c_3 > 0$ such that $\|R_{\lambda_0+1} f\|_{L_\infty(\Omega)} = \|R_{\lambda_0+1} f\|_{C_0(\Omega)} \leqslant c_3 \|f\|_{L_p(\Omega)}$ for all $f \in L_p(\Omega)$.

Let $\lambda \geqslant \lambda_0 + 1$ and $f \in L_2(\Omega)$. Write $u = R_\lambda f$ and $u_0 = R_{\lambda_0+1} f$. Then $\mathfrak{b}_\lambda(u, v) = \mathfrak{b}_{\lambda_0+1}(u_0, v)$ and $\mathfrak{b}_\lambda(u - u_0, v) = -(\lambda - \lambda_0 - 1)\,(u, v)_{L_2(\Omega)}$ for all $v \in H^1_0(\Omega)$. Hence $u - u_0 \in D(B^D)$ and $(B^D + \lambda I)(u - u_0) = -(\lambda - \lambda_0 - 1)\, u_0$. Consequently

$$R_\lambda = \big(I - (\lambda - \lambda_0 - 1)\,(B^D + \lambda I)^{-1}\big)\, R_{\lambda_0+1}$$

for all $\lambda \geqslant \lambda_0 + 1$. Since the semigroup generated by $-B^D$ has Gaussian bounds, there exists a $c_4 \geqslant 1$ such that $\|(B^D + \lambda I)^{-1}\|_{\infty\to\infty} \leqslant c_4 \lambda^{-1}$ for all $\lambda \geqslant \lambda_0 + 1$. Then $\|R_\lambda f\|_{L_\infty(\Omega)} \leqslant 2c_3 c_4 \|f\|_{L_p(\Omega)}$ for all $\lambda \geqslant \lambda_0 + 1$ and $f \in L_p(\Omega)$.

Finally let $p' \in (2, 4d)$ and let $q' \in (2, \infty)$ be such that $\frac{1}{q'} = \frac{1}{p'} - \frac{1}{4d}$. There exists a $\theta \in (0, 1)$ such that $\frac{1}{p'} = \frac{1-\theta}{2} + \frac{\theta}{p}$. Then $\frac{1}{q'} = \frac{1-\theta}{q}$, where $\frac{1}{q} = \frac{1}{2} - \frac{1}{4d}$. Let $c_1 > 0$ be as in Statement (2). The operator R_λ is bounded from $L_2(\Omega)$ into $L_q(\Omega)$ with norm at most c_1 by Statement (2), and we just proved that the operator R_λ is bounded from $L_p(\Omega)$ into $L_\infty(\Omega)$ with norm at most $2c_3 c_4$. Hence by interpolation the operator R_λ is bounded from $L_{p'}(\Omega)$ into $L_{q'}(\Omega)$ with norm bounded by $c_1^{1-\theta} (2c_3 c_4)^\theta \leqslant c_1 + 2c_3 c_4$, which gives Statement (3).

(4). This is a special case of Corollary 2.5. □

The main step in the proof of Proposition 2.6 is the next lemma.

LEMMA 2.8. — *There exist $\lambda > \lambda_0$ and $c > 0$ such that for all $\Phi \in C^1(\overline{\Omega}) \cap H^1(\Omega)$ there exists a unique $u \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $u|_\Gamma = \Phi|_\Gamma$ and $\mathcal{A}_\lambda u = 0$. Moreover,*

$$\|u\|_{C(\overline{\Omega})} \leqslant c \, \|\Phi|_\Gamma\|_{C(\Gamma)}.$$

Proof. — Choose $\delta = 0$ in Lemma 2.7. Let c_1 and c_2 be as in Lemma 2.7. Let $\lambda \in (\lambda_0 + 1, \infty)$ be such that $c_1 \, c_2^{2d-1} \, (\lambda - \lambda_0)^{-1/4} \, (1 + |\Omega|) \leqslant \frac{1}{2}$. Let R_λ be as in Lemma 2.7. Set $\varphi = \Phi|_\Gamma$.

There exist unique $w, \tilde{w} \in H_0^1(\Omega)$ such that $\mathfrak{a}_\lambda(w, v) = \mathfrak{a}_\lambda(\Phi, v)$ and $\mathfrak{b}_\lambda(\tilde{w}, v) = \mathfrak{b}_\lambda(\Phi, v)$ for all $v \in H_0^1(\Omega)$. Then $\tilde{w} \in C_0(\Omega)$ by Corollary 2.5. Define $u = \Phi - w$ and $\tilde{u} = \Phi - \tilde{w}$. Then $\tilde{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\tilde{u}|_\Gamma = \varphi$. Moreover, $\mathfrak{a}_\lambda(u, v) = 0$ and $\mathfrak{b}_\lambda(\tilde{u}, v) = 0$ for all $v \in H_0^1(\Omega)$, and $\|\tilde{u}\|_{C(\overline{\Omega})} \leqslant \|\varphi\|_{C(\Gamma)}$ by the result of Stampacchia mentioned in the introduction ([13, Théorème 3.8]).

Let $v \in H_0^1(\Omega)$. Then

$$\mathfrak{b}_\lambda(\tilde{u} - u, v) = \sum_{k=1}^{d} (b_k \, u, \partial_k v)_{L_2(\Omega)} + (c_0 \, u, v)_{L_2(\Omega)}$$

and $\tilde{u} - u = R_\lambda u$ by the definition of R_λ.

For all $n \in \{0, \ldots, 2d\}$ define $p_n = \frac{4d}{2d-n}$. Then $p_0 = 2$, $p_{2d-1} = 4d$, $p_{2d} = \infty$ and $\frac{1}{p_n} = \frac{1}{p_{n-1}} - \frac{1}{4d}$ for all $n \in \{1, \ldots, 2d\}$. So $\|\tilde{u} - u\|_{L_{p_n}(\Omega)} \leqslant c_2 \, \|u\|_{L_{p_{n-1}}(\Omega)}$ for all $n \in \{2, \ldots, 2d\}$ and

$$\|\tilde{u} - u\|_{L_{p_1}(\Omega)} \leqslant c_1 \, (\lambda - \lambda_0)^{-1/4} \, \|u\|_{L_2(\Omega)}$$

by Lemma 2.7 (3) and (2). Then

$$\|u\|_{L_{p_1}(\Omega)} \leqslant c_1 \, (\lambda - \lambda_0)^{-1/4} \, \|u\|_{L_2(\Omega)} + (1 + |\Omega|) \, \|\tilde{u}\|_{L_\infty(\Omega)}$$

and

$$\|u\|_{L_{p_n}(\Omega)} \leqslant c_2 \, \|u\|_{L_{p_{n-1}}(\Omega)} + (1 + |\Omega|) \, \|\tilde{u}\|_{L_\infty(\Omega)}$$

for all $n \in \{2, \ldots, 2d\}$. It follows by induction to n that

$$\|u\|_{L_{p_n}(\Omega)} \leqslant c_1 \, c_2^{n-1} \, (\lambda - \lambda_0)^{-1/4} \, \|u\|_{L_2(\Omega)} + (1 + |\Omega|) \sum_{k=0}^{n-1} c_2^k \, \|\tilde{u}\|_{L_\infty(\Omega)}$$

for all $n \in \{2, \ldots, 2d\}$. So $u \in L_{p_{2d-1}}(\Omega) = L_{4d}(\Omega)$ and $\tilde{u} - u = R_\lambda u \in C_0(\Omega)$ by Lemma 2.7 (4). In particular $u \in C(\overline{\Omega})$. Moreover,

$$\begin{aligned}
&\|u\|_{L_\infty(\Omega)} \\
&= \|u\|_{L_{p_{2d}}(\Omega)} \\
&\leqslant c_1\, c_2^{2d-1}\, (\lambda - \lambda_0)^{-1/4}\, \|u\|_{L_2(\Omega)} + 2d\,(1 + |\Omega|)\, c_2^{2d-1}\, \|\tilde{u}\|_{L_\infty(\Omega)} \\
&\leqslant c_1\, c_2^{2d-1}\, (\lambda - \lambda_0)^{-1/4}\, (1 + |\Omega|)\, \|u\|_{L_\infty(\Omega)} + 2d\,(1 + |\Omega|)\, c_2^{2d-1}\, \|\tilde{u}\|_{L_\infty(\Omega)} \\
&\leqslant \frac{1}{2}\, \|u\|_{L_\infty(\Omega)} + 2d\,(1 + |\Omega|)\, c_2^{2d-1}\, \|\tilde{u}\|_{L_\infty(\Omega)}
\end{aligned}$$

by the choice of λ. So

$$\|u\|_{L_\infty(\Omega)} \leqslant 4d\,(1 + |\Omega|)\, c_2^{2d-1}\, \|\tilde{u}\|_{L_\infty(\Omega)} \leqslant 4d\,(1 + |\Omega|)\, c_2^{2d-1}\, \|\varphi\|_{C(\Gamma)}$$

and the proof of the lemma is complete. □

We next wish to remove the λ in Lemma 2.8. For future purposes, we consider the full inhomogeneous problem.

Proposition 2.9. — *Let $p \in (d, \infty]$, $f_0 \in L_{p/2}(\Omega)$ and let $f_1, \ldots, f_d \in L_p(\Omega)$. Let $u \in H_0^1(\Omega)$ be such that $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$. Then $u \in C_0(\Omega)$.*

Proof. — Without loss of generality we may assume that $p \in (d, 4d)$. Choose $\lambda = \delta = \lambda_0 + 1$ in Lemma 2.7 and in Proposition 2.4. By Proposition 2.4 there exists a unique $\tilde{u} \in H_0^1(\Omega) \cap C_0(\Omega)$ such that $\mathcal{B}_\lambda \tilde{u} = f_0 + \sum_{k=1}^d \partial_k f_k$. If $v \in C_c^\infty(\Omega)$, then

$$\begin{aligned}
\mathfrak{b}_\lambda(\tilde{u}, v) &= \langle f_0 + \sum_{k=1}^d \partial_k f_k, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} \\
&= \mathfrak{a}(u, v) \\
&= \mathfrak{b}_\lambda(u, v) + \sum_{k=1}^d (b_k\, u, \partial_k v)_{L_2(\Omega)} + ((c_0 - \delta\, \mathbb{1}_\Omega)\, u, v)_{L_2(\Omega)}.
\end{aligned}$$

So

$$\mathfrak{b}_\lambda(\tilde{u} - u, v) = \sum_{k=1}^d (b_k\, u, \partial_k v)_{L_2(\Omega)} + ((c_0 - \delta\, \mathbb{1}_\Omega)\, u, v)_{L_2(\Omega)}$$

and by density for all $v \in H_0^1(\Omega)$. Hence $u - \tilde{u} = R_\lambda u$, where R_λ is as in Lemma 2.7. For all $n \in \{0, \ldots, 2d - 1\}$ define $p_n = \frac{4d}{2d-n}$. Then $u - \tilde{u} \in L_2(\Omega) = L_{p_0}(\Omega)$. It follows by induction to n that $u \in L_{p_{n-1}}(\Omega)$ and $u - \tilde{u} \in L_{p_n}(\Omega)$ for all $n \in \{1, \ldots, 2d-1\}$, where the last part follows from Lemma 2.7 (3). Hence $u - \tilde{u} \in L_{p_{2d-1}}(\Omega) = L_{4d}(\Omega)$ and $u \in L_p(\Omega)$. Then Lemma 2.7 (4) gives $u - \tilde{u} = R_\lambda u \in C_0(\Omega)$ and therefore $u \in C_0(\Omega)$. □

Corollary 2.10. — *Let $p \in (d,\infty]$. Then $(A^D)^{-1}(L_p(\Omega)) \subset C_0(\Omega)$.*

Corollary 2.11. — *There exists a $c' > 0$ such that $\|(A^D)^{-1} f\|_{L_\infty(\Omega)} \leqslant c' \, \|f\|_{L_\infty(\Omega)}$ for all $f \in L_\infty(\Omega)$.*

Proof. — Closed graph theorem. □

Proof of Proposition 2.6. — Let $c, \lambda > 0$ be as in Lemma 2.8 and let $c' > 0$ be as in Corollary 2.11. By Lemma 2.8 there exists a unique $\tilde{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\tilde{u}|_\Gamma = \Phi|_\Gamma$ and $\mathcal{A}_\lambda \tilde{u} = 0$. By Lemma 2.1 there exists a unique $w \in H^1_0(\Omega)$ such that $\mathfrak{a}(w,v) = \mathfrak{a}(\Phi|_\Omega, v)$ for all $v \in H^1_0(\Omega)$. Set $u = \Phi|_\Omega - w$ and $\tilde{w} = \Phi|_\Omega - \tilde{u}$. Then

$$\begin{aligned}\mathfrak{a}(w,v) = \mathfrak{a}(\Phi|_\Omega, v) &= \mathfrak{a}_\lambda(\Phi|_\Omega, v) - \lambda \, (\Phi, v)_{L_2(\Omega)} = \mathfrak{a}_\lambda(\tilde{w}, v) - \lambda \, (\Phi, v)_{L_2(\Omega)} \\ &= \mathfrak{a}(\tilde{w}, v) + \lambda \, (\tilde{w}, v)_{L_2(\Omega)} - \lambda \, (\Phi, v)_{L_2(\Omega)} = \mathfrak{a}(\tilde{w}, v) - \lambda \, (\tilde{u}, v)_{L_2(\Omega)}\end{aligned}$$

for all $v \in H^1_0(\Omega)$. So

$$\mathfrak{a}(\tilde{u} - u, v) = \mathfrak{a}(w - \tilde{w}, v) = -\lambda \, (\tilde{u}, v)_{L_2(\Omega)}.$$

Since $\tilde{u} - u \in H^1_0(\Omega)$ it follows that $A^D(\tilde{u} - u) = -\lambda \, \tilde{u}$. Consequently, $u = \tilde{u} + \lambda \, (A^D)^{-1} \tilde{u} \in C_0(\Omega)$ by Corollary 2.10. Moreover,

$$\begin{aligned}\|u\|_{C(\overline{\Omega})} = \|u\|_{L_\infty(\Omega)} &\leqslant \|\tilde{u}\|_{L_\infty(\Omega)} + \lambda \, \|(A^D)^{-1} \tilde{u}\|_{L_\infty(\Omega)} \\ &\leqslant (1 + c' \, \lambda) \, \|\tilde{u}\|_{L_\infty(\Omega)} \leqslant (1 + c' \, \lambda) \, c \, \|\Phi|_\Gamma\|_{C(\Gamma)}\end{aligned}$$

and the proof of Proposition 2.6 is complete. □

Define $|||\cdot|||\colon H^1_{\mathrm{loc}}(\Omega) \to [0,\infty]$ by

$$|||u||| = \sup_{\delta > 0} \sup_{\substack{\Omega_0 \subset \Omega \text{ open} \\ d(\Omega_0, \Gamma) = \delta}} \delta \left(\int_{\Omega_0} |\nabla u|^2 \right)^{1/2}.$$

Finally we need the following Caccioppoli inequality.

Proposition 2.12. — *There exists a $c' \geqslant 1$ such that $|||u||| \leqslant c' \, \|u\|_{L_2(\Omega)}$ for all $u \in H^1(\Omega)$ such that $\mathcal{A} u = 0$.*

Proof. — See [9, Theorem 4.4]. □

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. — The uniqueness is already proved in Proposition 2.3.

Let $c > 0$ and $c' \geqslant 1$ be as in Propositions 2.6 and 2.12. Let $\Phi \in C^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$. By Proposition 2.6 there exists a unique $u \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $u|_\Gamma = \Phi|_\Gamma$ and $\mathcal{A}u = 0$. Moreover,

$$\begin{aligned}\|u\|_{C(\overline{\Omega})} + |||u||| &\leqslant \|u\|_{C(\overline{\Omega})} + c' \,\|u\|_{L_2(\Omega)} \\ &\leqslant (2 + |\Omega|)\, c' \,\|u\|_{C(\overline{\Omega})} \\ &\leqslant (2 + |\Omega|)\, c\, c' \,\|\Phi|_\Gamma\|_{C(\Gamma)}. \end{aligned} \tag{2.3}$$

It follows from (2.3) that we can define a linear map $F\colon \{\Phi|_\Gamma : \Phi \in C^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)\} \to H^1(\Omega) \cap C(\overline{\Omega})$ by $F(\Phi|_\Gamma) = u$, where $u \in H^1(\Omega) \cap C(\overline{\Omega})$ is such that $u|_\Gamma = \Phi|_\Gamma$ and $\mathcal{A}u = 0$. Now let $\varphi \in C(\Gamma)$. By the Stone–Weierstraß theorem there are $\Phi_1, \Phi_2, \ldots \in C^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ such that $\lim \Phi_n|_\Gamma = \varphi$ in $C(\Gamma)$. Set $u_n = F(\Phi_n|_\Gamma)$ for all $n \in \mathbb{N}$. Then it follows from (2.3) that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C(\overline{\Omega})$. Let $u = \lim u_n$ in $C(\overline{\Omega})$. Also $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1_{\mathrm{loc}}(\Omega)$ by (2.3). So $u \in H^1_{\mathrm{loc}}(\Omega)$. Since $\mathcal{A}u_n = 0$ for all $n \in \mathbb{N}$, one deduces that $\mathcal{A}u = 0$. Moreover, $u|_\Gamma = \lim u_n|_\Gamma = \lim \Phi_n|_\Gamma = \varphi$. This proves existence. Finally,

$$\begin{aligned}\|u\|_{C(\overline{\Omega})} &= \lim \|u_n\|_{C(\overline{\Omega})} \\ &\leqslant \lim (2 + |\Omega|)\, c\, c' \,\|\Phi_n|_\Gamma\|_{C(\Gamma)} = (2 + |\Omega|)\, c\, c' \,\|\varphi\|_{C(\Gamma)}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Theorem 1.1 has the following extension.

Theorem 2.13. — *Adopt the notation and assumptions of Theorem 1.1. Let $\varphi \in C(\Gamma)$, $p \in (d, \infty]$, $f_0 \in L_{p/2}(\Omega)$ and let $f_1, \ldots, f_d \in L_p(\Omega)$. Then there exists a unique $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ such that $u|_\Gamma = \varphi$ and $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$.*

Proof. — The uniqueness follows as in the proof of Proposition 2.3.

By Lemma 2.1 there exists a $u_0 \in H^1_0(\Omega)$ such that $\mathcal{A}u_0 = f_0 + \sum_{k=1}^d \partial_k f_k$. Then $u_0 \in C_0(\Omega)$ by Proposition 2.9. By Theorem 1.1 there exists a $u_1 \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ such that $u_1|_\Gamma = \varphi$ and $\mathcal{A}u_1 = 0$. Define $u = u_0 + u_1$. Then $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$. Moreover, $u|_\Gamma = \varphi$ and $\mathcal{A}u = f_0 + \sum_{k=1}^d \partial_k f_k$. □

We conclude this section with some results for the classical solution. They will be used in Section 3 and are of independent interest. Recall that $P\colon C(\Gamma) \to C(\overline{\Omega})$ is given by $P\varphi = u$, where $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ is the classical solution, so $u|_\Gamma = \varphi$ and $\mathcal{A}u = 0$.

Proposition 2.14. — *Let $\Phi \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$. Suppose there exists a $w \in H^1_0(\Omega)$ such that $\mathcal{A}\Phi = \mathcal{A}w$. Then $w \in C(\overline{\Omega})$ and $P(\Phi|_\Gamma) = \Phi - w$.*

Proof. — Write $\tilde{w} = \Phi - P(\Phi|_\Gamma)$. Then $\tilde{w} \in C_0(\Omega) \cap H^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}\tilde{w} = \mathcal{A}\Phi = \mathcal{A}w = f_0 + \sum_{k=1}^d \partial_k f_k$, where $f_0 = c_0\, w + \sum_{l=1}^d c_l\, \partial_l w \in L_2(\Omega)$ and $f_k = -\sum_{l=1}^d a_{lk}\, \partial_l w - b_k\, w \in L_2(\Omega)$ for all $k \in \{1, \ldots, d\}$. So $\tilde{w} \in H^1_0(\Omega)$

by Lemma 2.2. Hence $\mathcal{A}(\tilde{w}-w) =$ and $\tilde{w}-w \in \ker A^D = \{0\}$. So $w = \tilde{w} = \Phi - P(\Phi|_\Gamma)$. □

We need the dual map of $\mathcal{A}$. Define the map $\mathcal{A}^t\colon H^1_{\rm loc}(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \mathcal{A}^t u, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \sum_{k,l=1}^d \int_\Omega a_{lk}\,(\partial_k u)\,\overline{\partial_l v} - \sum_{k=1}^d \int_\Omega \overline{c_k}\,u\,\overline{\partial_k v} - \sum_{k=1}^d \int_\Omega \overline{b_k}\,(\partial_k u)\,\overline{v} + \int_\Omega \overline{c_0}\,u\,\overline{v}$$

for all $u \in H^1_{\rm loc}(\Omega)$ and $v \in C_c^\infty(\Omega)$.

Corollary 2.15. — *Suppose that $a_{kl}, b_k, c_k \in W^{1,\infty}(\Omega)$ for all $k,l \in \{1,\dots,d\}$. Let $\Phi \in C(\overline{\Omega})$. Suppose there exists a $w \in H^1_0(\Omega)$ such that*

$$\langle \Phi, \mathcal{A}^t v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \mathfrak{a}(w,v)$$

for all $v \in C_c^\infty(\Omega)$. Then $w \in C(\overline{\Omega})$ and $P(\Phi|_\Gamma) = \Phi - w$.

Proof. — By assumption one has $\langle \Phi - w, \mathcal{A}^t v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0$ for all $v \in C_c^\infty(\Omega)$. Hence $\Phi - w \in H^1_{\rm loc}(\Omega)$ by elliptic regularity. So $\Phi \in H^1_{\rm loc}(\Omega)$ and

$$\langle \mathcal{A}\Phi, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \langle \Phi, \mathcal{A}^t v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \mathfrak{a}(w,v) = \langle \mathcal{A}w, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$$

for all $v \in C_c^\infty(\Omega)$. Therefore $\mathcal{A}\Phi = \mathcal{A}w$ and the result follows from Proposition 2.14. □

The last corollary takes a very simple form for the Laplacian.

Corollary 2.16. — *Let $\Phi \in C(\overline{\Omega})$. Suppose that $\Delta\Phi \in H^{-1}(\Omega)$. Let $w \in H^1_0(\Omega)$ be such that $\Delta\Phi = \Delta w$ as distribution. Then $w \in C(\overline{\Omega})$ and $P(\Phi|_\Gamma) = \Phi - w$.*

This corollary is a special case of [2, Theorem 1.1].

3. Variational and classical solutions: comparison

In this section we show that the variational and classical solutions of the Dirichlet problem are the same. For that we assume throughout this section that Ω is an open set with Lipschitz boundary. Moreover, we adopt the assumptions and notation of Theorem 1.1. Recall that for all $\varphi \in C(\Gamma)$ we denote by $P\varphi \in C(\overline{\Omega})$ the classical solution and for all $\varphi \in H^{1/2}(\Gamma)$, we denote by $\gamma\varphi \in H^1(\Omega)$ the variational solution of the Dirichlet problem. We shall prove in this section that they coincide if both are defined.

The fact that they coincide for restrictions to Γ of functions in $C(\overline{\Omega}) \cap H^1(\Omega)$ is a consequence of Proposition 2.14. We state this as a proposition.

Proposition 3.1. — *Let $\Phi \in C(\overline{\Omega}) \cap H^1(\Omega)$. Then $P(\Phi|_\Gamma) = \gamma(\Phi|_\Gamma)$ almost everywhere.*

So for the proof of Theorem 1.2 it suffices to show that the map $\Phi \mapsto \Phi|_\Gamma$ from $C(\overline{\Omega}) \cap H^1(\Omega)$ into $C(\Gamma) \cap H^{1/2}(\Gamma)$ is surjective. This is surprisingly difficult to prove. We first prove Theorem 1.2 for the Laplacian with the help of Proposition 3.1 and a deep result of Dahlberg. As a consequence we obtain the desired surjectivity result. Then as noticed earlier, Theorem 1.2 follows for our general elliptic operator.

Theorem 3.2. — *Assume that $a_{kl} = \delta_{kl}$ and $b_k = c_k = c_0 = 0$ for all $k, l \in \{1, \ldots, d\}$. Let $\varphi \in C(\Gamma) \cap H^{1/2}(\Gamma)$. Then $P\varphi = \gamma\varphi$ almost everywhere.*

Proof. — Let $x \in \Omega$. By Dahlberg [7, Theorem 1] there exists a unique $k_x \in L_1(\Gamma)$ such that $(P\varphi)(x) = \int_\Gamma k_x \, \varphi \, \mathrm{d}\sigma$ for all $\varphi \in C(\Gamma)$.

Now let $\varphi \in C(\Gamma) \cap H^{1/2}(\Gamma)$. Without loss of generality we may assume that φ is real valued. Then there exists a $u \in H^1(\Omega, \mathbb{R})$ such that $\varphi = \operatorname{Tr} u$. Since $H^1(\Omega) \cap C(\overline{\Omega})$ is dense in $H^1(\Omega)$, there exist $u_1, u_2, \ldots \in H^1(\Omega, \mathbb{R}) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$. Define $v_n = (-\|\varphi\|_{L_\infty(\Gamma)}) \vee u_n \wedge \|\varphi\|_{L_\infty(\Gamma)}$ for all $n \in \mathbb{N}$. Then $v_n \in H^1(\Omega) \cap C(\overline{\Omega})$. Write $\varphi_n = v_n|_\Gamma = \operatorname{Tr} v_n \in C(\Gamma) \cap H^{1/2}(\Gamma)$ for all $n \in \mathbb{N}$. Then $P\varphi_n = \gamma\varphi_n$ almost everywhere for all $n \in \mathbb{N}$ by Proposition 3.1.

Note that

$$\lim \varphi_n = \lim \operatorname{Tr} v_n = (-\|\varphi\|_{L_\infty(\Gamma)}) \vee \operatorname{Tr} u \wedge \|\varphi\|_{L_\infty(\Gamma)} = \varphi$$

in $H^{1/2}(\Gamma)$. So by continuity of γ one deduces that $\gamma\varphi = \lim \gamma\varphi_n$ in $H^1(\Omega)$ and in particular in $L_2(\Omega)$. Passing to a subsequence, if necessary, we may assume that

$$(\gamma\varphi)(x) = \lim(\gamma\varphi_n)(x)$$

for almost all $x \in \Omega$. Using again that $\lim \varphi_n = \varphi$ in $H^{1/2}(\Gamma)$ and therefore also in $L_2(\Gamma)$, we may assume that $\lim \varphi_n = \varphi$ almost everywhere on Γ. Hence if $x \in \Omega$, then

$$(P\varphi)(x) = \int_\Gamma k_x \, \varphi \, \mathrm{d}\sigma = \lim \int_\Gamma k_x \, \varphi_n \, \mathrm{d}\sigma = \lim(P\varphi_n)(x)$$

by the Lebesgue dominated convergence theorem. Since $P\varphi_n = \gamma\varphi_n$ almost everywhere for all $n \in \mathbb{N}$ one concludes that $(P\varphi)(x) = (\gamma\varphi)(x)$ for almost all $x \in \Omega$. $\square$

The desired surjectivity result is the following corollary of Theorem 3.2.

Corollary 3.3. — *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Let $\varphi \in C(\Gamma) \cap H^{1/2}(\Gamma)$. Then there exists a $u \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\varphi = u|_\Gamma$.*

Proof of Theorem 1.2. — This follows from Corollary 3.3 and Proposition 3.1. □

Corollary 3.4. — *Adopt the notation and assumptions of Theorem 1.1. Suppose that Ω has a Lipschitz boundary. Let $u \in C(\overline{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ and suppose that $\mathcal{A}u = 0$. Then $u \in H^1(\Omega)$ if and only if $u|_\Gamma \in H^{1/2}(\Gamma)$.*

Proof. — "$\Rightarrow$" is trivial.

"$\Leftarrow$". Suppose $u|_\Gamma \in H^{1/2}(\Gamma)$. Then $u = P(u|_\Gamma) = \gamma(u|_\Gamma) \in H^1(\Omega)$ by Theorem 1.2. □

4. Semigroup and holomorphy on $C_0(\Omega)$

In this section we prove Theorem 1.3. Throughout this section we adopt the notation and assumptions of Theorem 1.1. We need several lemmas.

Lemma 4.1. — *The operator A_c is invertible and, moreover, $(A_c)^{-1} = (A^D)^{-1}|_{C_0(\Omega)}$.*

Proof. — If $v \in C_0(\Omega)$, then $(A^D)^{-1}v \in C_0(\Omega)$ by Corollary 2.10. Moreover, $A^D((A^D)^{-1}v) = v$. So $(A^D)^{-1}v \in D(A_c)$ and $A_c((A^D)^{-1}v) = v$. Hence A_c is surjective. Since A^D is injective, also A_c is injective. Therefore A_c is invertible and $(A_c)^{-1} = (A^D)^{-1}|_{C_0(\Omega)}$. □

The next proof is inspired by arguments in [1, Theorem 4.4].

Lemma 4.2. — *The domain $D(A_c)$ of the operator A_c is dense in $C_0(\Omega)$.*

Proof. — Let $\rho \in M(\Omega)$, the Banach space of all complex measures on Ω and suppose that $\int_\Omega v \, d\rho = 0$ for all $v \in D(A_c)$. There exist $w_1, w_2, \ldots \in L_2(\Omega)$ such that $\sup \|w_n\|_{L_1(\Omega)} < \infty$ and $\lim \int_\Omega v \, \overline{w_n} = \int_\Omega v \, d\rho$ for all $v \in C_0(\Omega)$.

Choose $p = d + 2$ and let $q \in (1, 2)$ be the dual exponent of p. It follows from Proposition 2.9 that the operator $(A^D)^{-1}$ extends to a continuous operator from $W^{-1,p}(\Omega)$ into $C_0(\Omega)$. Hence the operator $(A^D)^{-1*}$ extends to a continuous operator from $M(\Omega)$ into $W^{1,q}_0(\Omega)$. In particular, there exists a $c > 0$ such that $\|(A^D)^{-1*}w\|_{W^{1,q}_0(\Omega)} \leqslant c \, \|w\|_{L_1(\Omega)}$ for all $w \in L_2(\Omega)$.

For all $n \in \mathbb{N}$ set $u_n = (A^D)^{-1*} w_n$. We emphasise that $u_n \in D((A^D)^*)$. Then $\sup \|u_n\|_{W_0^{1,q}(\Omega)} < \infty$. Note that $W_0^{1,q}(\Omega)$ is reflexive. Hence passing to a subsequence if necessary, there exists a $u \in W_0^{1,q}(\Omega)$ such that $\lim u_n = u$ weakly in $W_0^{1,q}(\Omega)$.

Let $v \in C_c^\infty(\Omega)$. Then $(A^D)^{-1} v \in D(A_c)$ by Lemma 4.1. Therefore

$$\begin{aligned} 0 &= \int_\Omega (A^D)^{-1} v \, \mathrm{d}\rho = \lim \int_\Omega \big((A^D)^{-1} v\big) \, \overline{w_n} \\ &= \lim \big(v, (A^D)^{-1*} w_n\big)_{L_2(\Omega)} = \lim (v, u_n)_{L_2(\Omega)} = \lim \int_\Omega v \, \overline{u_n} = \lim \int_\Omega v \, \overline{u}. \end{aligned}$$

Hence $u = 0$.

Again let $v \in C_c^\infty(\Omega)$. Then

$$\int_\Omega v \, \mathrm{d}\rho = \lim \int_\Omega v \, \overline{w_n} = \lim (v, (A^D)^* u_n)_{L_2(\Omega)} = \lim \mathfrak{a}(v, u_n) = 0,$$

where we used (1.4). So $\rho = 0$ and $D(A_c)$ is dense in $C_0(\Omega)$. □

Now we prove that $-A_c$ generates a holomorphic C_0-semigroup.

Proof of Theorem 1.3. — Let S be the semigroup generated by $-A^D$. Then S has a kernel with Gaussian upper bounds by [12, Theorem 6.10] (see also [8, Theorem 6.1] for operators with real valued coefficients and [3, Theorems 3.1 and 4.4]). Hence the semigroup S extends consistently to a semigroup $S^{(p)}$ on $L_p(\Omega)$ for all $p \in [1, \infty]$.

Choose $p \in (d, \infty]$. Let $t > 0$ and $u \in L_2(\Omega)$. Since S is a holomorphic semigroup, one deduces that $S_t u \in D(A^D)$ and $A^D \, S_t u \in L_2(\Omega)$. Next the Gaussian kernel bounds imply that S_t maps $L_2(\Omega)$ into $L_p(\Omega)$. So $A^D \, S_{2t} u = S_t \, A^D \, S_t u \in L_p(\Omega)$ and

$$S_{2t} u \in (A^D)^{-1}(L_p(\Omega)) \subset C_0(\Omega) \tag{4.1}$$

by Corollary 2.10. Hence $S_t C_0(\Omega) \subset C_0(\Omega)$ for all $t > 0$. For all $t > 0$ let $S_t^c = S_t|_{C_0(\Omega)} \colon C_0(\Omega) \to C_0(\Omega)$. Then $(S_t^c)_{t>0}$ is a semigroup on $C_0(\Omega)$. Moreover, using again the Gaussian kernel bounds there exists an $M \geqslant 1$ such that $\|S_t^c\| \leqslant \|S_t^{(\infty)}\| \leqslant M$ for all $t \in (0, 1]$.

Let $t \in (0, 1]$ and $u \in D(A_c)$. Then

$$\begin{aligned} \|(I - S_t^c) u\|_{C_0(\Omega)} &= \left\| \int_0^t S^s \, A_c u \, \mathrm{d}s \right\|_{C_0(\Omega)} \\ &\leqslant \int_0^t M \, \|A_c u\|_\infty \mathrm{d}s = M \, t \, \|A_c u\|_\infty. \end{aligned}$$

So $\lim_{t\downarrow 0} S^c_t u = u$ in $C_0(\Omega)$. Since $D(A_c)$ is dense in $C_0(\Omega)$ by Lemma 4.2, one deduces that $\lim_{t\downarrow 0} S^c_t u = u$ in $C_0(\Omega)$ for all $u \in C_0(\Omega)$. So S^c is a C_0-semigroup.

Finally, using once more the Gaussian kernel bounds, it follows that the semigroup S^c is holomorphic (see [3, Theorem 5.4]). □

We conclude this section by establishing Gaussian kernels which are continuous up to the boundary. For this we use the following special case of [4, Theorem 2.1].

Proposition 4.3. — *Suppose that $|\partial\Omega| = 0$. Let T be a semigroup in $L_2(\Omega)$ such that $T_t L_2(\Omega) \subset C(\overline{\Omega})$ and $T^*_t L_2(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$. Then for all $t > 0$ there exists a unique $k_t \in C(\overline{\Omega} \times \overline{\Omega})$ such that*

$$(T_t u)(x) = \int_\Omega k_t(x, y)\, u(y)\, \mathrm{d}y$$

for all $u \in L_2(\Omega)$ and $x \in \Omega$.

We continue to denote by S the semigroup generated by $-A^D$ and we also denote by S the holomorphic extension. For all $\theta \in (0, \pi]$ let $\Sigma(\theta) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ be the open sector with (half)angle θ.

Theorem 4.4. — *Adopt the notation and assumptions of Theorem 1.1. In addition assume that $|\partial\Omega| = 0$ and that b_k is real valued for all $k \in \{1, \dots, d\}$. Let θ be the holomorphy angle of S. Then for all $z \in \Sigma(\theta)$ there exists a unique $k_z \in C(\overline{\Omega} \times \overline{\Omega})$ such that the following is valid.*

1. *$(S_z u)(x) = \int_\Omega k_z(x, y)\, u(y)\, \mathrm{d}y$ for all $z \in \Sigma(\theta)$, $u \in L_2(\Omega)$ and $x \in \overline{\Omega}$.*
2. *$k_z(x, y) = 0$ for all $z \in \Sigma(\theta)$ and $x, y \in \overline{\Omega}$ with $x \in \partial\Omega$ or $y \in \partial\Omega$.*
3. *The map $z \mapsto k_z$ is holomorphic from $\Sigma(\theta)$ into $C(\overline{\Omega} \times \overline{\Omega})$.*
4. *For all $\theta' \in (0, \theta)$ there exist $b, c, \omega > 0$ such that*

$$|k_z(x, y)| \leqslant c\, |z|^{-d/2}\, \mathrm{e}^{\omega |z|}\, \mathrm{e}^{-b \frac{|x-y|^2}{|z|}}$$

for all $z \in \Sigma(\theta')$ and $x, y \in \overline{\Omega}$.

Proof. — It follows from (4.1) that $S_z L_2(\Omega) \subset C_0(\Omega)$ for all $z \in \Sigma(\theta)$. Since the coefficients b_k are real, also the adjoint operator satisfies the conditions of Theorem 1.1. Therefore $S^*_z L_2(\Omega) \subset C_0(\Omega)$ for all $z \in \Sigma(\theta)$. It follows from Proposition 4.3 that for all $z \in \Sigma(\theta)$ there exists a unique $k_z \in C(\overline{\Omega} \times \overline{\Omega})$ such that $(S_z u)(x) = \int_\Omega k_z(x, y)\, u(y)\, \mathrm{d}y$ for all $u \in L_2(\Omega)$ and $x \in \overline{\Omega}$. Since $S_z u \in C_0(\Omega)$ one deduces that $k_z(x, y) = 0$ for all

$z \in \Sigma(\theta)$, $x \in \partial\Omega$ and $y \in \overline{\Omega}$. Considering adjoints the same is valid with x and y interchanged. If $u, v \in C_0(\Omega)$, then the map

$$z \mapsto \langle k_z, u \otimes \overline{v} \rangle_{C(\overline{\Omega}\times\overline{\Omega})\times C(\overline{\Omega}\times\overline{\Omega})^*} = (S_z u, v)_{L_2(\Omega)}$$

is holomorphic on $\Sigma(\theta)$. Therefore Statement (3) is a consequence of [6, Theorem 3.1]. The Gaussian bounds of Statement (4) follow from [3, Theorem 5.4]. □

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