

# UC San Diego

## Recent Work

### **Title**

Starting Small and Commitment

### **Permalink**

<https://escholarship.org/uc/item/37p340fc>

### **Author**

Watson, Joel

### **Publication Date**

1999-04-01

99-08

**UNIVERSITY OF CALIFORNIA, SAN DIEGO**

DEPARTMENT OF ECONOMICS

STARTING SMALL AND COMMITMENT

BY

JOEL WATSON

**DISCUSSION PAPER 99-08  
APRIL 1999**

# Starting Small and Commitment

Joel Watson\*

April 19, 1999

## Abstract

I study a model of a long-term partnership with two-sided incomplete information. The partners jointly determine the stakes of their relationship and individually decide whether to cooperate with or betray each other over time. I characterize the extremal — *interim incentive efficient* — equilibria. In these equilibria, the partners generally “start small,” with the level of interaction growing over time. The types of players separate quickly. Further, cooperation between “good” types is viable regardless of how pessimistic the players are about each other initially. The quick nature of separation in an extremal equilibrium contrasts with the outcome selected by a strong renegotiation criterion (as studied in Watson [11]). *Journal of Economic Literature* Classification Numbers: C72, C73, D74.

At the beginning of a long-term relationship, prospective partners may be dubious about each others’ motives. To optimally encourage cooperation at low risk, the parties balance the opportunity for high returns with the possibility that one agent will take advantage of the other. Often the best way to structure the relationship involves “starting small,” whereby the stakes of the relationship grow over time.

I study the phenomenon of gradualism, or “starting small,” by analyzing a class of dynamic games with variable stakes. These games model a dynamic partnership in which (a) the two partners can jointly choose from among a variety of projects (levels of interaction) over time and (b) the partners also individually decide whether to cooperate at every instant. There is two-sided incomplete information about each partner’s incentive to behave opportunistically, an act which injures the other party. While “high” types generally prefer to cooperate over time as long as their partners cooperate, “low” types have an incentive to betray their partner’s trust. Interacting at a low level is less risky than at a high level because small stakes limit the detrimental consequences of opportunistic behavior. The

---

\*Department of Economics, UC San Diego and Cowles Foundation, Yale University. Internet: <http://weber.ucsd.edu/~jwatson/>. I appreciate the remarks of Vince Crawford, David Kreps, George Mailath, Darwin Neher, Clara Ponsati, Larry Samuelson, Joel Sobel, and numerous seminar participants. This project has been supported by grants from the National Science Foundation (SBR-9422196 and SBR-9630270).

players' incentives to cooperate or betray depend on how the stakes change over time. In particular, although low types favor betrayal, they may prefer delaying such an action if the stakes rise quickly enough.

My analysis of this strategic setting is divided between the current paper and a companion paper (Watson [11]). Both papers characterize *equilibrium regimes* in the partnership. An equilibrium regime is a specification of (a) how the stakes change over time and (b) an equilibrium in the players' decisions to cooperate or betray. These papers both demonstrate that cooperation between high types is viable, regardless of the players' initial beliefs about each other, as long as the relationship starts small enough.<sup>1</sup> Beyond the viability issue, the two papers focus on different genera of equilibrium regimes. Watson [11] characterizes a regime that is selected by a strong renegotiation condition. The present paper characterizes optimal regimes when the players have greater ability to commit to the way the stakes change over time.

Specifically, the present paper characterizes regimes that are interim incentive efficient (IIE). Such regimes are Pareto superior in the class of equilibrium regimes, treating different player-types separately. IIE regimes are shown to be "Q-regimes" ("Q" stands for "quick"), which are identified by two important properties. First, the relationship generally starts small. Second, separation of the types is concentrated on two points in time: at the beginning of the relationship and right when the level reaches its maximum.

In a Q-regime, two-sided incomplete information is resolved at time zero, when the low type of player 1 betrays with probability one and a weakly greater mass of the low type of player 2 also betrays. (At the outset of the game, player 1 is weakly more pessimistic about player 2 than is player 2 about player 1.) Then there is a period of time when neither player betrays with positive probability and the level of the relationship gradually rises. Just when the level reaches its maximum, the low type of player 2 betrays with his residual probability mass. Conditional on cooperation through this time, the players cooperate thereafter at the highest level of interaction.

I further analyze two special cases. First, I examine the symmetric case, where the players are equally pessimistic about each other at the beginning of the game. In this setting, the low and high types of both players separate completely in the first instant of interaction. The separation is achieved by the low types betraying with probability one. After the high types are made known, the level of the relationship rises for awhile before settling at its maximum. Second, I examine the one-sided case, where player 1 is known to be a high type. In this setting, the level of the relationship rises gradually over time and the low type of player 2 waits until the maximum level is reached before betraying. Under non-symmetric, two-sided incomplete information, IIE regimes generally share the features of both the symmetric and one-sided cases.

On a technical level, this paper offers a novel analysis of partnerships with two-sided incomplete information. The exercise delivers definitive conclusions in an inherently chal-

---

<sup>1</sup>This was first reported in Watson [10]. The present paper reports a version of the analysis on optimal regimes found in Watson [10].

lenging game-theoretic setting.<sup>2</sup> By design, my model focuses on how agents change the structure of their relationship and renegotiate due to revelation of information. Renegotiation in off-equilibrium path contingencies is not at issue.

The presentation begins in the following section with the description of the model. Some useful properties of equilibrium are developed in Section 2. Section 3 characterizes the set of interim incentive efficient regimes. Section 4 discusses the special cases noted above. The issue of renegotiation is discussed in Section 5. Section 6 concludes. The Appendix contains the formal proofs. Some related literature is discussed in the companion paper, Watson [11].

## 1 Dynamic Games with Variable Stakes

I consider a continuous-time model of a partnership between two agents. The model is the same as that of Watson [11], except here a more narrow class of games is studied for technical simplicity.<sup>3</sup> There is a *level function*  $\alpha : [0, \infty) \rightarrow [0, 1]$  specifying how the *stakes* of the partnership change over time (which runs from zero to infinity). At time  $t \in [0, \infty)$ ,  $\alpha(t)$  is the level of the relationship.<sup>4</sup> Prior to time zero, the players jointly select and commit to the function  $\alpha$ . I restrict attention to level functions that specify a positive level at some time. It will not be important how the selection of  $\alpha$  is modeled, so I shall consider  $\alpha$  exogenous for now. Issues regarding negotiation of, and commitment to, the level function are studied in Section 5.

Given the level function, the players interact over time in a prisoners' dilemma-like setting called the *partnership game*. At every instant of time, the players individually choose whether to continue cooperating or to take a selfish action that ends the partnership. In referring to selfish behavior, I use the terms "betray" and "quit." A player's strategy specifies when (if ever) the player will betray the other, ending the game. Cooperation entails a flow payoff  $z_i\alpha(t)$  to player  $i$  at time  $t$ . The selfish action at  $t$  brings about terminal payoffs. Player  $i$ 's terminal payoff at  $t$  is  $x_i\alpha(t)$  if player  $i$  betrays the other player at this time,  $y_i\alpha(t)$  if player  $i$ 's opponent betrays at  $t$ , and zero if both players betray at  $t$ . The numbers  $z_i$ ,  $x_i$ , and  $y_i$  are private information to the players, as discussed below.

To better understand the specification of payoffs, suppose the players cooperate until  $t$ , at which time the game ends due to betrayal by one or both players. Let  $t = \infty$  for the case of perpetual cooperation. Then player  $i$ 's payoff in the game is given by

$$\int_0^t z_i\alpha(s)e^{-rs}ds + g_i\alpha(t)e^{-rt},$$

---

<sup>2</sup>Generally, it is difficult to characterize equilibria, much less obtain cutting results, in models with two-sided incomplete information.

<sup>3</sup>Here I examine the class of dynamic games with variable stakes that are linear with respect to the stakes. In Watson [11] I study a more general class of games.

<sup>4</sup>For concreteness, think of  $\alpha(t)$  as the number of projects on which the players have agreed to work at time  $t$ .

	Cooperate	Betray
Cooperate	$z_1a/r, z_2a/r$	$y_1a, x_2a$
Betray	$x_1a, y_2a$	$0, 0$

Figure 1: An example with constant level over time.

where  $g_i \equiv x_i$  if player  $i$  alone betrayed at  $t$ ,  $g_i \equiv y_i$  if player  $j$  ( $i$ 's opponent) alone betrayed at  $t$ , and  $g_i \equiv 0$  if both players betrayed at  $t$ . Note that the level determines the scale of the cooperative flow and terminal payoffs at each moment of time; a low level implies that the benefits of ongoing cooperation, as well as the values due to betrayal, are small; a high level implies the opposite.

The players have private information about their flow and terminal payoff functions. Each player is of two possible types, low ( $L$ ) and high ( $H$ ). The low type has payoff parameters  $\underline{z}, \underline{x}, \underline{y}$ ; the high type has parameters  $\bar{z}, \bar{x}, \bar{y}$ . For  $i \in \{1, 2\}$  and  $K \in \{L, H\}$ , “ $iK$ ” denotes player  $i$  of type  $K$ . The ex ante probability that player  $i$  is the high type is  $p_i$ , which is common knowledge. Assume that  $p_1 \geq p_2$ . The model is otherwise symmetric, in that players of the same type  $K \in \{L, H\}$  have the same payoff parameters  $x, y$ , and  $z$ . As stipulated below, I associate with the high type a greater incentive to cooperate over time relative to the low type.

Before specifying how the types are assumed to differ, I make some assumptions common to the two types. I wish to study a setting in which the partners can benefit from long-term cooperation, each may have a short-term incentive to betray the other, and a player suffers if the other betrays him. These properties are captured by:

**Assumption 1**  $\underline{x}, \underline{z}, \bar{x}, \bar{z} \geq 0, \underline{y}, \bar{y} < 0$ .

Note that games with variable stakes are similar to wars of attrition in that they are played in continuous time and end when one or both players decides to quit.<sup>5</sup>

I next turn to distinctions between the high and low types, in particular regarding the incentive to cooperate in the partnership game. To develop intuition, take the example of a level function specifying  $\alpha(t) = a$  for all  $t$  and any  $a \in (0, 1]$ . The matrix in Figure 1 depicts the payoffs when the players are constrained to choose between betraying now or cooperating forever. Note that  $\int_0^\infty z_i a e^{-rs} ds = z_i a / r$ . As the figure indicates, perpetual cooperation can be sustained in the game only if  $z_1/r \geq x_1$  and  $z_2/r \geq x_2$ . Betrayal is the dominant choice for player  $i$  if  $z_i/r < x_i$ .

Evaluating incentives for games with non-constant level functions is more complicated than for those with constant level functions. In particular, if  $\alpha$  rises over time then player  $i$

---

<sup>5</sup>There are two substantive differences between these classes of games. First, in wars of attrition a player generally prefers that his opponent quit, while in the games studied here players may wish to cooperate perpetually. The second difference is that games with variable stakes include a level function, which the players jointly determine.

may be willing to cooperate for awhile even if  $z_i/r < x_i$ . Delaying betrayal can be optimal if the betrayal payoff rises fast enough over time to overcome discounting. However, since the level is bounded, this player must eventually prefer quitting. The following lemma from Watson [11] makes this formal.

**Lemma 1** *Suppose  $x_i > z_i/r$ . Take any level function  $\alpha$  such that  $\alpha(t) > 0$  for some  $t$ . There are times  $T, S \in [0, \infty)$ , with  $T < S$ , such that the following holds regardless of player  $j$ 's strategy. Conditional on reaching time  $T$  in the game, payoff-maximizing player  $i$  must quit prior to time  $S$  with probability one.*

I define the low and high types so that only the high types may have the incentive to cooperate perpetually in the partnership game.

**Assumption 2**  $\bar{x} < \bar{z}/r$ ,  $\underline{z} = 0 < \underline{x}$ , and  $\underline{x} + \underline{y} \leq 0$ .

This assumption yields the interesting class of games in which the low type players must eventually betray in every equilibrium. The high types would like to establish long-term cooperation with each other, but private information makes building the relationship difficult. Cooperation entails a risk if one's opponent may betray, since  $y < 0$ . The assumption  $\underline{z} = 0 < \underline{x}$  is stronger than is necessary ( $\underline{x} > \underline{z}/r$  would do), but it greatly simplifies some of the analysis.<sup>6</sup> The inequality  $\underline{x} + \underline{y} \leq 0$ , which is required for one of the components of the analysis, means no value is created between the low types from betrayal.

## Equilibrium

A level function and a strategy profile for the partnership game shall be denoted a *regime*. If the strategy profile is an equilibrium in the partnership game then I call the regime an *equilibrium regime*. Obviously, regardless of  $\alpha$ , there is always an equilibrium in which both types of both players betray at  $t = 0$ . Thus, every level function is associated with at least one equilibrium regime. However, I shall look for more interesting *cooperative equilibria*, where the partnership is viable between two high types. More precisely, I restrict attention to equilibria in which the high types cooperate perpetually. I assume the high types adopt a strategy of never betraying, subject to establishing later that this strategy is rational.<sup>7</sup> The low type players will betray at some point, but they may have an incentive to cooperate for a stretch of time before betraying. An important component of the analysis involves demonstrating that cooperative equilibria exist.

Each low type has a strategy defined by a cumulative distribution function  $F$ , where  $F(t)$  is the probability that this player betrays at or before time  $t$ . Since the low type must quit with probability one in bounded time,  $F(s) = 1$  for some time  $s$ . Assuming the high types cooperate perpetually, if player  $j$  of low type plays strategy  $F_j$  then player  $i$  effectively faces the strategy defined by  $(1 - p_j)F_j$ . We can then associate the low type's

---

<sup>6</sup>Those interested in a more general treatment of the basic model are invited to read Watson [10, 11].

<sup>7</sup>I therefore rule out regimes in which the high types cooperate perpetually with positive probability and quit with positive probability.

strategy  $F_j$  with the function  $\theta_j \equiv p_j + (1 - p_j)(1 - F_j)$ , where  $\theta_j(t)$  is the probability that player  $i$ 's opponent cooperates through time  $t$ . It will be easiest to work with the function  $\theta_j$  directly, with the understanding that  $\theta_j$  is weakly decreasing and continuous from the right,  $\theta_j(t) \geq p_j$  for all  $t$ , and  $\theta_j(s) = p_j$  for some  $s$ .

## 2 Cooperative Equilibrium Regimes

In this section, I study cooperative equilibrium in the partnership game for an arbitrary level function  $\alpha$  and initial beliefs  $p_1$  and  $p_2$ . I assume  $\theta_i$  is continuously differentiable, except perhaps on a closed set of measure zero. I also assume that  $\alpha$  has well-defined right- and left-hand limits and is differentiable except on a closed set of measure zero. The assumptions on  $\alpha$  are relaxed in subsequent sections.

For a useful benchmark, consider the partnership game with  $\alpha(t) = 1$  for all  $t$ . Obviously, with this level function, a low type has no incentive to delay betraying his partner, regardless of the partner's strategy. Therefore every equilibrium involves the low type players betraying at the beginning of the game. Next assess the high types' incentive to cooperate. Suppose player  $jH$  cooperates perpetually. If player  $iH$  is inclined to betray, the best time to do so is at  $t = 0$ . Player  $iH$  obtains  $p_j\bar{x}$  by quitting at the beginning of the game and  $(1 - p_j)\bar{y} + p_j\bar{z}/r$  from cooperating perpetually. Thus, cooperation by the high types is an equilibrium if and only if  $(1 - p_i)\bar{y} + p_i\bar{z}/r \geq p_i\bar{x}$ , for  $i = 1, 2$ . This condition simplifies to  $p_i \geq \bar{y}/[\bar{z}/r - \bar{x} - \bar{y}] \in (0, 1)$ . Observe that, in this constant  $\alpha(t)$  case, incomplete information denies any hope of cooperation between high types if either  $p_1$  or  $p_2$  is small.

Next I derive equilibrium conditions for more general level functions, where the low types may cooperate for some time before quitting. It is helpful to start by recognizing how a player's payoff depends on the level function, the time at which this player plans to quit, and the opponent's quit distribution. To be precise, if player  $i$  plans to quit at time  $t$  then her expected payoff is

$$\int_0^t z_i \alpha(s) e^{-rs} \theta_j(s) ds + \lim_{\tau \rightarrow t^-} \int_0^\tau y_i \alpha(s) e^{-rs} [-d\theta_j(s)] + x_i \alpha(t) e^{-rt} \theta_j(t). \quad (1)$$

The first integral is the expected payoff of cooperation with player  $j$ , the second integral is the expected (negative) payoff arising because of the chance that player  $j$  will quit before time  $t$ , and the last term is player  $i$ 's expected instantaneous payoff from quitting at  $t$ . The functions  $\theta_1$  and  $\theta_2$  define an equilibrium if the strategy  $F_i$  of the low type of player  $i$  is a best response to  $\theta_j$ , for  $i = 1, 2$ , and the strategy of cooperating perpetually is a best response for the high type players.

### Conditions Regarding Low Types

I shall examine how the low types' betrayal distribution relates to the level function in equilibrium. There may be times at which  $\theta_i$  is discontinuous and intervals in which

$\theta_i$  is continuously decreasing, reflecting atoms and continuous phases in the low types' betrayal distributions. In fact, in both cases, the level function  $\alpha$  constrains equilibrium strategies in a precise and easily identified way. Some additional notation will be helpful in demonstrating the connection. For all  $t > 0$ , define  $\alpha^-(t)$  and  $\alpha^+(t)$  as the directional limits of  $\alpha$  at  $t$  from the left and the right, respectively.<sup>8</sup> I use the convention that  $\alpha^-(0) \equiv 0$ . Define  $\theta_i^-(t)$  analogously, with  $\theta_i^-(0) \equiv 1$ , for  $i = 1, 2$ .

Consider first the implications of player  $iL$  weakly preferring to quit at some time  $t$ . Let  $q_j \equiv \theta_j(t)/\theta_j^-(t)$  denote the probability that player  $j$  cooperates at  $t$ , conditional on  $j$  cooperating prior to  $t$ . Note that  $q_j = 1$  if  $\theta_j$  is continuous at  $t$ . Conditional on reaching  $t$ , player  $iL$ 's normalized expected payoff from quitting at  $t$  is  $q_j \underline{x} \alpha(t)$ .<sup>9</sup> If  $iL$  waits until just after  $t$  to quit, her expected payoff can be made arbitrarily close to  $(1 - q_j) \underline{y} \alpha(t) + q_j \underline{x} \alpha^+(t)$ . Thus, given that he weakly prefers to quit at  $t$ , it must be that

$$q_j \underline{x} \alpha(t) \geq (1 - q_j) \underline{y} \alpha(t) + q_j \underline{x} \alpha^+(t). \quad (2)$$

Defining

$$D(q) \equiv \frac{q \underline{x}}{q \underline{x} - (1 - q) \underline{y}},$$

the inequality becomes  $\alpha(t) \geq \alpha^+(t) D(q_j)$ . This condition holds as an equality if player  $iL$  is indifferent between betraying at  $t$  and at times after, but arbitrarily close to,  $t$ .

If player  $iL$  quits just before time  $t$  then her payoff (conditional on player  $j$  cooperating before  $t$ ) is arbitrarily close to  $\underline{x} \alpha^-(t)$ , so the weak preference to wait until  $t$  implies

$$q_j \alpha(t) \geq \alpha^-(t). \quad (3)$$

As above, this condition holds as an equality if player  $iL$  is indifferent between betraying at  $t$  and just before  $t$ .

Next consider the implications of player  $iL$  quitting at a positive rate over an interval of time where  $\alpha$  is continuous. In particular, suppose that  $\alpha$  is continuous at some  $t \geq 0$  and  $\theta_i'(t) < 0$ . It must be that  $\theta_j^-(t) = \theta_j(t)$ , for otherwise the continuity of  $\alpha$  and the payoff parameters contradicts inequality (3). Thus the quit distribution of player  $i$ 's opponent has no mass point at  $t$ . Since player  $iL$  plans to betray at a positive rate in a neighborhood of  $t$ , this player must be indifferent between quitting at any time in the neighborhood. Using the low type's payoff parameters in expression (1), and recalling that  $\underline{z} = 0$ , his expected payoff from planning to betray at time  $w$  is

$$\int_0^w \underline{y} \alpha(s) e^{-rs} [-d\theta_j(s)] + \underline{x} \alpha(w) e^{-rw} \theta_j(w),$$

which must be constant in the neighborhood of  $t$ . Taking the derivative with respect to  $w$ , evaluating it at  $w = t$ , and rearranging terms yields:

$$\underline{x} \alpha'(t) - r \underline{x} \alpha(t) - \frac{\theta_j'(t)}{\theta_j(t)} [\underline{y} - \underline{x}] \alpha(t) = 0. \quad (4)$$

<sup>8</sup>For  $t > 0$ ,  $\alpha^-(t) \equiv \lim_{s \rightarrow t^-} \alpha(s)$ . For  $t \geq 0$ ,  $\alpha^+(t) \equiv \lim_{s \rightarrow t^+} \alpha(s)$ .

<sup>9</sup>Recall that players earn a terminal payoff of zero if they quit at the same time.

The left side of this equation is the value at  $t$  of player  $iL$  waiting before quitting, relative to quitting at  $t$ . This value must be zero if player  $iL$  is indifferent. The first two terms are the increase in the betrayal payoff from changes in the level and the decrease in the betrayal payoff due to discounting. Remember that the cooperative flow payoff for the low type is zero. The last term is the potential loss if the opponent quits before player  $i$  quits, times the rate that the opponent quits conditional on reaching  $t$  (the hazard rate of the opponent's quit distribution).

In equilibrium,  $\alpha$  must gradually rise in any interval in which a player quits at a positive rate. To see this, note that the last term on the left side of (4) is nonpositive. Observe that  $\alpha$  must rise enough to offset discounting in order for the low type to be indifferent. Furthermore, if the opponent betrays at a positive rate then  $\alpha$  must rise even faster in order for the low type to be indifferent between quitting and waiting. That is, the “prize” of betrayal in the future must be large enough to counter the chance that the opponent will betray first.

## Phases of Equilibrium Regimes

Other general properties of equilibrium regimes can be derived. Given  $\theta_i$ , define  $T_i \equiv \min\{t \geq 0 \mid \theta_i(t) = p_i\}$  as the time by which the low type of player  $i$  betrays with probability one.<sup>10</sup> I shall pay special attention to regimes with continuous level functions. As the following lemma from Watson [11] establishes, these regimes have a three-phase structure in equilibrium.

**Lemma 2** *Consider any level function  $\alpha$  that is continuous at every  $t > 0$ . Suppose  $\theta_1$  and  $\theta_2$  define a cooperative equilibrium in the partnership game. Then  $T_1 \leq T_2$  and  $\theta_2(T_1) \leq \theta_1(T_1)$ . Furthermore,  $\theta_1(t) = \theta_2(t)$  for all  $t < T_1$  and these functions are continuous on  $(0, T_1)$ .*

To interpret Lemma 2, remember that  $p_1 \geq p_2$ . The three phases of a cooperative equilibrium regime are  $[0, T_1]$ ,  $[T_1, T_2]$ , and  $[T_2, \infty)$ . In the first phase the low types behave symmetrically, betraying with the same probability over time. This phase ends when the probability mass on the low type of player 1 has “run out” (when  $\theta_1(t) = p_1$ ) and, conditional on cooperation, the updated probability that player 1 is the high type equals one. In the second phase, only the low type of player 2 quits with positive probability. Just after  $T_2$ , the updated probability that player 2 is the high type has reached one as well. The high types cooperate through the third phase, where only they remain.

The proof of Lemma 2 builds on the following intuition about the incentives of the low types. Suppose there is an interval of time (perhaps a point) in which player 1 betrays with higher probability than does player 2, and also suppose that player 2 betrays later in the game with positive probability. Note that player 2 is willing to quit later even though there is a chance that player 1 will quit in the interval. Since player 1 is at less risk in the interval, he should strictly prefer to wait, which contradicts that he quits with positive probability in the interval.

---

<sup>10</sup>This is well-defined since  $\theta_i$  is right-continuous.

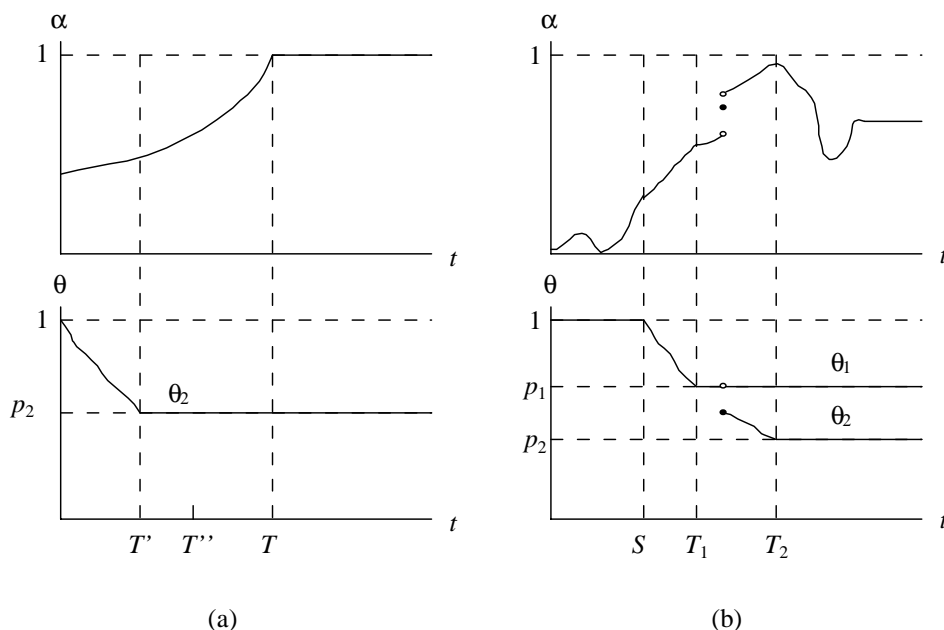


Figure 2: Equilibrium regimes.

## Examples and Existence

It is not difficult to construct a variety of different equilibrium regimes. For example, consider a setting with one-sided incomplete information, where  $p_1 = 1$ . For every  $T > 0$ , define  $\alpha_T$  to be the continuous function satisfying  $\alpha_T(t) = 1$  for all  $t \geq T$  and  $\alpha'_T(t)/\alpha_T(t) = r$  for all  $t \in (0, T)$ . This differential condition is equation (4) for the case in which  $\theta'_j = 0$ . If the high types cooperate perpetually, such a level function makes player 2L indifferent between betraying at all times from 0 to  $T$ . Any quit distribution satisfying  $\theta_2(T) = p_2$  characterizes an equilibrium, as long as player 1H weakly prefers to cooperate perpetually. (Obviously, player 2H prefers to cooperate.) In fact, take any  $\theta_2$  such that  $\theta_2(T') = p_2$  for some  $T'$ . One can show that there is a number  $T'' \geq T'$  such that, for every  $T \geq T''$ , the regime defined by  $\theta_2$  and  $\alpha_T$  is an equilibrium. Figure 2(a) illustrates this class of regimes.

Equilibria of the same flavor can be constructed for games with two-sided incomplete information. For example, consider a symmetric game, where  $p_1 = p_2$ . Look for a symmetric equilibrium by taking any differentiable  $\theta_1 = \theta_2$  and, for  $T$  such that  $\theta_1(T) = p_1$ , selecting  $\alpha$  to satisfy (4) for all  $t \in (0, T)$ . As in the one-sided case, one can prove that this is an equilibrium if the level starts small enough (so that  $T$  is large enough). There are also symmetric equilibria with atoms in the low types' quit distribution. Such equilibria generally involve discontinuities in  $\alpha$ , because inequalities (2) and (3) must hold where  $\theta_1 = \theta_2$  is discontinuous. In addition,  $\alpha$  may be non-monotone before or after the low types betray. Figure 2(b) shows an example.

As the examples confirm, a cooperative equilibrium regime always exists.

**Theorem 1** *Regardless of  $p_1 > 0$  and  $p_2 > 0$ , there is a cooperative equilibrium regime.*

Given the example with constant  $\alpha$  discussed at the beginning of this section, Theorem 1 reveals the value of starting small. In the fixed  $\alpha$  case, the partnership is not viable when the players are initially very pessimistic about one another. However, *cooperation between two high type players can always be sustained if the players start small enough in their relationship.*

### 3 Incentive Efficient Regimes

The examples indicate there is a plethora of equilibrium regimes. However, to the extent that the players jointly determine the regime, it seems reasonable to expect them to settle on one that is *interim incentive efficient* (abbreviated IIE). An equilibrium regime is interim incentive efficient if, within the class of equilibrium regimes, there is no way of making one type of one player better off without making another player-type worse off. In this section I characterize the IIE regimes. I constrain attention to cooperative regimes whose level functions are continuous at positive times.

I shall prove that IIE regimes have two noteworthy properties. First, they entail starting small, which should come as no surprise given the discussion following Theorem 1 in the previous section. Second, and more significant, IIE regimes involve betrayal by low types that is concentrated on two times: at the beginning of the relationship and just as the level reaches its maximum. In other words, betrayal occurs quickly and is lumpy over time.

To state the result, a bit more notation and terminology is useful. Given an equilibrium regime defined by  $\alpha$ ,  $\theta_1$ , and  $\theta_2$ , let the function  $v_{iH} : \mathbf{R}_+ \rightarrow \mathbf{R}$  associate with every time in the game the equilibrium continuation payoff for player  $iH$ , for  $i = 1, 2$ . That is,  $v_{iH}(t)$  is the continuation payoff for player  $iH$  from time  $t$ , conditional on reaching this time in the game (neither player betraying prior to  $t$ ). I use the convention that the continuation value at  $t$  is evaluated before the parties interact at this time; thus,  $v_{iH}$  reflects any mass point in the betrayal distribution of player  $i$ 's opponent at  $t$ . Let  $v_{iL}$  be defined analogously.<sup>11</sup> Further, define by  $\hat{v}_{iH}(t)$  the payoff player  $iH$  would obtain at time  $t$  if he were to betray at this time; we have  $\hat{v}_{iH}(t) = \bar{x}\alpha(t)\theta_j(t)/\theta_j^-(t)$ .

For any interval of time  $M \subset \mathbf{R}_+$ , I say a regime has *maximal delay on  $M$*  if for  $t, t' \in M$ ,  $\theta_2(t) < \theta_2^-(t')$  implies  $v_{1H}(s) = \hat{v}_{1H}(s)$  for every  $s \in M$  with  $s > t$ . In words, if betrayal occurs in  $M$  by time  $t$  then the high type of player 1 is indifferent between quitting and cooperating after  $t$  until the right endpoint of  $M$ . In this case, the mass of player  $2L$ 's quit distribution on  $M$  is pushed as close to the right endpoint of  $M$  as is possible while still maintaining player  $1H$ 's incentive to cooperate. I feature the following type of equilibrium.

---

<sup>11</sup>Note that, since the high types cooperate perpetually,  $v_{iH}$  is continuous wherever  $\theta_j$  is continuous; the function is everywhere left continuous and has well-defined right-hand limits. Where  $\alpha$  is continuous,  $v_{iL}$  has well-defined right- and left-hand limits.

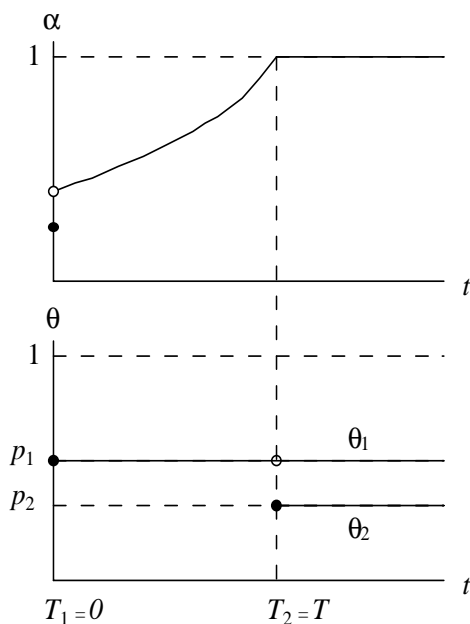


Figure 3: Q-regime.

**Definition 1** A **Q-regime** is a cooperative equilibrium regime in which (a)  $\theta_1(0) = p_1$  (so  $T_1 = 0$ ); (b)  $\alpha(t) = \min\{\alpha(0)e^{rt}/D(p_1), 1\}$  for all  $t > 0$ ; and (c) with  $T \equiv \inf\{t \mid \alpha(t) = 1\}$ , the regime has maximal delay on  $(0, T]$ .

Here, Q stands for “quick” (referring to the nature of betrayal by the low types). A Q-regime is pictured in Figure 3. Note that the entire mass of quitting by player 1 is concentrated at time zero, along with at least the same mass of quitting by player 2 (who has a weakly greater chance of being a low type). The remainder of player 2L’s betrayal mass occurs at  $T$ . Thus, if  $\theta_2(0) > p_2$  then  $T_2 = T$ ; otherwise,  $T_2 = T_1 = 0$ . The level of the relationship jumps up just after  $t = 0$  and then gradually rises until  $T$ , after which it stays at its maximum. The discontinuity at  $t = 0$  makes the low types weakly prefer to quit at this time. The gradual rise after time zero obeys  $\alpha'/\alpha = r$ , which coincides with equation (4) for  $\theta'_j = 0$ . Observe that the class of Q-regimes is parameterized by the numbers  $\alpha(0) \in [0, 1]$  and  $\theta_2(0) \in [p_2, p_1]$ .

Here is the main result of this paper.

**Theorem 2** *Among the class of cooperative regimes whose level functions are continuous on  $(0, \infty)$ , every interim incentive efficient regime is a Q-regime. Furthermore, there exists an interim incentive efficient regime.*

The proof of this theorem has several complicated components, which I attempt to summarize in more intuitive form here. The general idea is to start with an arbitrary regime and show how it can be transformed into a Q-regime, improving the payoffs of

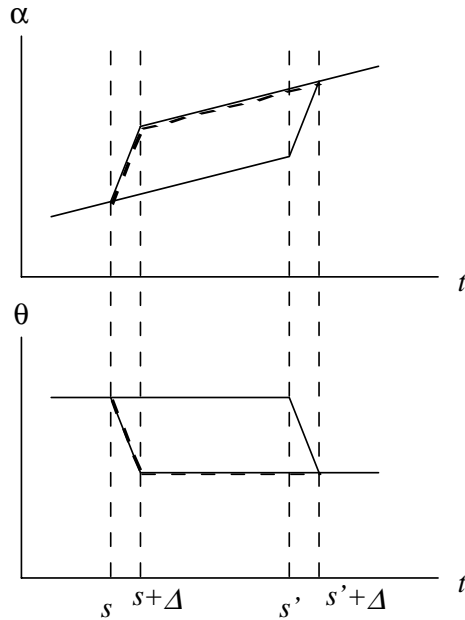


Figure 4: Shifting probability mass in the first phase.

all player-types in the process. The proof first establishes that efficiency requires  $\alpha$  to eventually reach the maximum level (one) and remain there forever. Further, defining  $S \equiv \min\{s \mid \alpha(s) = 1\}$ , it must be that player  $2L$  is indifferent between quitting at every time between zero and  $S$ . Otherwise, one can raise  $\alpha$  at times where player  $2L$  strictly prefers not to quit, forming a new equilibrium that all player-types prefer (player  $1H$  strictly so).<sup>12</sup> I show that the indifference condition implies that equation (4) holds on the interval  $(0, T_1)$ . The proof then addresses separately the first two phases of an equilibrium regime (that is,  $[0, T_1)$  and  $[T_1, T_2]$ ).

On the first phase, I show that it is optimal to shift toward time zero the probability mass of quitting by the low types. The following heuristic argument provides the intuition behind this claim. Suppose one starts with an equilibrium regime in which a section of the first phase of interaction is described by the solid lines in Figure 4. In this region, the low types quit with positive probability between  $s'$  and  $s' + \Delta$ . Note that  $\theta_1 = \theta_2 \equiv \theta$  in the first phase of the equilibrium. Dividing equation (4) by  $\alpha$  and integrating reveals that  $\alpha(t) = e^{rt}\theta(t)^{-b}c$  for an open set containing  $[s, s' + \Delta]$ , where  $b \equiv (\underline{x} - \underline{y})/\underline{x}$  and  $c$  is a constant.

Consider another equilibrium regime constructed by shifting the betrayal mass on  $(s', s' + \Delta)$  to instead occur on  $(s, s + \Delta)$ . The rest of the regime remains the same. The new regime is represented by the lines with dashed highlights. It is not difficult to

<sup>12</sup>This step is simplified by the assumption  $\underline{z} = 0$ . Under this assumption, the incentives of the low types are preserved when the level function is modified at times where the low types strictly prefer not to quit.

verify that the low types are indifferent between the old and new regimes. However, the high types strictly prefer the new regime. To see this, first observe that the component of the high types' payoff due to betrayal by the low types is the same under the two regimes. This is because, although betrayal occurs earlier in the new regime, it occurs at a lower level which exactly offsets the change in discounting between  $s$  and  $s'$ .<sup>13</sup> The difference between the two regimes, in terms of the high types' expected payoff, thus amounts to the difference on the interval  $(s + \Delta, s')$ .

Cooperation occurs on the interval  $(s + \Delta, s')$  in both the new and old regimes. The old regime has a higher probability of cooperation here (since the betrayal mass under consideration occurs after this interval), but cooperation occurs at a lower level. Relative to the old regime, the level under the new regime is higher by the multiplicative factor  $\theta(s' + \Delta)^{-b}/\theta(s')^{-b}$ , where  $\theta$  is given by the old regime. The probability of cooperation over the interval in the new regime is lower than in the old regime by the multiplicative factor  $\theta(s' + \Delta)/\theta(s')$ . Putting these factors together, the ratio of high types' payoff from the interval  $(s + \Delta, s')$  in the two regimes is  $\theta(s' + \Delta)^{1-b}/\theta(s')^{1-b}$ . Since  $1 - b = \underline{y}/\underline{x} < 0$  and  $\theta(s' + \Delta) < \theta(s')$ , the high types like the new regime better.

In short, shifting probability mass forward in the first phase of interaction is preferred because the reduced probability of cooperation in the interim is more than offset by the gain of cooperation at a higher level. Thus, IIE regimes must have the first phase of interaction occur arbitrarily quickly. Of course, there is a difference between "arbitrarily quickly" associated with (i) the limiting payoff of a sequence of continuous  $\theta^k$  and (ii) the payoff of the limiting  $\theta$ , which has a mass point at time zero. I show that, given  $\underline{x} + \underline{y} \leq 0$ , the latter is preferred by the players.<sup>14</sup> The intuition behind this last claim has to do with whether it is better to have signaling occur in several small stages or one big stage. Times of quitting by low types can be interpreted as signaling stages, where high types send a signal of their type by not quitting. Multiple stages of signaling may be valued if the signaling technology creates value. This would be the case if  $\underline{x} + \underline{y} > 0$ , meaning betrayal between low types yields a surplus.

The final component of the proof involves showing that the betrayal mass of player  $2L$  in the second phase of interaction is optimally shifted later in time. Note that  $\alpha'/\alpha = r$  between  $T_1$  and the time  $S$  at which  $\alpha$  reaches one. This follows from the facts that player  $1L$  does not quit in this interval of time and player  $2L$  is indifferent here. Delaying the betrayal of player  $2L$  in this interval does not alter the payoff of this player-type. However, it does *increase* the payoff of player  $1H$ . Player  $1H$  obtains the same negative payoff from the cheating event, since changes in the timing are exactly offset by changes in the level. But this player enjoys a higher probability of cooperation in the interim.

---

<sup>13</sup>This follows from the  $\alpha(t) = e^{rt}\theta(t)^{-b}c$  identity.

<sup>14</sup>This is why I allow the level function to be discontinuous at  $t = 0$ ; efficiency demands it.

## 4 Two Special Cases

Recall that Q-regimes are parameterized by the numbers  $\alpha^0 = \alpha(0) \in [0, 1]$  and  $\theta_2^0 = \theta_2(0) \in [p_2, p_1]$ . Theorem 2 establishes that IIE regimes are Q-regimes, but it does not indicate the values  $\alpha^0$  and  $\theta_2^0$  precisely or how they relate to the payoffs of the player-types. To further characterize the set of IIE regimes and generate simple comparative statics conclusions, it is helpful to examine two important special cases of the model: the symmetric case and the one-sided case.

Consider first the symmetric case, distinguished by  $p_1 = p_2 = p$ . Under symmetry, we obviously have  $\theta_1(0) = \theta_2(0)$  in any Q-regime. Therefore  $\theta_2^0 = p_2$  and so  $T_2 = T_1 = 0$ . IIE regimes are thus symmetric and are characterized by the number  $\alpha^0$ . To learn how  $\alpha^0$  relates to the payoffs of the players, consider maximizing the welfare function  $w^{\text{sym}}(m) \equiv mu_H + (1 - m)u_L$ , where  $u_H$  is the equilibrium expected payoff of the high types and  $u_L$  is the same for the low types. Let  $\alpha_{\text{sym}}^0(m, p)$  denote the initial level characterizing the regime that maximizes  $w^{\text{sym}}$  over all equilibrium regimes.

**Lemma 3**  $\alpha_{\text{sym}}^0(m, p)$  is weakly decreasing in  $m$  and weakly increasing in  $p$ . Also,  $m = 1$  implies  $T > 0$ .

Equilibria that favor the high types have lower initial levels; the initial level is also lower when players are more pessimistic about each other. Remember that  $T$  defines the time in the regime at which  $\alpha$  first reaches one. Note that  $T > 0$  means the level of the relationship rises gradually even *after* time zero, where both low type players quit with probability one.

Next consider the one-sided case, distinguished by  $p_1 = 1$ . In this setting, there is maximum delay from the beginning of the game, so player 2L only quits at  $t = 0$  if  $\alpha^0$  is large.

**Lemma 4** In the one-sided case, every incentive efficient regime has maximum delay on  $[0, T]$ , where  $T \equiv \inf\{t \mid \alpha(t) = 1\}$ .

Regarding the players' preferences over IIE regimes, note that players 1L, 2H and 2L fancy  $\alpha^0$  as large as possible; player 1H is more cautious and generally prefers starting small. Consider a welfare function  $w^{\text{one}}(m) \equiv mu_{1H} + (1 - m)u_{2H}$ , where  $u_{iH}$  is the equilibrium expected payoff of player  $iH$ . Let  $\alpha_{\text{one}}^0(m, p_2)$  denote the initial level characterizing the regime that maximizes  $w^{\text{one}}$  over all equilibrium regimes.

**Lemma 5**  $\alpha_{\text{one}}^0(m, p)$  is weakly decreasing in  $m$  and weakly increasing in  $p$ . Also,  $m = 1$  implies  $T > 0$ .

## 5 Negotiation and Commitment

The level function  $\alpha$  is interpreted as jointly selected by the players. However, in the analysis thus far I have treated  $\alpha$  as exogenous. In this section, I address whether IIE

regimes would result from negotiation between the players before the partnership game begins. I also comment on the issue of renegotiation by the players in the course of play.

## Initial Selection of the Level Function

Consider the joint selection of  $\alpha$  prior to play of the partnership game. There are three natural settings in which negotiation may take place: (a) the players determine the level function before obtaining their private information (*ex ante*), (b) the players negotiate the level function with the knowledge of their own types (*interim*), and (c) the players select  $\alpha$  *ex ante* but then can alter  $\alpha$  after learning their own types. Some of the relevant issues in contracting with asymmetric information have been studied for general mechanism design problems.<sup>15</sup> The notion of “durability” — advanced by Holmström and Myerson (1983) and Crawford (1985)) — provides a way of assessing whether a mechanism survives renegotiation once players obtain private information (case (c)) and it provides a consistency criterion for a mechanism to arise from initial interim negotiation (case (b)). Crawford (1985) proves that interim incentive efficient mechanisms are durable in that there is a reasonable model of negotiation that supports these regimes as contracted in equilibrium. The regimes identified in the previous section here are IIE. Thus, they are supported by a reasonable model of negotiation between the players.

## Renegotiation in the Course of Play

I have focused in this paper on commitment to the level function initially selected by the players. Watson [11] concentrates on renegotiation and studies a particular strong renegotiation criterion. I wish to point out, though, that there is a sense in which the regimes identified in the present paper are also renegotiation-proof. Consider the following notion of renegotiation-proofness, which is a dynamic version of Maskin and Tirole’s (1992) “weak renegotiation-proofness.” Suppose some  $\alpha$  has been set by contract and the partnership game begins. At some time  $s$  ( $s$  may equal zero) play is temporarily suspended and player  $i$  has the option of suggesting an alternative level function  $\hat{\alpha}$  that will be in effect for the remainder of the game. If  $\hat{\alpha} \neq \alpha$  then player  $j$  decides whether to accept or reject this new level function. If player  $j$  accepts  $\hat{\alpha}$  then play continues with this level function; otherwise, play resumes with  $\alpha$ . Call this renegotiation phase and the ensuing partnership game from  $s$  the “renegotiation game.” The original level function  $\alpha$  is *weakly renegotiation-proof* if there exists an equilibrium in the renegotiation game in which both types of player  $i$  suggest level function  $\alpha$ .<sup>16</sup>

It is not difficult to see that optimal Q-regimes are weakly renegotiation-proof. This is because player  $j$  may learn about player  $i$  on the basis of player  $i$ ’s suggested level function at time  $s$ . Take a Q-regime with level function  $\alpha$  and I will construct an equilibrium in

---

<sup>15</sup>Standard references include Holmström and Myerson (1983), Myerson (1983), and Crawford (1985), and more recent work includes Maskin and Tirole (1990,1992) and Cramton and Palfrey (1995).

<sup>16</sup>Maskin and Tirole (1992) also define “strongly renegotiation-proof” contracts as those that are renegotiated in no equilibrium of the renegotiation game. In the partnership game, no regimes are strongly renegotiation-proof.

which the level function is not renegotiated at  $s$ . Prescribe that at time  $s$  both types of player  $i$  suggest continuing with  $\alpha$ . When  $\alpha$  is suggested, then, player  $j$ 's belief about  $i$ 's type does not change and we assume that play resumes with the high types cooperating perpetually. Suppose further that if player  $i$  suggests any other level function then  $j$ 's conditional belief about player  $i$ 's type puts probability one on the low type. (Player  $j$  believes that player  $i$  is the low type trying to fool him.) In this case, player  $j$  either accepts or rejects  $\hat{\alpha}$  and both players quit immediately as the partnership game resumes. (When one player is known to be the low type, the best equilibrium involves both players quitting immediately.) Given these beliefs, player  $i$  has no incentive to suggest a level function other than  $\alpha$  and so the original contract on  $\alpha$  is not renegotiated. Furthermore, one obtains the same conclusion with other renegotiation procedures, such as ones in which the players simultaneously offer new contracts.

The difference between the renegotiation criterion in Watson [11] and the one discussed above hinges on the whether beliefs about types can be altered contingent on messages sent in the renegotiation game. In Watson [11], beliefs are not allowed to change directly due to play in the message game, while here the beliefs may be influenced so. The freedom to design beliefs in an equilibrium implies that, in the present context, the players have better ability to commit to their initial selection of  $\alpha$ . This commitment has two manifestations. First, it allows the players to engineer their relationship so that simultaneous betrayal by the low types occurs rapidly in the first phase of equilibrium, with the level function gradually rising afterward. Without commitment, the players would hasten the rise of the level once they became confident of each others' types. Second, commitment allows the players to push "residual" betrayal by player  $2L$  as far into the future as is possible, even though player  $1H$  may wish to change this plan once the time comes. Watson [11] shows that, without this commitment, the *G-regime* is selected; this regime features gradual rise of the level and gradual separation of types throughout the first two phases of equilibrium.<sup>17</sup>

## 6 Conclusion

The class of games with variable stakes permits a tractable treatment of the partnership problem, leading to intuitive results on the nature of starting small and the manner in which the players learn about one another. Possible extensions of the model include (a) incorporating a multi-dimensional "level" parameter and (b) enlarging the scope for signaling between the players by including other signaling technologies. Both extensions are worthwhile pursuits; both present some challenging technical problems. Extension (b) is attempted in a preliminary and limited way in Watson [10]. Regarding (a), I simply note that the present analysis relies heavily on symmetries between the types of different players

---

<sup>17</sup>The reader is referred to Watson [11] for the details. On a technical note, in Watson [11], my stated assumptions include  $\underline{z} > 0$ . However, one can verify that all of the analysis goes through with  $\underline{z} = 0$ , as I assume here. Also, the other paper makes the additional regularity assumptions  $\bar{y}/\underline{y} > \bar{z}/r\underline{x}$  and  $(\underline{x} + \underline{y})\bar{z}/r\underline{x} > -\underline{y}(\bar{z}/r\underline{x} - \bar{y}/\underline{y})$ , which are satisfied for  $\bar{y}$  great enough in magnitude.

and with the way the level affects the payoffs of the players. In the least, the present paper and Watson [11] together provide a reasonably complete analysis of a class of partnership problems, which highlights the virtue of starting small.<sup>18</sup>

---

<sup>18</sup>Rauch and Watson [8] offer a complementary analysis of starting small in an environment with symmetric information and joint uncertainty.

# A Proofs

## Proof of Lemma 1

This is proved in Watson [11], where the same Lemma appears as Lemma 1. Although  $\underline{z} > 0$  is assumed in Watson [11], none of the analysis is altered when  $\underline{z} = 0$ . *Q.E.D*

## Proof of Lemma 2

This is proved in Watson [11], where the same Lemma also appears as Lemma 2. *Q.E.D*

## Proof of Theorem 1

This is an immediate consequence of Lemma 3 in Watson [11]. *Q.E.D.*

## Proof of Theorem 2

Note that, by Lemma 2, any equilibrium regime satisfying (b) of the definition of a  $Q$ -regime is fully characterized by  $(\alpha, \theta_2)$ , with the understanding that  $\theta_1 \equiv \max\{\theta_2, p_1\}$ . I begin by proving the following result.

**Lemma 6** *Take as given a cooperative equilibrium regime  $(\alpha, \theta_2)$  for which  $\alpha$  is continuous on  $(0, \infty)$ . Then there exists a  $Q$ -regime  $(\hat{\alpha}, \hat{\theta}_2)$  in which the expected payoff of every player-type weakly exceeds that of  $(\alpha, \theta_2)$ .*

I prove the lemma in four steps. I shall focus on the case in which  $T_2 > 0$ . The case of  $T_2 = 0$  is simpler, requiring just part of steps 1 and 4.

*Step 1. Claim: For the regime  $(\alpha, \theta_2)$ ,  $\theta_1$  is continuous on  $(0, T_1]$ .* To prove this claim, suppose it does not hold and let  $t \in (0, T_1]$  be such that  $\theta_1^-(t) > \theta_1(t)$ . (Remember that  $\theta$  is right-continuous.) From Lemma 2, we know that  $\theta_2^-(t) > \theta_2(t)$  as well, so player 2L quits with positive probability at  $t$ . But then inequality (3) must hold, implying that  $\alpha$  is discontinuous at  $t$ , which is a contradiction.

*Step 2.* I next define a new equilibrium regime, based on  $(\alpha, \theta_2)$ . Let  $\tilde{\alpha}$  be defined by: (i)  $\tilde{\alpha}(0) = \alpha(0)$ , (ii)  $\tilde{\alpha}(t)\underline{x} \equiv v_{2L}(t)$  for all  $t \in (0, T_2]$  (where  $v_{2L}$  is the continuation value function associated with  $(\alpha, \theta_2)$ ), (iii)  $\tilde{\alpha}(t) \equiv \min\{1, \alpha(T_2)e^{r(t-T_2)}\}$  for all  $t > T_2$ . We have  $\tilde{\alpha}(t) \in [0, 1]$  for part (ii) since  $v_{2L}(t) \in [0, \underline{x}]$ . Furthermore, from step 1 we know  $v_{2L}$  is continuous on  $(0, T_1)$ , which implies  $\tilde{\alpha}$  is continuous on this interval. In fact, one can easily verify that  $\tilde{\alpha}$  is continuous on  $(0, \infty)$ .

Note that  $\tilde{\alpha}$  is defined so that, first,  $\tilde{\alpha}(t) = \alpha(t)$  where the low type quit with positive probability in the original regime and, second,  $\tilde{\alpha}$  is set to make player 2L indifferent everywhere else (until  $\tilde{\alpha} = 1$ ). It should be obvious that, because  $\underline{z} = 0$ , the incentives of players 1L and 2L are the same under  $(\tilde{\alpha}, \theta_2)$  as they are under  $(\alpha, \theta_2)$ . That is, for the low types,  $(\tilde{\alpha}, \theta_2)$  is incentive compatible. It is also not difficult to show that players 1H and 2H have no incentive to quit under  $(\tilde{\alpha}, \theta_2)$ ; at times where low types quit, the high types' value of cooperating is weakly greater relative to the value of quitting; at other

times, the high types strictly prefer to wait. Therefore,  $(\tilde{\alpha}, \theta_2)$  is a cooperative equilibrium regime. By construction, every player-type fares weakly better under  $(\tilde{\alpha}, \theta_2)$  than under  $(\alpha, \theta_2)$ .

*Step 3.* I define yet another regime by modifying  $(\tilde{\alpha}, \theta_2)$ . This modification involves shifting the probability mass for which the low types quit in  $(0, T_1)$  to a mass point at time zero. The new regime shall be denoted  $(\bar{\alpha}, \bar{\theta}_2)$ .

I begin the construction with some general analysis. Take any cooperative equilibrium regime  $(\beta, \mu)$  and suppose these functions are continuous on  $[0, s)$ , with  $\mu(s) \geq p_1$ . Here  $\mu$  directly defines the cooperation distribution for player  $2L$  and  $\max\{\mu, p_1\}$  defines the same for player  $1L$ . Further suppose the low types are indifferent between quitting at every time in  $[0, s)$ . Then, using equation (4), we have

$$\underline{x}\beta'(t) - r\underline{x}\beta(t) - \frac{\mu_j'(t)}{\mu_j(t)}[\underline{y} - \underline{x}]\beta(t) = 0.$$

Integrating, we obtain

$$\beta(t) = e^{rt}\mu(t)^{-b}c \quad (5)$$

for all  $t \in (0, s)$ , where  $b \equiv (\underline{x} - \underline{y})/\underline{x}$  and  $c = \beta(s)e^{-rs}\mu^-(s)^{-1}$ . Further, using inequality (2) and low type indifference, we have

$$\beta(0) = q^{-b}D(q)c, \quad (6)$$

where  $q \equiv \mu(0)$ . Equations (5) and (6) define  $\beta$  as a function of  $\mu$ ; they represent conditions from the incentives of the low types.

I next use the relationship of equations (5) and (6) to write  $\tilde{\alpha}$  as a function of  $\theta_2$  for the regime  $(\tilde{\alpha}, \theta_2)$ . The expected payoff of both players  $1H$  and  $2H$  on the interval  $[0, T_1)$  (that is, not including the payoff from time  $T_1$ ) is

$$\bar{y}(1 - q)\tilde{\alpha}(0) + \int_0^{T_1} \bar{z}\tilde{\alpha}(t)e^{-rt}\theta(t)dt + \int_0^{T_1} \bar{y}\tilde{\alpha}(t)e^{-rt}(-\theta'(t))dt,$$

where  $\theta \equiv \theta_1 = \theta_2$  on  $[0, T_1)$  and  $q \equiv \theta(0)$ . Substituting for  $\tilde{\alpha}$  using (5) and (6), and using  $c = \tilde{\alpha}(T_1)e^{-rT_1}\mu^-(T_1)^{-1}$  we obtain

$$c \left\{ \bar{y}(1 - q)D(q)q^{-b} + \int_0^{T_1} \bar{z}\theta(t)^{1-b}dt - \int_0^{T_1} \bar{y}\theta(t)^{-b}\theta'(t)dt \right\}.$$

Evaluating the second integral and combining the terms containing  $q$ , this becomes

$$c \left\{ \bar{y}[(1 - q)D(q)q^{-b} + q^{\underline{y}/\underline{x}}\underline{x}/\underline{y}] + \int_0^{T_1} \bar{z}\theta(t)^{\underline{y}/\underline{x}}dt - \bar{y}(\underline{x}/\underline{y})\theta^-(T_1)^{\underline{y}/\underline{x}} \right\}. \quad (7)$$

Here I write  $b$  where it simplifies the expression; otherwise I have substituted its value  $(\underline{x} - \underline{y})/\underline{x}$ . Note that  $1 - b = \underline{y}/\underline{x}$ .

Next I examine H's payoff on  $[0, T_1]$  for modified versions of regime  $(\tilde{\alpha}, \theta_2)$ . Specifically, I look at a regime  $(\beta, \mu)$  constructed to satisfy equations (5) and (6) on  $[0, T_1]$  and also to be equivalent to  $(\tilde{\alpha}, \theta_2)$  on  $[T_1, \infty)$ . Such a regime is consistent with rationality of the low types, as the analysis above establishes. Let  $(\beta, \mu)$  be defined by a parameter  $m \in [0, 1]$ , such that  $\mu(t) \equiv m\theta_2(t) + (1 - m)\theta_2^-(T_1)$  for every  $t \in [0, T_1]$ . The level function  $\beta$  is then defined by (5) and (6). Observe that the regime is exactly  $(\tilde{\alpha}, \theta_2)$  when  $m = 1$ ; as  $m$  decreases, "quitting" probability mass is shifted from the interval  $(0, T_1)$  to a mass point at time zero.

Rewriting expression (7) for regime  $(\beta, \mu)$ , we see that the expected payoff of the high types in the interval of time  $[0, T_1]$  is  $c(A + B + C)$ , where

$$A = \bar{y}(1 - q)D(q)q^{-b} + \bar{y}q^{\underline{x}/\underline{y}}\underline{x}/\underline{y},$$

$$B = \int_0^{T_1} \bar{z}[m\theta_2(t) + (1 - m)\theta_2^-(T_1)]^{\underline{y}/\underline{x}} dt,$$

$$C = -\bar{y}(\underline{x}/\underline{y})\theta^-(T_1)^{\underline{y}/\underline{x}},$$

$$c = \beta(T_1)e^{-rT_1}\mu^-(T_1)^{-1} = \tilde{\alpha}(T_1)e^{-rT_1}\theta_2^-(T_1)^{-1},$$

and

$$q = \mu(0) = m\theta_2(0) + (1 - m)\theta_2^-(T_1).$$

I shall demonstrate that the regime specified by  $m = 0$  is an equilibrium and that each player-type prefers it to the one specified by  $m = 1$ , in terms of expected payoffs.

To prove the regime with  $m = 0$  is an equilibrium, I only need to demonstrate that the high types do not wish to quit at  $t = 0$ . (The preferences of the low types were addressed above; the high types' preferences from  $T_1$  are captured in the original regime  $(\tilde{\alpha}, \theta_2)$ ; and by construction, the high types do not wish to quit on  $(0, T_1)$ .) We know  $(\tilde{\alpha}, \theta_2)$  is an equilibrium and the component of the high types' payoffs from  $T_1$  are independent of  $m$ . Also note that  $\bar{x}q^{1-b}D(q)c$  is the high types' payoff from quitting at the beginning of the game. It is sufficient to show that  $E \equiv c(A + B + C) - \bar{x}q^{1-b}D(q)c$  is decreasing as a function of  $m$ . Let  $F \equiv A - \bar{x}q^{1-b}D(q)$ . We have  $dE/dm = c(dF/dq)dq/dm + cdB/dm + cdC/dm$ . Note that  $c$  is not a function of  $m$ ,  $dq/dm > 0$ ,  $dC/dm = 0$ , and  $dB/dm \leq 0$  since  $\underline{y} \leq 0$ . Thus, I must establish that  $dF/dq \leq 0$ . Taking the derivative and performing some algebraic manipulation, we find that  $dF/dq$  has the same sign as  $(\underline{x} + \underline{y})[D(q) - \bar{D}(q)]$ , where

$$\bar{D}(q) \equiv \frac{q\bar{x}}{q\bar{x} - (1 - q)\bar{y}}.$$

Recall that  $\underline{x} + \underline{y} \leq 0$  is assumed. Thus, we obtain the desired conclusion under the condition  $D(q) \geq \bar{D}(q)$ , which simplifies to  $-\bar{y}/\bar{x} \geq -\underline{y}/\underline{x}$ .

In fact, the regime with  $m = 0$  is also an equilibrium in the case in which  $D(q) < \bar{D}(q)$ . To see this, recall that  $D(q)$  is defined so that the low type is indifferent between quitting at  $t = 0$  and just after this time;  $D(q)$  is a scaling factor on the relation between the level

at zero and just after zero. We can compute the corresponding scaling factor for the high types; it is  $\bar{D}(q)$ . Thus, if the level is scaled according to  $\bar{D}(q)$ , then the high type player is indifferent between quitting at zero and just after this time. When  $D(q) < \bar{D}(q)$  and  $D(q)$  is used to scale the level, the high type strictly prefers waiting to quitting at  $t = 0$ .

Next I prove that all player-types fair weakly better under the regime with  $m = 0$  than they do if  $m = 1$ . To demonstrate that the high types prefer the  $m = 0$  regime, I need to show that  $dA/dq \leq 0$ . Taking the derivative and performing some algebraic manipulation, we find that  $dA/dq$  has the same sign as  $\underline{x} + \underline{y}$ , which is nonpositive given Assumption 2. To see that the low types prefer the  $m = 0$  regime, note that the payoff of the low types as a function of  $q$  is

$$q^{\underline{y}/\underline{x}} D(q) c \underline{x}.$$

We require the derivative of this expression with respect to  $q$  to be nonpositive. Evaluating the derivative, we find that it has the same sign as  $\underline{x} + \underline{y}$ , which is indeed nonpositive.

Let  $(\bar{\alpha}, \bar{\theta}_2)$  denote the regime specified by  $m = 0$ . To be precise,  $\bar{\theta}_2(t) = \theta_2^-(T_1)$  for every  $t \in [0, T_1)$ ;  $\bar{\theta}_2(t) = \theta_2(t)$  for every  $t \in [T_1, \infty)$ ;  $\bar{\alpha}(t) = e^{rt} \theta_2^-(T_1)^{-b} c$  for all  $t \in (0, T_1)$ ;  $\bar{\alpha}(t) = \tilde{\alpha}(t)$  for all  $t \in [T_1, \infty)$ ; and  $\bar{\alpha}(0) = D(\theta_2^-(T_1)) \theta_2^-(T_1)^{-b} c$ . Recall that this proof formally covers the case in which  $T_1 > 0$ . If  $T_1 = 0$  then this entire step is not required.

*Step 4.* Define  $T \equiv \min\{t \mid \bar{\alpha}(t) = 1\}$ . The final step involves constructing a new regime by modifying  $(\bar{\alpha}, \bar{\theta}_2)$  to delay as much as possible the quit mass of player 2L in the interval  $[T_1, T]$ . Let  $\bar{T}_1$  and  $\bar{T}_2$  be the times that define the phases of regime  $(\bar{\alpha}, \bar{\theta}_2)$ . We have  $\bar{T}_1 = 0$  and  $\bar{T}_2 = T_2$ . By the construction of  $(\bar{\alpha}, \bar{\theta}_2)$ , we know player 2L is indifferent between quitting at all times in  $[0, T]$ . It is not difficult to see that there is a unique regime  $(\hat{\alpha}, \hat{\theta}_2)$  satisfying: (i)  $\hat{\alpha} \equiv \bar{\alpha}$  (for all  $t$ ); (ii)  $\hat{\theta}_2(0) = \bar{\theta}_2(0)$ ; and (iii) the regime has maximal delay on  $(\hat{T}_1, T) = (0, T)$ .

The regime  $(\hat{\alpha}, \hat{\theta}_2)$  is an equilibrium by construction. In addition, it is the case that  $\hat{\theta}_2 \geq \bar{\theta}_2$ , meaning that player 2L cooperates for a longer period of time in this regime than with  $(\bar{\alpha}, \bar{\theta}_2)$ . The payoffs of players 1L, 2L, and 2H are identical between regimes  $(\hat{\alpha}, \hat{\theta}_2)$  and  $(\bar{\alpha}, \bar{\theta}_2)$ . The payoff of player 1H is weakly greater under  $(\hat{\alpha}, \hat{\theta}_2)$ . To see this, note that delaying when player 2L quits does not change the discounted (terminal) payoff of the quit event to player 1H (since  $\hat{\alpha}'/\hat{\alpha} = r$  on  $(0, T)$ ); however, the delay gives player 1H an additional payoff due to cooperation by player 2L.

To summarize the steps, I have identified an equilibrium regime  $(\hat{\alpha}, \hat{\theta}_2)$  under which the payoff of every player-type is weakly greater than with the original regime  $(\alpha, \theta_2)$ . By construction,  $(\hat{\alpha}, \hat{\theta}_2)$  is a Q-regime. This proves the first assertion of the theorem. To prove that an IIE regime exists, recall that the class of Q-regimes is parameterized by the numbers  $\alpha^0 = \alpha(0) \in [0, 1]$  and  $\theta_2^0 = \theta_2(0) \in [p_2, p_1]$ . It is obvious that payoffs are continuous in these parameters. Furthermore, the set of parameter values yielding equilibrium regimes (a subset of  $[0, 1]^2$ ) is closed; this follows from continuity of the relevant continuation payoff expressions. Thus, for each welfare function  $w(v_{1H}(0), v_{1L}(0), v_{2H}(0), v_{2L}(0))$  that is strictly increasing, there is a Q-regime that maximizes  $w$  over all equilibrium regimes with level functions continuous on  $(0, \infty)$ . This proves the second assertion. *Q.E.D.*

### Proof of Lemma 3

Consider the region of  $(m, p)$  for which  $\alpha_{\text{sym}}^0(m, p)$  is “interior,” meaning  $\alpha_{\text{sym}}^0$  is less than  $D(p)$  so  $T > 0$ . (The other case is easy to handle.) It is not difficult to show that an interior solution is unique and continuous in  $m$  and  $p$ . To prove  $\alpha_{\text{sym}}^0$  is weakly increasing in  $p$ , by the implicit function theorem it is enough to demonstrate that  $\partial^2 w^{\text{sym}} / \partial \alpha^0 \partial p \geq 0$ . We thus need to show that  $\partial^2 u_L / \partial \alpha^0 \partial p \geq 0$  and  $\partial^2 u_H / \partial \alpha^0 \partial p \geq 0$ . Equivalently, we need  $\partial^2 u_L / \partial T \partial p \leq 0$  and  $\partial^2 u_H / \partial T \partial p \leq 0$ , where  $T$  and  $\alpha^0$  are related by the identity  $\alpha^0 = D(p)e^{-rT}$  in an interior solution. Note that

$$u_L = \frac{p^2 \underline{x}^2}{p\underline{x} + (1-p)\underline{y}} e^{-rT}$$

and

$$u_H = \bar{y}D(p)(1-p)e^{-rT} + pe^{-rT}T\bar{z} + pe^{-rT}\bar{z}/r. \quad (8)$$

Differentiating these expressions, we obtain the desired cross partial conditions. Regarding  $u_H$ , the calculations are simplified by first noting that

$$\partial u_H / \partial T = -ru_H + pe^{-rT}\bar{z}. \quad (9)$$

To prove  $\alpha_{\text{sym}}^0$  is weakly decreasing in  $m$ , note that

$$mu'_H(\alpha_{\text{sym}}^0(m, p)) + (1-m)u'_L(\alpha_{\text{sym}}^0(m, p)) = 0$$

in an interior solution. Clearly  $u'_L(\alpha_{\text{sym}}^0(m, p)) > 0$ , so it must be that  $u'_H(\alpha_{\text{sym}}^0(m, p)) < 0$ . The result then follows. To prove that  $T > 0$  in the case of  $m = 1$ , first note that if  $T = 0$  then the high types' payoff is maximized with  $\alpha^0 = D(p)$ . Then it is enough to demonstrate that, treating the expression for  $u_H$  in the previous paragraph as a function of  $T$ ,  $du_H(0)/dT > 0$ . Using equations (8) and (9), we have

$$du_H(0)/dT = -r\bar{y}D(p)(1-p) - p\bar{z} + p\bar{z} > 0,$$

which completes the proof. *Q.E.D.*

### Proof of Lemma 4

This follows from the argument used in the last step of the proof of Theorem 2.

### Proof of Lemma 5

As in the proof of Lemma 3, we can write the payoffs as a function of  $T$ . Note that

$$u_{1H} = \bar{y}(1-p)e^{-rT} + \rho e^{-rT}T\bar{z} + pe^{-rT}\bar{z}/r$$

and

$$u_{2H} = e^{-rT}T\bar{z} + e^{-rT}\bar{z}/r,$$

where  $\rho \in (0, 1]$  captures the fact that it may be necessary to have player 2L betray in a region  $(s, T]$  for some  $s < T$ . If  $T = 0$  defines an equilibrium regime, then it must be the case that  $\rho = 1$ . One can then follow the steps used in the proof of Lemma 3 to complete this proof. *Q.E.D.*

## References

- [1] P. Cramton and T. Palfrey, Ratifiable mechanisms: learning from disagreement, *Games Econ. Behavior* **10** (1995), 255-283.
- [2] V. Crawford, Efficient and durable decision rules: a reformulation, *Econometrica* **53** (1985), 817-835.
- [3] O. Hart and J. Tirole, Contract renegotiation and Coasian dynamics, *Rev. Econ. Studies* **55** (1988), 509-540.
- [4] B. Holmström and R. Myerson, Efficient and durable decision rules with incomplete information, *Econometrica* **51** (1983), 1799-1819.
- [5] E. Maskin and J. Tirole, The principal-agent relationship with an informed principal, I: the case of private values, *Econometrica* **58** (1990), 379-410.
- [6] E. Maskin and J. Tirole, The principal-agent relationship with an informed principal, II: common values, *Econometrica* **60** (1992), 1-42.
- [7] R. Myerson, Mechanism design by an informed principal, *Econometrica* **51** (1983), 1767-1797.
- [8] J. Rauch and J. Watson, Starting small in an unfamiliar environment, UCSD WP 96-28r (1999).
- [9] J. Sobel, A theory of credibility, *Rev. Econ. Studies* **52** (1985), 557-573.
- [10] J. Watson, Building a relationship, draft, 1996.
- [11] J. Watson, Starting small and renegotiation, *J. Econ. Theory* **85** (1999), 52-90.