

SOME ORTHOGONALITY PRESERVING OPERATORS

W. A. AL-SALAM AND A. VERMA

1. Introduction. Let $\{p_n(x)\}$ be a set of orthogonal polynomials (s.o.p.), i.e., $p_0(x), p_1(x), p_2(x), \dots$ is a sequence of polynomials which satisfy the recurrence relation

$$(1.1) \quad \begin{aligned} p_{n+1}(x) &= (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) & (n \geq 0) \\ p_{-1}(x) &= 0, & p_0(x) = 1 \end{aligned}$$

for some A_n, B_n and C_n where $A_n C_{n+1} \neq 0$ ($n \geq 0$).

It is clear that if $\{p_n(x)\}$ is s.o.p. then $\{\alpha p_n(x + \beta)\}$ is also s.o.p. Let E^β be the shift operator $E^\beta f(x) = f(x + \beta)$. We then note that the operator $Lp_n(x) = \alpha p_n(x + \beta)$ can be written, on the set of all polynomials, as a differential operator of infinite order, namely,

$$(1.2) \quad L = \alpha \sum_{n=0}^{\infty} \frac{\beta^n}{n!} D^n, \quad D = \frac{d}{dx}.$$

Let us now consider the set S of all differential operators

$$(1.3) \quad J(D) = \sum_{k=0}^{\infty} a_k D^k, \quad a_0 \neq 0,$$

where $a_0, a_1, \dots$ are constants independent of x . Thus $L \in S$. With no loss of generality we shall take in the following $a_0 = 1$.

Since L is orthogonality preserving, i.e. if $\{p_n(x)\}$ is s.o.p. then $\{Lp_n(x)\}$ is also s.o.p., it is of interest to see what operators in S preserve the orthogonality property of all s.o.p.

We are also interested in elements of S which preserve orthogonality of particular s.o.p. In this connection let an s.o.p. $\{p_n(x)\}$ be given. We say that $\{p_n(x)\}$ possesses an orthogonality preserving operator if there is $J \in S$ such that $\{p_n(x)\}$ and $\{Jp_n(x)\}$ are sets of orthogonal polynomials. We shall prove below the following theorems:

THEOREM 1. *An operator $J \in S$ is orthogonality preserving for all sets of orthogonal polynomials if and only if $J = L$.*

THEOREM 2. *A set of orthogonal polynomials $\{p_n(x)\}$ possesses an orthogonality preserving operator if and only if $\{p_n(x)\}$ is one of the Meixner [1] polynomial sets, i.e., if and only if $\{p_n(x)\}$ is s.o.p. and has a generating relation of the form*

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$$(1.4) \quad \sum_0^{\infty} p_n(x) t^n / n! = A(t) \exp(xH(t))$$

where

$$A(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots, \quad H(t) = t + h_1 t^2 + h_2 t^3 + \cdots.$$

Sheffer [2] proved that (1.4) holds if and only if there is an operator $B(D)$ of the form $B(D) = \sum_{k=0}^{\infty} b_k D^{k+1}$, $b_0 \neq 0$ such that

$$(1.5) \quad B(D)p_n(x) = np_{n-1}(x).$$

He calls such sets "sets of A -type 0."

2. Proof of Theorems 1 and 2. Let $J \in S$ where $J(D) = \sum_0^{\infty} a_k D^k$, $a_0 \neq 0$. Let us assume that $\{p_n(x)\}$ and $\{Jp_n(x)\}$ are sets of orthogonal polynomials. We then have

$$(2.1) \quad p_n(x) = (x - \alpha_n)p_{n-1}(x) - \beta_n p_{n-2}(x),$$

and

$$(2.2) \quad Jp_n(x) = (x - \bar{\alpha}_n)Jp_{n-1}(x) - \bar{\beta}_n Jp_{n-2}(x).$$

If we operate on (2.1) by J we get

$$(2.3) \quad J'(D)p_{n-1}(x) = (\alpha_n - \bar{\alpha}_n)Jp_{n-1}(x) + (\beta_n - \bar{\beta}_n)Jp_{n-2}(x).$$

Comparing coefficients of x^{n-1} and x^{n-2} in (2.3) we get

$$\alpha_n - \bar{\alpha}_n = a_1 \quad \text{and} \quad \beta_n - \bar{\beta}_n = (n-1)(2a_2 - a_1^2).$$

Hence (2.3) may be written in the form

$$(2.4) \quad \bar{B}(D)p_n(x) = n(2a_2 - a_1^2)p_{n-1}(x), \quad (n \geq 0),$$

where $\bar{B}(t) = J'(t)/J(t) - a_1$.

We now consider the two cases:

Case (i). If $2a_2 = a_1^2$, it follows that $J'(D)p_n(x) = a_1 J(D)p_n(x)$ which implies that $J(t) = c \exp(a_1 t)$, c an arbitrary constant. Thus $J(D)$ is the operator (1.2).

Case (ii). If $2a_2 - a_1^2 = b_0 \neq 0$. We may write (2.4) as

$$B(D)p_n(x) = b\bar{B}(D)p_n(x) = np_{n-1}(x) \quad (n \geq 0)$$

where $b = 1/b_0 \neq 0$ and

$$B(D) = \sum_{k=0}^{\infty} b_k D^{k+1}, \quad b \neq 0.$$

Hence, $\{p_n(x)\}$ is of Sheffer A -type 0 [2] and must satisfy (1.5) with $H(t)$ defined by $H(B(t)) = t$.

This finishes the proof of Theorem 1 and we only need to prove that if $\{p_n(x)\}$ is one of the orthogonal sets of polynomials and is of A -type 0 (i.e., satisfies (1.4)) then $\{p_n(x)\}$ possesses an orthogonality preserving operator. Indeed let $\{p_n(x)\}$ satisfy (1.5) and $J(t)$ be the solution of the differential equation $J'(t) = \gamma B(t)J(t)$, $J(0) = 1$ and γ is arbitrary. Thus we must have

$$(2.5) \quad J'(D)p_n(x) = \gamma n J(D)p_{n-1}(x).$$

Hence we obtain from (2.1) and (2.5) that

$$Jp_n(x) = (x - \alpha_n)Jp_{n-1}(x) - [\beta_n - (n-1)\gamma]Jp_{n-2}(x) \quad (n \geq 1)$$

which is of the form (1.1). Hence $\{Jp_n(x)\}$ is a set of orthogonal polynomials.

Meixner [1] and Sheffer [2] determined all orthogonal polynomials which are of A -type 0. We give them below together with their orthogonality preserving operator $J(D)$ and evaluate $Jp_n(x)$.

(i) *The Laguerre polynomials* $L_n^{(\alpha)}(x)$.

$$H(t) = B(t) = t/(t-1), \quad J(t) = (1-t)^\gamma,$$

so that

$$(1-D)^\gamma L_n^{(\alpha)}(x) = L_n^{(\alpha+\gamma)}(x)$$

which is valid for arbitrary γ .

(ii) *The Hermite polynomials* $H_n(x)$. $H(t) = 2t$, $B(t) = t/2$ so that $J(t) = \exp(\gamma t^2)$ where γ is an arbitrary constant. We have

$$(2.7) \quad \exp(\gamma D^2)H_n(x) = (1-4\gamma)^{-n/2}H_n(x(1-4\gamma)^{1/2}).$$

(iii) *The Charlier polynomials* $c_n(x; a)$. $H(t) = \log(1-t/a)$, $B(t) = a(1-e^t)$ so that $J(t) = \exp(-\gamma(1-e^t))$ where γ is an arbitrary constant.

The relation

$$(2.8) \quad e^{\gamma \Delta} c_n(x; a) = (1 + \gamma/a)^n c_n(x; a + \gamma)$$

can be easily verified.

(iv) *The Meixner polynomials* $M_n(x; \mu, d)$ defined by means of

$$\frac{(1-\mu t)^{x+d}}{(1-t)^x} = \sum_{n=0}^{\infty} M_n(x; \mu, d) t^n / n!,$$

$$H(t) = \log \frac{1-\mu t}{1-t}, \quad B(t) = \frac{e^t - 1}{e^t - \mu}.$$

As before we have $J(t) = e^{\gamma t}(e^t - \mu)^{\gamma(\mu-1)}$ where e^γ is an arbitrary constant. Hence we have

$$(2.9) \quad E^\gamma(E - \mu)^{\gamma(\mu-1)} M_n(x; \mu, d) \\ = (1 - \mu)^{\gamma(\mu-1)} M_n(x + \gamma\mu; \mu, d + \gamma - \gamma\mu).$$

In the above examples E is the shift operator $Ef(x) = f(x+1)$ and Δ is the difference operator $\Delta = E - 1$. We also note that $j(D)$ was determined up to a shift operator because if $J(D)$ is orthogonality preserving then $E^c J(D)$ has the same property. Thus we shall consider two operators $J_1(D)$ and $J_2(D)$ equal if there is a constant c such that $J_1(D) = E^c J_2(D)$.

REFERENCES

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UNIVERSITY OF ALBERTA