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Solving Linear Fractional Programming Problems with Interval Coefficients in the Objective Function. A New Approach

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Abstract

In the recent years we have seen many approaches to solve fractional programming problems. In this paper, the linear fractional programming problem with interval coefficients in objective function is solved by the variable transformation. In this method a convex combination of the first and the last points of the intervals are used in place of the intervals and consequently the problem is reduced to a nonlinear programming problem. Finally, the nonlinear problem is transformed into a linear programming problem with two more constraints and one more variable compare to the initial problem. Numerical examples are illustrated to show the efficiency of the method.

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1. Introduction

The fractional programming (FP) is a special case of a nonlinear programming, which is generally used for modeling real life problems with one or more objective(s) such as profit/cost,
actual cost/standard, output/employee,etc, and it is applied to different disciplines such as
engineering, business, finance, economics, etc.
Ref .[3] gives a survey on fractional programming which covers applications as well as major theoretical and algorithmic developments.
The linear fractional programming (LFP) is a special class of fractional programming which can
be transformed into a linear programming problem by the method of Charnes and Cooper[2].
The same problem can also be solved by adopting the updated objective function method which was discussed by Bitran and Novaes[1]. Here we briefly illustrate the technique of Charnes and Cooper to reduce an LFP into an LP.
The general extended form of a linear fractional programming problem is as follows:

Minimize $\frac{a_1x_{1+\cdots+}a_kx_k+a_{k+1}}{c_1x_1+\cdots+c_kx_k+c_{k+1}}$
s.t

$$A_1x_1+\cdots+A_kx_k \leq b,$$
$$x_1 \geq 0, \ldots, x_k \geq 0, \qquad P(1)$$

where A_i , for $i = 1, \ldots, k$ and b are m-dimensional constant column vectors.
Moreover we assume that:

$c_1x_1+\cdots+c_kx_k+c_{k+1} > 0 \; for \; all \; x^T = (x_1, \ldots, x_k) \in X$, where X is the compact feasible region of Problem (1).
For solving problem (1) by Charnes and Cooper, we set: $z = \frac{1}{c_1x_1+\cdots+c_kx_k+c_{k+1}}$, which transforms problem (1) into the following linear programming problem:

Minimize $a_1x_1z + \cdots + a_kx_kz + a_{k+1}z$

s.t

$$c_1x_1z + \cdots + c_kx_kz + c_{k+1}z = 1,$$
$$A_1x_1z + \cdots + A_kx_kz - bz \leq 0,$$
$$x_1 \geq 0, \ldots, x_k \geq 0, z \geq 0. \qquad P(2)$$

By introducing variables $y_i = x_iz$ for $i = 1, \ldots, k$, the problem (2) can be further reduced to:

Minimize $a_1y_1 + \cdots + a_ky_k + a_{k+1}z$

s.t

$$c_1y_1 + \cdots + c_ky_k + c_{k+1}z = 1,$$
$$A_1y_1 + \cdots + A_ky_k - bz \leq 0,$$
$$y_1 \geq 0, \ldots, y_k \geq 0, z \geq 0. \qquad P\ (3)$$

As we know, there are many phenomena in the real physical world in which the coefficients are
not certain when they are modeled mathematically. So in such cases, it is much better to select
coefficients as the intervals instead of fixed numbers. For example one of these situations occurs when the coefficients are fuzzy numbers. In these cases if decision makers specify a α-level of satisfactory, then the fuzzy numbers are transformed into intervals [4]. Therefore in such situations we deal with interval mathematical programming. Under these circumstances, In this paper, the linear fractional programming problem with interval coefficients in the objective function is considered . For solving the problem, a method based on variable transformation by Charnes and Cooper and convex combination of intervals is used.

2. Formulation of the problem

The general extended form of a linear fractional programming problem with interval coefficients in the objective function is as follows:

Minimize $\frac{[a_1,b_1]x_1+\cdots+[a_k,b_k]x_k+[a_{k+1},b_{k+1}]}{[c_1,d_1]x_1+\cdots+[c_k,d_k]x_k+[c_{k+1},d_{k+1}]}$

s.t

$$A_1x_1 + \dots + A_kx_k \le b,$$
$$x_1 \ge 0, \dots, x_k \ge 0. \qquad P\ (4)$$

it has also been assumed that
$[c_1, d_1]x_1 + \dots + [c_k, d_k]x_k + [c_{k+1}, d_{k+1}] > 0$ for all $x^T = (x_1, \dots, x_k) \in X$, where X is
the compact feasible region of problem (4).
For solving problem (4), we introduce variable
$z = \frac{1}{[c_1,d_1]x_1+\dots+[c_k,d_k]x_k+[c_{k+1},d_{k+1}]}$
and then we have:

Minimize $[a_1, b_1]x_1z + \dots + [a_k, b_k]x_kz + [a_{k+1}, b_{k+1}]z$
s.t
$$[c_1, d_1]x_1z + \dots + [c_k, d_k]x_kz + [c_{k+1}, d_{k+1}]z = 1,$$
$$A_1x_1z + \dots + A_kx_kz - bz \le 0,$$
$$x_1 \ge 0, \dots, x_k \ge 0, z \ge 0. \qquad P\ (5)$$

By introducing variables $y_i = x_iz$ for $i = 1, \dots, k$ the problem (5) is transformed into the
following equivalent problem:

Minimize $[a_1, b_1]y_1 + \dots + [a_k, b_k]y_k + [a_{k+1}, b_{k+1}]z$
s.t
$$[c_1, d_1]y_1 + \dots + [c_k, d_k]y_k + [c_{k+1}, d_{k+1}]z = 1,$$
$$A_1y_1 + \dots + A_ky_k - bz \le 0,$$
$$y_1 \ge 0, \dots, y_k \ge 0, z \ge 0. \qquad P\ (6)$$

The linear combination of each interval yields to the following problem:

Min $[\lambda_1a_1 + (1-\lambda_1)b_1]y_1 + \dots + [\lambda_ka_k + (1-\lambda_k)b_k]y_k + [\lambda_{k+1}a_{k+1} + (1-\lambda_{k+1})b_{k+1}]z$

s.t $A_1y_1 + \dots + A_ky_k - bz \le 0,$
$[\beta_1c_1 + (1-\beta_1)d_1]y_1 + \dots + [\beta_kc_k + (1-\beta_k)d_k]y_k + [\beta_{k+1}c_{k+1} + (1-\beta_{k+1})d_{k+1}]z = 1,$

$$y_1 \geq 0, \dots, y_k \geq 0,\ z \geq 0,\ 0 \leq \lambda_i \leq 1, \dots, 0 \leq \beta_i \leq 1 \ \text{for}\ i = 1, \dots, k+1 . \qquad P\,(7)$$

The equality constraint in $P(7)$ can be further reduced to

$$[\beta_1 y_1(c_1 - d_1) + \dots + \beta_k y_k(c_k - d_k) + \beta_{k+1} z(c_{k+1} - d_{k+1})] + d_1 y_1 + \dots + d_k y_k + d_{k+1} z = 1, \quad (8)$$

since
$y_j \geq 0 \quad for\ j = 1, \dots, k\ , z \geq 0, 0 \leq \beta_i \leq 1, (d_i - c_i) \geq 0$ for $i = 1, \dots, k+1$.

Therefore (8) can be written as:

$$1 \leq 1 + [\beta_1 y_1(d_1 - c_1) + \dots + \beta_k y_k(d_k - c_k) + \beta_{k+1} z(d_{k+1} - c_{k+1}) \leq 1 + y_1(d_1 - c_1) + \dots + y_k(d_k - c_k) + z(d_{k+1} - c_{k+1}) \quad (9)$$

Combining (8) and (9) results:

$$1 \leq d_1 y_1 + \dots + d_k y_k + d_{k+1} z \leq 1 + y_1(d_1 - c_1) + \dots + y_k(d_k - c_k) + z(d_{k+1} - c_{k+1}) \quad (10)$$

which further reduced to:

$$d_1 y_1 + \dots + d_k y_k + d_{k+1} z \geq 1 \quad (11)$$

and

$$c_1 y_1 + \dots + c_k y_k + c_{k+1} z \leq 1 \quad (12)$$

Therefore on using (11) and (12), the problem (7) is transformed into the following equivalent
problem.

Minimize $[\lambda_1 a_1 + (1 - \lambda_1) b_1] y_1 + \dots + [\lambda_k a_k + (1 - \lambda_k) b_k] y_k + [\lambda_{k+1} a_{k+1} + (1 - \lambda_{k+1}) b_{k+1}] z$

s.t

$$c_1y_1 + \cdots + c_ky_k + c_{k+1}z \leq 1,$$
$$d_1y_1 + \cdots + d_ky_k + d_{k+1}z \geq 1,$$
$$A_1y_1 + \cdots + A_ky_k - bz \leq 0,$$
$$y_1 \geq 0, \ldots, y_k \geq 0,\ z \geq 0\ , 0 \leq \lambda_i \leq 1 \text{ for } i = 1, \ldots, k+1\ . \qquad P\ (13)$$

In addition, if we let $(\bar{y}_1, \ldots, \bar{y}_k, \bar{z})$ be a point of feasible region of problem (13), with $0 \leq \lambda_i \leq 1\ , (a_i - b_i) \leq 0$ for $i = 1, \ldots, k+1$, then the objective function in problem (13) can be written as:

$$\lambda_1(a_1 - b_1)\bar{y}_1 + \cdots + \lambda_k(a_k - b_k)\bar{y}_k + \lambda_{k+1}(a_{k+1} - b_{k+1})\bar{z} + b_1\bar{y}_1 + \cdots + b_k\bar{y}_k + b_{k+1}\bar{z} \geq$$

$$(a_1 - b_1)\bar{y}_1 + \cdots + (a_k - b_k)\bar{y}_k + (a_{k+1} - b_{k+1})\bar{z} + b_1\bar{y}_1 + \cdots + b_k\bar{y}_k + b_{k+1}\bar{z} =$$

$$a_1\bar{y}_1 + \cdots + a_k\bar{y}_k + a_{k+1}\bar{z}\ .$$

The right hand side of the above equality can be considered as a lower bound for the objective function of the problem (13). Therefore, the problem (13) can be equivalently written as:

Minimize $\quad a_1y_1 + \cdots + a_ky_k + a_{k+1}z$

s.t

$$c_1y_1 + \cdots + c_ky_k + c_{k+1}z \leq 1,$$
$$d_1y_1 + \cdots + d_ky_k + d_{k+1}z \geq 1,$$
$$A_1y_1 + \cdots + A_ky_k - bz \leq 0,$$
$$y_1 \geq 0, \ldots, y_k \geq 0, z \geq 0. \qquad P\ (14)$$

The optimal solution $({y_1}^*, \ldots, {y_k}^*, z^*)$ of the above linear programming problem is same as the
optimal solution of the problem (3) which can be easily obtained by

$$({x_1}^*, \ldots, {x_k}^*) = (\frac{{y_1}^*}{z^*}, \ldots, \frac{{y_k}^*}{z^*}).$$

Proposition: *According to the fact that each fixed number* a *can be equivalently written as interval* [*a*, *a*], *we can claim that linear fractional programming is a special case of linear fractional programming with interval coefficients in the objective function.*

Proof: Problem (1) can be equivalently written as follows:

Minimize $\frac{[a_1,a_1]x_1+\cdots+[a_k,a_k]x_k+[a_{k+1},a_{k+1}]}{[c_1,c_1]x_1+\cdots+[c_k,c_k]x_k+[c_{k+1},c_{k+1}]}$

s.t

$$A_1x_1+\cdots+A_kx_k \le b,$$
$$x_1 \ge 0,\ldots,x_k \ge 0. \qquad P\,(15)$$

The problem (15) is a linear fractional programming problem with interval coefficients in the objective function which is transformed into the following problem:

Minimize $a_1y_1+\cdots+a_ky_k+a_{k+1}z$

s.t

$$c_1y_1+\cdots+c_ky_k+c_{k+1}z \le 1,$$
$$c_1y_1+\cdots+c_ky_k+c_{k+1}z \ge 1,$$
$$A_1y_1+\cdots+A_ky_k-bz \le 0,$$
$$y_1 \ge 0,\ldots,y_k \ge 0, z \ge 0.$$

The combination of the first two constraints causes the following problem which is same as $P(3)$:

Minimize $a_1y_1+\cdots+a_ky_k+a_{k+1}z$

s.t

$$c_1y_1+\cdots+c_ky_k+c_{k+1}z = 1,$$
$$A_1y_1+\cdots+A_ky_k-bz \le 0,$$
$$y_1 \ge 0,\ldots,y_k \ge 0,\ z \ge 0.$$

3. Numerical example

Consider the following linear fractional programming problem with interval coefficients in the objective function:

Minimize $\frac{[-3,-1]x_1+[2,4]x_2+[-2,-.5]}{[.5,1.5]x_1+[.5,1.5]x_2+[3,5]}$

s.t

$$-x_1+x_2 \le 2,$$

$$2x_1 + 3x_2 \leq 14,$$
$$x_1 - x_2 \leq 5,$$
$$x_1 \geq 0\,, x_2 \geq 0.$$

First of all, the above problem is transformed into the problem (14). Therefore we have:

Minimize $\quad -3y_1 + 2y_2 - 2z$

s.t

$$.5y_1 + .5y_2 + 3z \leq 1,$$
$$1.5y_1 + 1.5y_2 + 5z \geq 1,$$
$$-y_1 + y_2 - 2z \leq 0,$$
$$2y_1 + 3y_2 - 14z \leq 0,$$
$$y_1 - y_2 - 5z \leq 0,$$
$$y_1 \geq 0\,,\ y_2 \geq 0\,,\ z \geq 0.$$

The optimum solution of the above problem is $y_1{}^* = .9091\ , y_2{}^* = 0\ \ , z^* = .1818$ with

The optimum objective function value = -3.0909.

For the sake of surety of the methodology, we select an arbitrary point within each interval and solve the problem, we will show that the solution of these arbitrary points is not as well as the optimum solution of the problem.

See the next numerical example:

Minimize $\quad \dfrac{-2x_1+3x_2-1.25}{x_1+x_2+4}$

s.t

$$-x_1 + x_2 \leq 2,$$
$$2x_1 + 3x_2 \leq 14,$$
$$x_1 - x_2 \leq 5,$$
$$x_1 \geq 0\,,\ x_2 \geq 0.$$

The above problem is transformed into the problem (3) as below:

Minimize $-2y_1 + 3y_2 - 1.25z$
s.t

$$y_1 + y_2 + 4z = 1,$$
$$-y_1 + y_2 - 2z \leq 0,$$
$$2y_1 + 3y_2 - 14z \leq 0,$$
$$y_1 - y_2 - 5z \leq 0,$$
$$y_1 \geq 0\ ,\ y_2 \geq 0\ ,\ z \geq 0.$$

Now on solving above LPP we get the optimum Solution of the problem as:
$y_1{}^* = 0.5556\ , y_2{}^* = 0\ , z^* = 0.1111$
With the optimum objective function = -1.25.

4. Conclusion

In this paper, we introduced a method to solve a linear fractional programming problem with interval coefficients in the objective function. In the proposed method, on using convex combination of the first and the last points of intervals instead of intervals and also using variable transformation, the initial problem is transformed into a nonlinear programming problem which finally is changed into a linear programming problem which has two more constraints and one more variable compare to the initial problem. The method is designed in such a way that each an every points of the intervals examined for achieving the best possible solution for the problem.

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