

Portfolio optimization with transaction costs

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AMaMeF Mid-Term Conference, 18.09.2007, Wien

Outline

The Merton problem with transaction costs

A general principle

Application to Merton problem with transaction costs

References

The Merton problem with transaction costs

- ▶ **Goal:** Maximize expected utility from consumption

$$\mathbb{E} \left(\int_0^{\infty} e^{-\delta t} u(c_t) dt \right)$$

- ▶ Here: $u(x) = \log(x)$
- ▶ c admissible consumption rate (no debts)
- ▶ Bank account: 1 (no interest paid)
- ▶ Stock price S : Modeled as geometric Brownian motion
- ▶ Proportional transaction costs μ, λ (e.g. 1%)

The Merton problem with transaction costs

Without transaction costs (Merton [1971]):

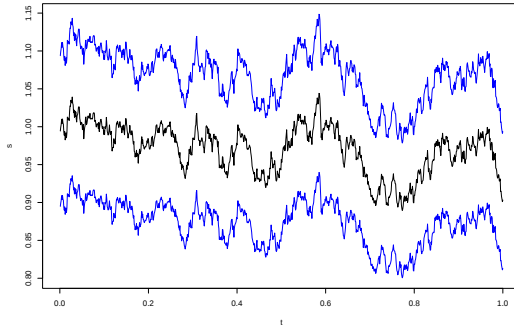
- ▶ Fixed fraction of wealth in stock (e.g. 31%)
- ▶ Consumption rate is fixed proportion of wealth
- ▶ Both numbers explicitly known

With transaction costs (Magill and Constantinides [1976], Davis and Norman [1990]):

- ▶ Fraction of wealth in stock in fixed corridor (e.g. 20-40%)
- ▶ Consumption rate is function of wealth in cash and stock
- ▶ Corridor known only as solution to free boundary problem

A general principle

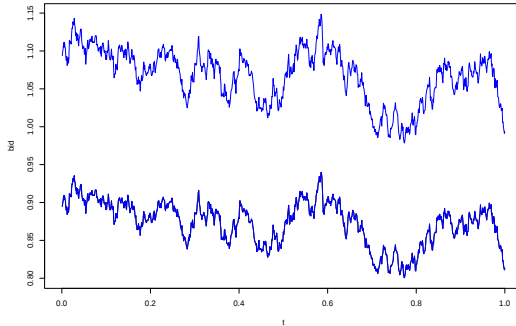
Shadow prices



Optimal portfolio **with transaction costs?**

A general principle

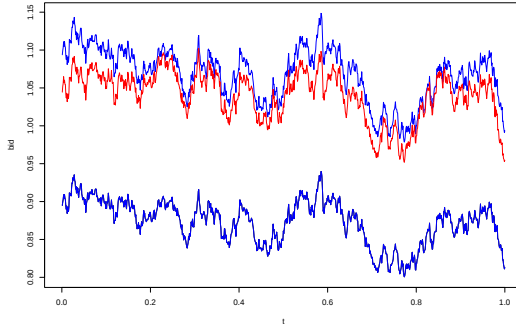
Shadow prices



Optimal portfolio **with transaction costs?**

A general principle

Shadow prices



Optimal portfolio **with transaction costs**



Optimal portfolio **without transaction costs** for **shadow price**

A general principle

Shadow prices

- ▶ **Idea:** Problem with transaction costs as problem without transaction costs for different price process
- ▶ Shadow price at boundary when optimal strategy transacts

Appearances in various fields:

- ▶ Jouini and Kallal [1995]: No-arbitrage
- ▶ Lamberton et al. [1998]: Local risk minimization
- ▶ Cvitanić and Karatzas [1996], Loewenstein [2000]: Portfolio optimization

Useful for computations?

Application to Merton problem with transaction costs

Real price processes:

- ▶ **Stock price**(discounted): $dS_t/S_t = \alpha dt + \sigma dW_t$
- ▶ **Bid price**: $(1 - \mu)S_t$
- ▶ **Ask price**: $(1 + \lambda)S_t$

Shadow price process $\tilde{S} \in [(1 - \mu)S, (1 + \lambda)S]$:

- ▶ $\tilde{S}_t = \exp(C_t)S_t$
- ▶ $C_t = \log(\tilde{S}_t/S_t)$ deviation from real price
- ▶ $C_t \in [\log(1 - \mu), \log(1 + \lambda)]$

Application to Merton problem with transaction costs

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Dynamics of C ?

Application to Merton problem with transaction costs

Ansatz:

- ▶ Itô process $dC_t = \tilde{\alpha}(C_t)dt + \tilde{\sigma}(C_t)dW_t$

$$\Rightarrow d\tilde{S}_t/\tilde{S}_t = \text{Drift}(C_t)dt + \text{Diffusion}(C_t)dW_t$$

Optimal strategy (without transaction costs):

- ▶ Consumption: $\delta\tilde{V}_t$
- ▶ Fraction of stocks: $\pi(C_t) = \frac{\text{Drift}(C_t)}{\text{Diffusion}(C_t)^2}$
- ▶ Use transformation $\frac{1}{1+\exp(f(C_t))} = \pi(C_t)$

$\Rightarrow$ Need to determine 3 functions: $\tilde{\alpha}$, $\tilde{\sigma}$, f

$\Rightarrow f(\log(1 - \mu)), f(\log(1 + \lambda))$ determine corridor

Application to Merton problem with transaction costs

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Application to Merton problem with transaction costs

- **Optimality:**

$$\frac{1}{1 + \exp(-f)} = \frac{\text{Drift}}{\text{Diffusion}^2} \quad (\text{I})$$

- **No trading within bounds:** $d\varphi_t = 0$ for optimal φ

- Itô's formula:

$$\begin{aligned} d\varphi_t = & \text{somefunction}(f, f', f'', \tilde{\alpha}, \tilde{\sigma})dt \\ & + \text{anotherfunction}(f, f', \tilde{\alpha}, \tilde{\sigma})dW_t \end{aligned}$$

- Hence

$$0 = \text{somefunction}, \quad (\text{II})$$

$$0 = \text{anotherfunction} \quad (\text{III})$$

- 3 conditions

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Solution to Equations I-III:

$$\tilde{\sigma} = \frac{\sigma}{f' - 1}$$

$$\tilde{\alpha} = -\alpha + \sigma^2 \left(\frac{f'}{f' - 1} \right) \left(\frac{1}{1 + e^{-f}} \right)$$

f satisfies the ODE

$$\begin{aligned} f''(x) = & \left(\frac{2\delta}{\sigma^2} (1 + e^{f(x)}) \right) + \left(\frac{2\alpha}{\sigma^2} - 1 - \frac{4\delta}{\sigma^2} (1 + e^{f(x)}) \right) f'(x) \\ & + \left(\frac{4\alpha}{\sigma^2} + 2 - \frac{2\delta}{\sigma^2} (1 + e^{f(x)}) + \frac{1 - e^{-f(x)}}{1 + e^{-f(x)}} \right) (f'(x))^2 \\ & + \left(\frac{2\alpha}{\sigma^2} + \frac{2}{1 + e^{-f(x)}} \right) (f'(x))^3 \end{aligned}$$

Still missing:

Boundary conditions for $x = \log(1 - \mu)$ and $x = \log(1 + \lambda)$

Application to Merton problem with transaction costs

Heuristics for boundary conditions:

- ▶ Optimal fraction $\pi(C_t)$: Reflected diffusion (e.g. between 20% and 40%) $\Rightarrow$ local time at boundary
- ▶ Hence $f(C_t) = \log\left(\frac{1-\pi(C_t)}{\pi(C_t)}\right)$ has local time
- ▶ Our Ansatz: $\tilde{S}_t$ (and hence C_t) Itô process, i.e. no local time at boundary
- ▶ Intuition: otherwise infinite position optimal at boundary

Consequence: Contradiction unless $f' = \infty$ on boundary

- ▶ Boundary conditions

$$f'(\log(1 - \mu)) = \infty$$

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Application to Merton problem with transaction costs

Numerical solution: Consider $g = f^{-1}$

- ▶ ODE for g :

$$\begin{aligned} g''(y) = & \left(\frac{1-e^{-y}}{1+e^{-y}} + 1 - \frac{2\alpha}{\sigma^2} \right) \\ & + \left(\frac{4\alpha}{\sigma^2} - 2 - \frac{1-e^{-y}}{1+e^{-y}} - \frac{2\delta}{\sigma^2}(1+e^y) \right) g'(y) \\ & + \left(-\frac{2\alpha}{\sigma^2} + 1 - \frac{4\delta}{\sigma^2}(1+e^y) \right) (g'(y))^2 \\ & - \left(\frac{2\delta}{\sigma^2}(1+e^y) \right) (g'(y))^3 \end{aligned}$$

- ▶ Free boundary: y_1, y_2 with

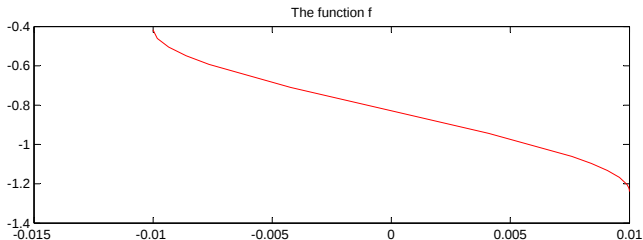
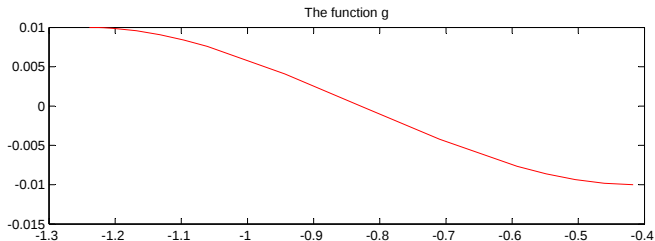
$$g(y_1) = \log(1 - \mu), \quad g'(y_1) = 0$$

$$g(y_2) = \log(1 + \lambda), \quad g'(y_2) = 0$$

- ▶ Free boundaries y_1, y_2 determine corridor, $g = f^{-1}$ determines dynamics of C and hence $\tilde{S} = \exp(C)S$

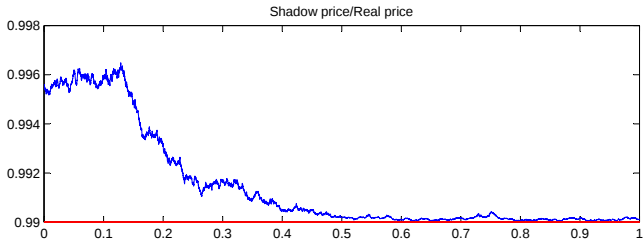
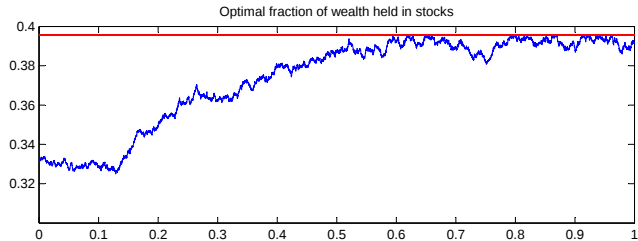
Application to Merton problem with transaction costs

Numerical solution ct'd



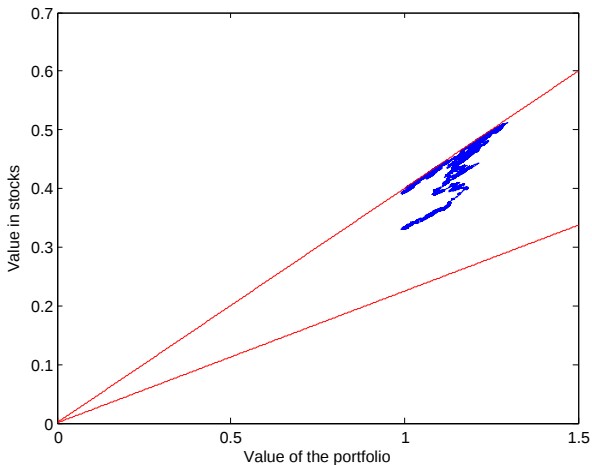
Application to Merton problem with transaction costs

Simulation



Application to Merton problem with transaction costs

Simulation ct'd



Summary

Computation of conditions:

1. Optimality without transaction costs,
2. Constant trading strategy within bounds,
3. Boundary conditions via Itô process assumption.

Verification:

1. Prove existence of a solution to free boundary problem.
2. Prove existence of corresponding processes $\tilde{S}$ etc.
3. Show that optimal investment in $\tilde{S}$ trades only at boundary.

References

- J. Cvitanić and I. Karatzas. Hedging and portfolio optimization under transaction costs: a martingale approach. *Mathematical Finance*, 6(2): 133–165, 1996.
- M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.
- E. Jouini and H. Kallal. Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1):178–197, 1995.
- D. Lamberton, H. Pham, and M. Schweizer. Local risk-minimization under transaction costs. *Mathematics of Operations Research*, 23(3):585–612, 1998.
- M. Loewenstein. On optimal portfolio trading strategies for an investor facing transactions costs in a continuous trading market. *Journal of Mathematical Economics*, 33(2), 2000.
- M. J. P. Magill and G. M. Constantinides. Portfolio selection with transactions costs. *Journal of Economic Theory*, 13(2):245–263, 1976.
- R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.