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On *VT*-harmonic maps

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Abstract

VT-harmonic maps generalize the standard harmonic maps, with respect to the structure of both domain and target. These can be manifolds with natural connections other than the Levi-Civita connection of Riemannian geometry, like Hermitian, affine or Weyl manifolds. The standard harmonic map semilinear elliptic system is augmented by a term coming from a vector field V on the domain and another term arising from a 2-tensor T on the target. In fact, this geometric structure then also includes other geometrically defined maps, for instance magnetic harmonic maps. In this paper, we treat VT-harmonic maps and their parabolic analogues with PDE tools. We establish a Jäger–Kaul type maximum principle for these maps. Using this maximum principle, we prove an existence theorem for the Dirichlet problem for VT-harmonic maps. As applications, we obtain results on Weyl/affine/Hermitian harmonic maps between Weyl/affine/Hermitian manifolds, as well as on magnetic harmonic maps from two-dimensional domains. We also derive gradient estimates and obtain existence results for such maps from noncompact complete manifolds.

Keywords VT-harmonic maps · Maximum principle · Uniqueness · Existence

Mathematics Subject Classification 58E20 · 53C43

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1 Introduction

Let (M^m, g) be a compact Riemannian manifolds with nonempty boundary ∂M and $(N^n, \tilde{g})$ a complete Riemannian manifold without boundary. Let $d : N \times N \to \mathbb{R}$ be the distance function on N and $B_{(1+\sigma)R}(p) := \{q \in N : d(p, q) \leq (1+\sigma)R\}$ a regular ball in N, that is, disjoint from the cut locus of its center p and of radius $(1+\sigma)R < \frac{\pi}{2\sqrt{\kappa}}$, where $\kappa = \max\{0, \sup_{B_{(1+\sigma)R}(p)} K_N\}$ and $\sup_{B_{(1+\sigma)R}(p)} K_N$ is an upper bound of the sectional curvature K of N on $B_{(1+\sigma)R}(p)$, and $\sigma > 0$ is any given constant.

Let $V \in \Gamma(TM)$, $T \in \Gamma(\otimes^{1,2}TN)$. We call a map $u : M \to N$ a *VT-harmonic map* if u satisfies

$$\tau(u) + du(V) + \mathrm{Tr}_g T(du, du) = 0, \tag{1.1}$$

where $\tau(u) = \mathrm{tr}Ddu$ is the tension field of the map u. This is a generalization the notion of a V-harmonic map that has been studied in recent years as a common framework including Hermitian, affine and Weyl harmonic maps into Riemannian manifolds, that is, the domain possessed a connection different from the Levi-Civita connection, but the target was a Riemannian manifold with its Levi-Civita connection. This generalized the standard harmonic map system $\tau(u) = 0$ to a system of the form $\tau(u) + du(V) = 0$ with a vector field V on the domain. Here, we want to consider targets that are of the same type as the domain. That leads to the system (1.1) with an additional term arising from a 2-tensor T on the target. As this new term $\mathrm{Tr}_g T(du, du)$, in contrast to the term $du(V)$, is analytically of the same weight as the elliptic operator $\tau(u)$ (which includes the Laplace–Beltrami operator of the domain), this makes the analysis more difficult and subtle. This is the problem that we are addressing in this paper.

In local coordinates $\{x^\alpha\}$ on M and $\{y^i\}$ on N, respectively, we can write (1.1) as

$$\Delta_M u^i + \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} + T^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} = 0, \tag{1.2}$$

where Δ_M is the Laplacian on (M, g), Γ^i_{jk} stands for the Christoffel symbols of $(N, \tilde{g})$, $V := V^\alpha \frac{\partial}{\partial x^\alpha}$ and $T := T^i_{jk} \frac{\partial}{\partial y^i} \otimes dy^j \otimes dy^k$. This is a second-order semilinear elliptic system on the manifold (M, g).

As is already the case for V-harmonic maps, in general, (1.1) is neither in divergence form, nor has a variational structure. Chen et al. [5] established a Jäger–Kaul type maximum principle for V-harmonic maps by using the method of [8], and combining this with the continuity method, the existence of V-harmonic maps into a regular ball could be proved. Therefore, it is natural to ask whether a maximum principle holds for VT-harmonic maps. However, the case of VT-harmonic maps is harder to deal with than V-harmonic maps since we now have an additional quadratic term arising from the tensor T. Due to this additional structure, the construction of the elliptic operator in [5] is no longer valid in our case. To overcome this difficulty, we use another construction as in [7] to compensate this term and obtain the following maximum principle for VT-harmonic maps:

Theorem 1 *Let $u_1, u_2 \in C^0(M, N)$ be two VT-harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max |\nabla T| + \max |T| \leq C_0, \tag{1.3}$$

then the function $\Theta : M \to \mathbb{R}$ *defined by*

$$\Theta := \frac{q_{\frac{\kappa}{4}}(\rho)}{(q_\kappa((1+\sigma)R) - q_\kappa(\rho_1))^{\frac{1}{2}} \cdot (q_\kappa((1+\sigma)R) - q_\kappa(\rho_2))^{\frac{1}{2}}} \tag{1.4}$$

satisfies the maximum principle, namely

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

Here the expression of q_κ *is given in Sect. 2, and* $\rho := d(u_1, u_2)$, $\rho_i := d(p, u_i)$, $i = 1, 2$.

In particular, if $u_1 = u_2$ *on the boundary* ∂M*, then* $u_1 \equiv u_2$ *on* M.

Remark The explicit expression of the constant C_0 in the above and in the subsequent results can be seen in (3.5). Importantly, $C_0 \to \infty$ for $R \to 0$. Thus, we can also satisfy the condition on T by making the target ball sufficiently small.

For the heat flow of VT-harmonic maps, an analogous result holds. For $T > 0$, we set

$$M_T := M \times [0, T]$$

and denote the parabolic boundary of M_T by

$$\partial_p M_T := (M \times \{0\}) \cup (\partial M \times [0, T]).$$

We consider the heat flow of VT-harmonic maps

$$\partial_t u = \tau(u) + du(V) + \mathrm{Tr}_g T(du, du) \tag{1.5}$$

and have

Theorem 2 *Let* $u_1, u_2 \in C^0(M, N)$ *be two solutions of heat flow Eq. (1.5) for* VT*-harmonic maps into a geodesic ball* $B_R(p)$*. For appropriate* σ *and* R*, there exists a constant* C_0 *depending only on* κ, σ, R *and the geometry of* N*, such that if*

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the function $\Theta : M_T \to \mathbb{R}$ *defined by (1.4) with* M *replaced by* M_T *satisfies the maximum principle:*

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

In particular, if $u_1 = u_2$ *on the boundary* $\partial_p M_T$*, then* $u_1 \equiv u_2$ *on* M_T.

As an application of the above maximum principle, we obtain the existence of VT-harmonic maps into a geodesic ball.

Theorem 3 *Let* $M, N, V, T, B_R(p)$ *be as in Theorem 2. Suppose* $u_0 \in H^{2,q}(M, N) (q > m)$ *with* $u_0(M) \subset B_R(p)$*. For appropriate* σ *and* R*, there exists a constant* C_0 *depending only on* κ, σ, R *and the geometry of* N*, such that if*

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the initial boundary value problem

$$\begin{cases} \partial_t u = \tau(u) + du(V) + \mathrm{Tr}_g T(du, du), \\ u - u_0 \in H_0^{2,q}(M, N), \quad u(0) = u_0, \quad u(M \times [0, \infty)) \subset B_R(p), \end{cases} \tag{1.6}$$

admits a unique global solution u which subconverges to a unique solution $u \in H^{2,q}(M, N)$ *of the Dirichlet problem*

$$\begin{cases} \tau(u) + du(V) + \mathrm{Tr}_g T(du, du) = 0, \\ u - u_0 \in H_0^{2,q}(M, N), \end{cases} \tag{1.7}$$

such that $u(M) \subset B_R(p)$.

Furthermore, based on Theorem 3, we shall also establish the existence of VT-harmonic maps from complete noncompact Riemannian manifolds by using a gradient estimate and the compact exhaustion method.

Theorem 4 *Let* (M^m, g) *be a complete noncompact Riemannian manifold and* $(N^n, \tilde{g})$ *be a complete Riemannian manifold with sectional curvature bounded above by a positive constant* κ. *Let* $B_R(p)$ *be a geodesic ball with radius* $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$ *and* $u_0 : M \to N$ *a smooth map with* $u_0(M) \subset B_R(p)$. *Suppose* $\|V\|_{L^\infty} < +\infty$.

For appropriate σ *and* R, *there exists a constant* C_0' *depending only on* κ, σ, R *and the geometry of* N, *such that if*

$$\max |\nabla T| + \max |T| \le C_0',$$

then there exists a VT*-harmonic map* $u \in C^\infty(M, N)$ *homotopic to* u_0 *such that* $u(M) \subset B_R(p)$.

2 Preliminaries

Let us first give some notations:

$$s_\kappa(t) := \begin{cases} t & \kappa = 0 \\ \dfrac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t & \kappa > 0, \end{cases} \qquad q_\kappa(t) := \begin{cases} \dfrac{t^2}{2} & \kappa = 0 \\ \dfrac{1}{\kappa}(1 - \cos \sqrt{\kappa} t) & \kappa > 0. \end{cases}$$

$$a_\kappa(t) := \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{1 - s_\kappa'(t)}{s_\kappa(t)} & \text{if } t > 0, \end{cases} \qquad b_\kappa(t) := \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{1 - s_\kappa'(t)}{2 s_\kappa(t)} \left(1 + \dfrac{t}{s_\kappa(t)}\right) & \text{if } t > 0. \end{cases}$$

In local coordinates $\{x^\alpha\}$ on M and $\{y^i\}$ on N, respectively, the energy density of u is

$$e(u) := g^{\alpha\beta} \tilde{g}_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

Assume the metric of N satisfies:

$$0 < \widetilde{\lambda}(y)(\delta_{ij}) \le (\widetilde{g}_{ij}(y)) \le \widetilde{\Lambda}(y)(\delta_{ij}), \quad \forall y \in N.$$

Denote $\lambda := \min_N \widetilde{\lambda}$ and $\Lambda := \max_N \widetilde{\Lambda}$

$\forall y_1, y_2 \in B_R(p)$, there exists a unit speed geodesic $\gamma : [0, \rho] \to B_R(p) \subset N$ with $\gamma(0) = y_1, \gamma(\rho) = y_2$, where $\rho = \mathrm{dist}(y_1, y_2)$. For any $v_j \in T_{y_j} N$, $j = 1, 2$, let X be the unique Jacobi field along γ with $X(0) = v_1$, $X(\rho) = v_2$. Then, we define a pseudo-distance

$$\delta(v_1, v_2) := \begin{cases} \left(\rho \displaystyle\int_0^\rho |\dot{X}|^2\right)^{\frac{1}{2}} & \text{if } \rho > 0, \\ |v_1 - v_2| & \text{if } \rho = 0. \end{cases}$$

Another pseudo-distance is given by

$$\delta_0(v_1, v_2) := \left|v_1 - \bar{\bar{v}}_2\right|,$$

where $\bar{\bar{v}}_2 \in T_{y_1}N$ stands for the vector obtained by parallel displacement of $v_2 \in T_{y_2}N$ along γ. Let $L(T_xM, T_yN)$ be the space of all linear maps from T_xM to T_yN. The pseudo-distance δ on the tangent bundle can be extended to a pseudo-distance on the fibers, that is, for $q_1, q_2 \in \cup_{y\in B_R(p)} L(T_xM, T_yN)$ (disjoint union), we define their pseudo-distance as

$$\delta(q_1, q_2) := \left(\sum_{\alpha=1}^{m} \delta^2(q_1(e_\alpha), q_2(e_\alpha))\right)^{\frac{1}{2}},$$

where $\{e_1, \ldots, e_m\}$ is an orthonormal base for T_xM.

We have the following relationship between these two pseudo-distances:

Lemma 1 ([4]) *There is a positive constant C depending only on $B_R(p)$ and the geometry of N such that for any $y_j \in B_R(p)$ and $v_j \in T_{y_j}N$, $j = 1, 2$, we have*

$$\delta_0^2(v_1, v_2) - C(|v_1|^2 + |v_2|^2)\rho^2 \le \delta^2(v_1, v_2) \le \delta_0^2(v_1, v_2) + C(|v_1|^2 + |v_2|^2)\rho^2.$$

Remark 1 In fact, by the proof in [4] and using a well-known expression of the curvature operator (see, e.g., Lemma 4.3.3 in [12]), it is not hard to see that if the sectional curvature K on $B_R(p)$ satisfies $\theta \le K|_{B_R(p)} \le \kappa$ for a constant $\theta < 0$, then the constant C can be expressed as $14(\kappa - \theta)$.

The following estimates will also be important for us:

Lemma 2 ([7]) *Let (M, g) be a compact Riemannian manifolds with nonempty boundary ∂M and $(N, \tilde{g})$ a complete manifold without boundary and $B_R(p)$ a regular ball in N. Let*

$$\begin{aligned} g_1 :=& q_\kappa \circ d(p, \cdot) : B_R(p) \to \mathbb{R}, \\ h :=& q_{\frac{\kappa}{4}} \circ d : B_R(p) \times B_R(p) \to \mathbb{R}. \end{aligned}$$

Then,

$$\nabla^2 g_1(u, u) \ge s_\kappa'(\tau)|u|^2 \tag{2.1}$$

hold for $u \in T_xN$, $x \in B_R(p)$ and $\tau := d(p, x)$.

$$\nabla^2 h(v, v) \ge -s_{\frac{\kappa}{4}}(\rho)a_\kappa(\rho)\sum_{i=1}^{2}|v_i|^2 \tag{2.2}$$

and

$$\nabla^2 h(v, v) \ge s_{\frac{\kappa}{4}}'(\rho)\delta^2(v_1, v_2) - s_{\frac{\kappa}{4}}(\rho)b_\kappa(\rho)\sum_{i=1}^{2}|v_i|^2 \tag{2.3}$$

holds for $v = v_1 \oplus v_2$, $v_j \in T_{y_j}N$, $y_j \in B_R(p)$, $j = 1, 2$, $\rho = \mathrm{dist}(y_1, y_2)$.

3 The maximum principle

Proof of Theorem 1 Let

$$\begin{aligned}\psi(x) :=& q_{\frac{\kappa}{4}} \circ d(u_1(x), u_2(x)),\\ \psi_i(x) :=& q_\kappa \circ d(p, u_i(x)), \quad i = 1, 2,\\ \Phi(x) :=& \frac{1}{2}\sum_{i=1}^{2} \omega \circ \psi_i(x), \quad \text{where} \quad \omega(t) := -\log(q_\kappa((1+\sigma)R) - t).\end{aligned}$$

We consider the operator

$$\mathcal{L}_V(\cdot) := e^{\Phi} \cdot \operatorname{div}(e^{-2\Phi}\nabla\cdot) + e^{-\Phi} \cdot V(\cdot).$$

By direct computation, we obtain

$$\begin{aligned}\mathcal{L}_V(\Theta) =& \mathcal{L}_V(e^{\Phi} \cdot \psi) = \Delta\psi + \psi(\Delta\Phi - |\nabla\Phi|^2) + \psi V(\Phi) + V(\psi)\\ =& \Delta_V \psi + \psi(\Delta_V\Phi - |\nabla\Phi|^2).\end{aligned}$$

Define $U, U_1, U_2 : M \to N \times N$ by

$$U(x) := (u_1(x), u_2(x)), \quad U_i(x) := (p, u_i(x)), \quad i = 1, 2.$$

Let $\nu := \frac{\kappa}{4}$, $h := q_\nu \circ d$, $\phi := q_\kappa \circ d$, then $\psi = h \circ U$, $\psi_i = \phi \circ U_i$.

For any $x \in M$, we let $\tilde{\gamma}$ be the unique geodesic connecting $u_1(x)$ and $u_2(x)$. Choosing a parallel orthonormal frame $\{E_i(t)\}$ along $\tilde{\gamma}$ with $E_1 = \tilde{\gamma}'$, and a local orthonormal frame $\{e_\alpha\}|_{\alpha=1}^{m}$ around x, assuming that $\frac{\partial}{\partial y^i} := a_i^j E_j$, we have

$$\begin{aligned}&\delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))\\ &= \delta_0\left((u_1)^i_\alpha (u_1)^j_\alpha T^r_{ij}(u_1)\frac{\partial}{\partial y^r}(u_1), (u_2)^i_\alpha (u_2)^j_\alpha T^r_{ij}(u_2)\frac{\partial}{\partial y^r}(u_2)\right)\\ &= \delta_0\left((u_1)^i_\alpha (u_1)^j_\alpha T^r_{ij}(u_1) a^\mu_r(u_1) E_\mu(u_1), (u_2)^i_\alpha (u_2)^j_\alpha T^r_{ij}(u_2) a^\mu_r(u_2) E_\mu(u_2)\right)\\ &\le \sum_\mu \left| T^r_{ij}(u_1) a^\mu_r(u_1)(u_1)^i_\alpha (u_1)^j_\alpha - T^r_{ij}(u_2) a^\mu_r(u_2)(u_2)^i_\alpha (u_2)^j_\alpha \right|\\ &\le \sum_\mu \left| (T^r_{ij}(u_1) - T^r_{ij}(u_2)) a^\mu_r(u_1)(u_1)^i_\alpha (u_1)^j_\alpha \right|\\ &\quad + \sum_\mu \left| T^r_{ij}(u_2)(a^\mu_r(u_1) - a^\mu_r(u_2))(u_1)^i_\alpha (u_1)^j_\alpha \right|\\ &\quad + \sum_\mu \left| T^r_{ij}(u_2) a^\mu_r(u_2)(u_1)^j_\alpha \left((u_1)^i_\alpha - (u_2)^i_\alpha\right) \right|\\ &\quad + \sum_\mu \left| T^r_{ij}(u_2) a^\mu_r(u_2)(u_2)^i_\alpha \left((u_1)^j_\alpha - (u_2)^j_\alpha\right) \right|.\end{aligned}$$

Denote $A = (a^k_i)$, then $AA^T = G := (\tilde{g}_{ik})$. Since

$$\sum_{i,k} |a^k_i|^2 = \|A\|^2 = \operatorname{tr}(AA^T) = \operatorname{tr}(G) \le n\Lambda, \qquad \left|(u_1)^i_\alpha (u_1)^j_\alpha\right| \le \frac{1}{\lambda}|du_1|^2,$$

we then obtain

$$\begin{aligned}
&\delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))\\
&\quad \le \sum_{\mu=1}^{n}\left(\frac{\sqrt{n\Lambda}}{\lambda}\rho \max|\nabla T||du_1|^2 + \frac{\widetilde{C}_1}{\lambda}\rho\max|T||du_1|^2\right.\\
&\qquad \left. + \frac{\sqrt{n\Lambda}}{\lambda}\max|T|(|du_1|+|du_2|)|du_1-du_2|\right)\\
&\quad = \frac{n\sqrt{n\Lambda}}{\lambda}\rho\max|\nabla T||du_1|^2 + \frac{n\widetilde{C}_1}{\lambda}\rho\max|T||du_1|^2\\
&\qquad + \frac{n\sqrt{n\Lambda}}{\lambda}\max|T|(|du_1|+|du_2|)|du_1-du_2|,
\end{aligned}$$

where $\widetilde{C}_1 > 0$ is a constant depending only on the bound of (da_r^μ) on $B_{\frac{\pi}{2\sqrt{\kappa}}}(p)$. By Lemma 1, we get

$$\begin{aligned}
&\delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))\\
&\quad \le \delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) + \sqrt{C}\rho(|T(du_1(e_\alpha), du_1(e_\alpha))|\\
&\qquad + |T(du_2(e_\alpha), du_2(e_\alpha))|)\\
&\quad \le \frac{n\sqrt{n\Lambda}}{\lambda}\rho\max|\nabla T|\sum_{i=1}^{2}|du_i|^2 + \left(\frac{n\widetilde{C}_1}{\lambda}+\sqrt{C}\right)\rho\max|T|\sum_{i=1}^{2}|du_i|^2\\
&\qquad + \frac{n\sqrt{n\Lambda}}{\lambda}\max|T|(|du_1|+|du_2|)|du_1-du_2|,
\end{aligned}$$

where $C = 14(\kappa - \theta)$, and the constant θ is a lower bound of the sectional curvature of N on $B_{\frac{\pi}{2\sqrt{\kappa}}}(p)$. The Cauchy inequality implies that

$$\begin{aligned}
\frac{n\sqrt{n\Lambda}}{\lambda}\max|T||du_i||du_1-du_2|s_\nu(\rho) &\le \frac{\varepsilon_1 n^2}{2}|du_1-du_2|^2\\
&\quad + \frac{n\Lambda}{2\varepsilon_1\lambda^2}s_\nu^2(\rho)\max|T|^2|du_i|^2.
\end{aligned}$$

By using the formula (2.13) in [7], it follows that

$$\begin{aligned}
&\langle(\nabla h)\circ U, -T(dU(e_\alpha), dU(e_\alpha))\rangle\\
&\quad = -s_\nu(\rho)\langle(\nabla d)\circ U, T(dU(e_\alpha), dU(e_\alpha))\rangle\\
&\quad = -s_\nu(\rho)\langle\tilde{e}_1(U)\oplus\tilde{e}_2(U), T(du_1(e_\alpha), du_1(e_\alpha))\oplus T(du_2(e_\alpha), du_2(e_\alpha))\rangle\\
&\quad \ge -s_\nu(\rho)\delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))\\
&\quad \ge -s_\nu(\rho)\left\{\frac{n\sqrt{n\Lambda}}{\lambda}\rho\max|\nabla T|\sum_{i=1}^{2}|du_i|^2 + \left(\frac{n\widetilde{C}_1}{\lambda}+\sqrt{C}\right)\rho\max|T|\sum_{i=1}^{2}|du_i|^2\right\}\\
&\qquad - \varepsilon_1 n^2|du_1-du_2|^2 - \frac{n\Lambda}{2\varepsilon_1\lambda^2}s_\nu^2(\rho)\max|T|^2\sum_{i=1}^{2}|du_i|^2,
\end{aligned}$$

where $\tilde{e}_1 = -\tilde{\gamma}'(0), \tilde{e}_2 = \tilde{\gamma}'(\rho)$. Since

$$\begin{aligned}
&\delta_0^2(du_1(e_\alpha), du_2(e_\alpha)) = \delta_0^2\left((u_1)^i_\alpha \frac{\partial}{\partial y^i}(u_1), (u_2)^i_\alpha \frac{\partial}{\partial y^i}(u_2)\right)\\
&\quad = \delta_0^2\left((u_1)^i_\alpha a^j_i(u_1)E_j(u_1), (u_2)^i_\alpha a^j_i(u_2)E_j(u_2)\right) = \sum_j \left[(u_1)^i_\alpha a^j_i(u_1) - (u_2)^i_\alpha a^j_i(u_2)\right]^2\\
&\quad = \sum_j \left[(u_1)^i_\alpha (a^j_i(u_1) - a^j_i(u_2)) + a^j_i(u_2)((u_1)^i_\alpha - (u_2)^i_\alpha)\right]^2\\
&\quad \geq \frac{1}{2}\sum_j \left[a^j_i(u_2)((u_1)^i_\alpha - (u_2)^i_\alpha)\right]^2 - \sum_j \left[(u_1)^i_\alpha (a^j_i(u_1) - a^j_i(u_2))\right]^2\\
&\quad = \frac{1}{2}\sum_j \sum_{ik} \tilde{g}_{ik}(u_2)((u_1)^i_\alpha - (u_2)^i_\alpha)((u_1)^k_\alpha - (u_2)^k_\alpha) - \sum_j \left[(u_1)^i_\alpha (a^j_i(u_1) - a^j_i(u_2))\right]^2\\
&\quad \geq \frac{n\lambda}{2}|(du_1 - du_2)(e_\alpha)|^2 - n\widetilde{C}_1\rho^2|du_1(e_\alpha)|^2,
\end{aligned}$$

then by Lemma 1, we have

$$\begin{aligned}
\delta^2(du_1(e_\alpha), du_2(e_\alpha)) &\geq \delta_0^2(du_1(e_\alpha), du_2(e_\alpha)) - C\left(|du_1(e_\alpha)|^2 + |du_2(e_\alpha)|^2\right)\rho^2\\
&\geq \frac{n\lambda}{2}|(du_1 - du_2)(e_\alpha)|^2 - \left(n\widetilde{C}_1 + C\right)\rho^2 \sum_{i=1}^{2} |du_i(e_\alpha)|^2.
\end{aligned}$$

Namely,

$$\delta^2(du_1, du_2) \geq \frac{n\lambda}{2}|du_1 - du_2|^2 - \left(n\widetilde{C}_1 + C\right)\rho^2 \sum_{i=1}^{2} |du_i|^2. \tag{3.1}$$

Therefore,

$$\begin{aligned}
&\langle (\nabla h)\circ U, -T(dU(e_\alpha), dU(e_\alpha))\rangle\\
&\quad \geq -s_\nu(\rho)\left\{\frac{n\sqrt{n\Lambda}}{\lambda}\rho \max|\nabla T| \sum_{i=1}^{2}|du_i|^2 + \left(\frac{n\widetilde{C}_1}{\lambda} + \sqrt{C}\right)\rho \max|T| \sum_{i=1}^{2}|du_i|^2\right\}\\
&\qquad - \frac{2n\varepsilon_1}{\lambda}\left[\delta^2(du_1, du_2) + \left(n\widetilde{C}_1 + C\right)\rho^2 \sum_{i=1}^{2}|du_i|^2\right] - \frac{n\Lambda}{2\varepsilon_1\lambda^2}s_\nu(\rho)\rho \max|T|^2| \sum_{i=1}^{2} du_i|^2\\
&\quad = -s_\nu(\rho)\rho\left\{\frac{n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| + \left(\frac{n\widetilde{C}_1}{\lambda} + \sqrt{C}\right)\max|T| + \frac{n\Lambda}{2\varepsilon_1\lambda^2}\max|T|^2\right\}\sum_{i=1}^{2}|du_i|^2\\
&\qquad - \frac{2n\varepsilon_1}{\lambda}\left[\delta^2(du_1, du_2) + \left(n\widetilde{C}_1 + C\right)\rho^2 \sum_{i=1}^{2}|du_i|^2\right].
\end{aligned}$$

The above inequality and (2.3) imply that

$$\begin{aligned}\Delta_V\psi =&\Delta_V(h\circ U)=\sum_\alpha \nabla^2 h(dU(e_\alpha),dU(e_\alpha))+\langle(\nabla h)\circ U,\tau_V(U)\rangle.\\ &=\sum_\alpha \nabla^2 h(dU(e_\alpha),dU(e_\alpha))+\sum_\alpha\langle(\nabla h)\circ U,-T(dU(e_\alpha),dU(e_\alpha))\rangle\\ &\geq\left(s'_\nu(\rho)-\frac{2n\varepsilon_1}{\lambda}\right)\delta^2(du_1,du_2)-s_\nu(\rho)b_\kappa(\rho)\sum_{i=1}^2|du_i|^2\\ &\quad -s_\nu(\rho)\rho\left\{\frac{n\sqrt{n\Lambda}}{\lambda}\max|\nabla T|+\left(\frac{n\widetilde{C}_1}{\lambda}+\sqrt{C}\right)\max|T|\right.\\ &\quad\left.+\frac{n\Lambda}{2\varepsilon_1\lambda^2}\max|T|^2\right\}\sum_{i=1}^2|du_i|^2\\ &\quad -\frac{2n\varepsilon_1}{\lambda}\left(n\widetilde{C}_1+C\right)\rho^2\sum_{i=1}^2|du_i|^2.\end{aligned}$$

Choosing $\varepsilon_1=\frac{\lambda}{2n}\cos(\sqrt{\kappa}R)>0$, then we obtain

$$\begin{aligned}\Delta_V\psi\geq&-s_\nu(\rho)b_\kappa(\rho)\sum_{i=1}^2|du_i|^2-\cos(\sqrt{\kappa}R)\left(n\widetilde{C}_1+C\right)\rho^2\sum_{i=1}^2|du_i|^2\\ &-s_\nu(\rho)\rho\left\{\frac{n\sqrt{n\Lambda}}{\lambda}\max|\nabla T|+\left(\frac{n\widetilde{C}_1}{\lambda}+\sqrt{C}\right)\max|T|\right.\\ &\left.+\frac{n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2\right\}\sum_{i=1}^2|du_i|^2.\end{aligned}\tag{3.2}$$

It follows from (2.1) that

$$\begin{aligned}\Delta_V\psi_i&=\Delta_V(\phi\circ U_i)=\sum_\alpha\nabla^2\phi(dU_i(e_\alpha),dU_i(e_\alpha))+\langle(\nabla\phi)\circ U_i,\tau_V(U_i)\rangle\\ &\geq s'_\kappa(\rho_i)|du_i|^2+s_\kappa(\rho_i)\langle(\nabla d)\circ U_i,0\oplus(-T(du_i(e_\alpha),du_i(e_\alpha)))\rangle\\ &\geq s'_\kappa(\rho_i)|du_i|^2-s_\kappa(\rho_i)\,|T(du_i(e_\alpha),du_i(e_\alpha))|\\ &\geq s'_\kappa(\rho_i)|du_i|^2-s_\kappa(\rho_i)\max|T||du_i|^2.\end{aligned}\tag{3.3}$$

It is easy to check that

$$\omega''=\omega'^2,\qquad \omega'\circ\psi_i=\frac{1}{q_\kappa((1+\sigma)R)-q_\kappa(\rho_i)}.$$

Therefore,

$$
\begin{aligned}
\mathcal{L}_V(\Theta) &= \Delta_V \psi + \psi(\Delta_V \Phi - |\nabla \Phi|^2) \\
&= \Delta_V \psi + \frac{1}{2}\psi \sum_{i=1}^{2}(\omega' \circ \psi_i)\Delta_V \psi_i + \sum_{i=1}^{2}\left[\frac{1}{2}\psi(\omega'' \circ \psi_i) - \frac{1}{4}\psi(\omega' \circ \psi_i)^2\right]|\nabla \psi_i|^2 \\
&= \Delta_V \psi + \frac{1}{2}\psi \sum_{i=1}^{2}(\omega' \circ \psi_i)\Delta_V \psi_i + \frac{\psi}{4}\sum_{i=1}^{2}(\omega' \circ \psi_i)^2 |\nabla \psi_i|^2 \\
&\geq - s_\nu(\rho) b_\kappa(\rho) \sum_{i=1}^{2}|du_i|^2 - \cos(\sqrt{\kappa}R)\left(n\widetilde{C}_1 + C\right)\rho^2 \sum_{i=1}^{2}|du_i|^2 \\
&\quad - s_\nu(\rho)\rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| + \left(\frac{n\widetilde{C}_1}{\lambda} + \sqrt{C}\right)\max|T| \right. \\
&\quad \left. + \frac{n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2 \right\} \sum_{i=1}^{2}|du_i|^2 \\
&\quad + \frac{1}{2}\psi \sum_{i=1}^{2}\frac{1}{q_\kappa((1+\sigma)R) - q_\kappa(\rho_i)}\left[s'_\kappa(\rho_i)|du_i|^2 - s_\kappa(\rho_i)\max|T||du_i|^2\right] \\
&= \psi \sum_{i=1}^{2}\left\{ \frac{s'_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} - \frac{b_\kappa(\rho)s_\nu(\rho)}{q_\nu(\rho)} \right. \\
&\quad - \frac{s_\nu(\rho)\rho}{q_\nu(\rho)}\left[\frac{n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| + \left(\frac{n\widetilde{C}_1}{\lambda} + \sqrt{C}\right)\max|T| \right. \\
&\quad \left. + \frac{n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2\right] - \frac{s_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))}\max|T| \\
&\quad \left. - \frac{\rho^2}{q_\nu(\rho)}\cos(\sqrt{\kappa}R)\left(n\widetilde{C}_1 + C\right)\right\}|du_i|^2 \qquad (3.4) \\
&\geq \psi \sum_{i=1}^{2}\left\{ \frac{s'_\kappa(\rho_i)}{2(q_\kappa((1+\sigma)R) - q_\kappa(\rho_i))} - \frac{b_\kappa(\rho)s_\nu(\rho)}{q_\nu(\rho)} \right. \\
&\quad - \left[\frac{2n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| + \left(\frac{2n\widetilde{C}_1}{\lambda} + 2\sqrt{C}\right)\max|T| \right. \\
&\quad \left. + \frac{2n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2\right] - \frac{\kappa R^2}{1 - \cos(\sqrt{\kappa}R)}\cos(\sqrt{\kappa}R)\left(n\widetilde{C}_1 + C\right) \\
&\quad \left. - \frac{\sqrt{\kappa}\sin(\sqrt{\kappa}R)}{2\left[\cos(\sqrt{\kappa}R) - \cos((1+\sigma)\sqrt{\kappa}R)\right]}\max|T|\right\}|du_i|^2,
\end{aligned}
$$

where we have used the fact that $\frac{s_\nu(\rho)\rho}{q_\nu(\rho)}$ is nonincreasing in (0, 2R] and $\frac{\rho^2}{q_\nu(\rho)}$ is increasing in (0, 2R]. Direct computation gives us

$$
\frac{b_\kappa(\rho)s_\nu(\rho)}{q_\nu(\rho)} = \frac{\kappa}{4}(1 + \frac{1}{s'_{\frac{\kappa}{4}}(\rho)})(1 + \frac{\rho}{s_\kappa(\rho)}) =: \alpha(\rho),
$$

and $\alpha(t)$ is increasing in [0, 2R] and $\beta_R(t) = \frac{s_\kappa'(t)}{2(q_\kappa((1+\sigma)R)-q_\kappa(t))}$ is increasing in [0, R]. Hence, we obtain

$$\begin{aligned}
&\frac{s_\kappa'(\rho_i)}{2(q_\kappa((1+\sigma)R)-q_\kappa(\rho_i))} - \frac{b_\kappa(\rho)s_\nu(\rho)}{q_\nu(\rho)} \\
&\quad\ge \frac{1}{2q_\kappa((1+\sigma)R)} - \frac{\kappa}{4}\left(1+\frac{1}{s_{\frac{\kappa}{4}}'(2R)}\right)\left(1+\frac{2R}{s_\kappa(2R)}\right) \\
&\quad= \frac{\kappa}{2[1-\cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1+\cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R+\sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{L}_V(\Theta) \ge{}& \psi\sum_{i=1}^{2}\Bigg\{\frac{\kappa}{2[1-\cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1+\cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R+\sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \\
&- \frac{2n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| \\
&- \left(\frac{2n\widetilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{\kappa}\sin(\sqrt{\kappa}R)}{2\left[\cos(\sqrt{\kappa}R)-\cos((1+\sigma)\sqrt{\kappa}R)\right]}\right)\max|T| \\
&- \frac{2n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2 - \frac{\left(n\widetilde{C}_1+C\right)\kappa R^2\cos(\sqrt{\kappa}R)}{1-\cos(\sqrt{\kappa}R)}\Bigg\}|du_i|^2.
\end{aligned}$$

Clearly, by choosing appropriate σ and R, we obtain

$$\begin{aligned}
&\frac{\kappa}{2[1-\cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1+\cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R+\sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \\
&\quad - \frac{\left(n\widetilde{C}_1+C\right)\kappa R^2\cos(\sqrt{\kappa}R)}{1-\cos(\sqrt{\kappa}R)} > 0.
\end{aligned}$$

Hence, if

$$\begin{aligned}
&\frac{2n\sqrt{n\Lambda}}{\lambda}\max|\nabla T| + \left(\frac{2n\widetilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{\kappa}\sin(\sqrt{\kappa}R)}{2\left[\cos(\sqrt{\kappa}R)-\cos((1+\sigma)\sqrt{\kappa}R)\right]}\right)\max|T| \\
&\quad + \frac{2n^2\Lambda}{\lambda^3\cos(\sqrt{\kappa}R)}\max|T|^2 \\
&\le \frac{\kappa}{2[1-\cos((1+\sigma)\sqrt{\kappa}R)]} - \frac{\kappa[1+\cos(\sqrt{\kappa}R)][2\sqrt{\kappa}R+\sin(2\sqrt{\kappa}R)]}{4\cos(\sqrt{\kappa}R)\sin(2\sqrt{\kappa}R)} \\
&\quad - \frac{\left(n\widetilde{C}_1+C\right)\kappa R^2\cos(\sqrt{\kappa}R)}{1-\cos(\sqrt{\kappa}R)},
\end{aligned} \tag{3.5}$$

then we have

$$\mathcal{L}_V(\Theta) \ge 0.$$

(For $\sqrt{\kappa}R \to 0$, we use the Taylor expansions of sin and cos to obtain positive values on the right-hand side of (3.5).) It is easy to see that there exists a constant C_0 depending only on κ, σ, R and the geometry of N, so that if

$$\max|\nabla T| + \max|T| \le C_0, \tag{3.6}$$

then (3.5) holds true; consequently, $\mathcal{L}_V(\Theta) \geq 0$. Applying the ordinary maximum principle, we have

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

□

Proof of Theorem 2 We consider a parabolic operator of the form

$$\widetilde{\mathcal{L}}_V := \mathcal{L}_V - e^{-\Phi}\partial_t.$$

By using

$$\begin{aligned}(\Delta_V - \partial_t)\,\psi &= (\Delta_V - \partial_t)\,(h \circ U) = \sum_\alpha \nabla^2 h(dU(e_\alpha), dU(e_\alpha)) \\ &\quad + \langle (\nabla h) \circ U, \tau_V(U) - \partial_t U \rangle\end{aligned}$$

and

$$\begin{aligned}(\Delta_V - \partial_t)\,\psi_i &= (\Delta_V - \partial_t)\,(\phi \circ U_i) = \sum_\alpha \nabla^2 \phi(dU_i(e_\alpha), dU_i(e_\alpha)) \\ &\quad + \langle (\nabla \phi) \circ U_i, \tau_V(U_i) - \partial_t U_i \rangle,\end{aligned}$$

as in the proof of Theorem 1, we can conclude that $\widetilde{\mathcal{L}}_V(\Theta) \geq 0$ on M_T. From the parabolic maximum principle, we have

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

4 Existence results

Using the maximum principle obtained in the last section, we shall prove the existence of solutions of the Dirichlet problem for $VT-$harmonic maps.

Proof of Theorem 3 Let us choose normal coordinates $\{y^i\}_{i=1,2,\ldots,n}$ centered at p, then any $VT-$harmonic map $u : M \to B_R(p) \subset N$ can be written as

$$u = (u^1, \ldots, u^n) \in (H^{2,q}(M))^n$$

which satisfies the elliptic system

$$\Delta_M u^i + \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} + T^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} = 0, \quad i = 1, 2, \ldots, n.$$

For simplicity of notation, we write it in a concise form

$$\Delta u + \Gamma(du, du) + du(V) + \mathrm{Tr}_g T(du, du) = 0.$$

Now we consider the initial boundary value problem for the heat flow of $VT-$harmonic maps

$$\begin{cases} \partial_t u = \Delta u + \Gamma(du, du) + du(V) + \mathrm{Tr}_g T(du, du), \\ u - u_0 \in H^{2,q}_0(M, N), \quad u(0) = u_0, \\ u(M \times [0, +\infty)) \subset B_R(p). \end{cases} \tag{4.1}$$

As in the proof of Theorem 3 in [5], by a continuity method that rests on the maximum principle Theorem 2, we can conclude the global existence of a solution $u(x,t)$ of the above flow (4.1). This solution satisfies

$$\|u(\cdot,t)\|_{1+\alpha} \le C(q, M, V, T, N, u_0, R), \quad \forall t \in (0, +\infty)$$

for some $\alpha > 0$. Consequently, by the parabolic regularity theory, we have the uniform estimate

$$\|u\|_{C^{1+\alpha,2+\alpha}(M)} \le C. \tag{4.2}$$

For $u_1(x,t) = u(x,t), u_2(x,t) = u(x, t+\sigma_1), \sigma_1 > 0, \forall (x,t) \in M \times (0, +\infty)$, as in the proof of Theorem 2, the function Θ satisfies

$$\begin{cases} (\Delta - \partial_t)(\frac{\Theta}{\sigma_1^2}) + \langle V - 2\nabla\Phi, \nabla(\frac{\Theta}{\sigma_1^2})\rangle \ge 0, \\ \Theta|_{\partial M} = 0. \end{cases}$$

By the ordinary maximum principle for functions, it follows that (see pp.178–179 in [17])

$$\left(\frac{\Theta}{\sigma_1^2}\right) \le C(t - t_0)^{-k}, \quad \forall t \ge t_0$$

for any positive integer k and some $t_0 > 0$. Letting $\sigma_1 \to 0$, then we obtain $|u_t| \to 0$ as $t \to +\infty$, from which together with (4.2), we have u subconverges to a $VT-$harmonic map u_∞ satisfying (1.7) and $u_\infty(M) \subset B_R(p)$. □

With the Schauder and higher regularity estimates, we can improve Theorem 3 to the following

Theorem 5 *Let $M, N, V, T, B_R(p)$ be as in Theorem 1. Suppose $u_0 \in C^0(M, N)$ with $u_0(M) \subset B_R(p)$. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max |\nabla T| + \max |T| \le C_0, \tag{4.3}$$

then the Dirichlet problem

$$\begin{cases} \tau(u) + du(V) + \mathrm{Tr}_g T(du, du) = 0, \\ u|_{\partial M} = u_0|_{\partial M} \end{cases}$$

admits a unique solution $u \in C^\infty(M, N) \cap C^0(\overline{M}, N)$ such that $u(M) \subset B_R(p)$.

5 Applications

5.1 Weyl harmonic maps (c.f. [14])

Let $(M, [g], {}^W\nabla)$ be a Weyl manifold. According to the definition, there exists a 1-form Θ such that ${}^W g = \Theta \otimes g$ for any $g \in [g]$. Equivalently, ${}^W\nabla$ is defined by

$${}^W\nabla_X Y = \nabla_X Y - \frac{1}{2}\Theta(X)Y - \frac{1}{2}\Theta(Y)X + \frac{1}{2}g(X,Y)\Theta^\sharp, \quad \forall X, Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection and $\Theta^\sharp$ the vector field dual to Θ w.r.t. g. Let $\Gamma^\gamma_{\alpha\beta}$, ${}^W\Gamma^\gamma_{\alpha\beta}$ be the Christoffel symbols corresponding to ∇ and ${}^W\nabla$, respectively.

Let $(N, [\tilde{g}], {}^W\tilde{\nabla})$ be also a Weyl manifold, and correspondingly, we denote by $\widetilde{\Theta}$ the 1-form, and Γ^k_{ij}, ${}^W\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols for the Levi-Civita connection $\tilde{\nabla}$ and Weyl connection ${}^W\tilde{\nabla}$, respectively. Let $u : (M, [g], {}^W\nabla) \to (N, [\tilde{g}], {}^W\tilde{\nabla})$ be the usual smooth map.

Let

$$\begin{aligned} V :=& (\Gamma^\gamma_{\alpha\beta} - {}^W\Gamma^\gamma_{\alpha\beta})g^{\alpha\beta}\frac{\partial}{\partial x^\gamma}, \\ T(\tilde{X}, \tilde{Y}) :=& -\frac{1}{2}\widetilde{\Theta}(\tilde{X})\tilde{Y} - \frac{1}{2}\widetilde{\Theta}(\tilde{Y})\tilde{X} + \frac{1}{2}g(\tilde{X}, \tilde{Y})\widetilde{\Theta}^\sharp, \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(TN). \end{aligned}$$

Then, we have

$$\begin{aligned} \tau(g, {}^W\nabla, {}^W\tilde{\nabla}) =& g^{\alpha\beta}(u^k_{\alpha\beta} + {}^W\tilde{\Gamma}^k_{ij}u^i_\alpha u^j_\beta - {}^W\Gamma^\sigma_{\alpha\beta}u^k_\sigma)\partial_{y^k} \\ =& [g^{\alpha\beta}(u^k_{\alpha\beta} + \Gamma^k_{ij}u^i_\alpha u^j_\beta + T^k_{ij}u^i_\alpha u^j_\beta - \Gamma^\sigma_{\alpha\beta}u^k_\sigma) + V^\sigma u^k_\sigma]\partial_{y^k} \\ =& \tau(g, \nabla, \tilde{\nabla}) + du(V) + \mathrm{Tr}_g(du, du). \end{aligned}$$

Hence,

$$\tau(g, {}^W\nabla, {}^W\tilde{\nabla}) = 0 \quad \text{iff} \quad \tau(u) + du(V) + \mathrm{Tr}_g T(du, du) = 0.$$

Corollary 1 *Let $(M, [g], {}^W\nabla)$ be a compact Wey manifold with nonempty boundary ∂M and $(N, [\tilde{g}], {}^W\tilde{\nabla})$ a complete Weyl manifold with sectional curvature bounded from above by $\kappa \geq 0$. Let $u_0 : M \to N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max|\nabla\widetilde{\Theta}| + \max|\widetilde{\Theta}| \leq C_0,$$

then there exists a unique Weyl harmonic map $u : M \to B_R(p) \subset N$ with $u = u_0$ on ∂M.

5.2 Affine harmonic maps (c.f. [9,10])

Let $(M, g, \tilde{\nabla})$, $(N, h, \tilde{\nabla}')$ both be affine manifolds, where $\tilde{\nabla}$ is a global flat and torsion-free connection on M and $\tilde{\nabla}'$ is a torsion-free connection on N. Then, we have

$$\tau(g, \tilde{\nabla}, \tilde{\nabla}') = g^{\alpha\beta}(u^k_{\alpha\beta} + \tilde{\Gamma}'^k_{ij}u^i_\alpha u^j_\beta)\partial_{y^k},$$

where $\tilde{\Gamma}'^k_{ij}$ are the Christoffel symbols of $\tilde{\nabla}'$.

Regarding (M, g) and (N, h) as Riemannian manifolds, let $\Gamma^\gamma_{\alpha\beta}$ and Γ^i_{jk} be the Christoffel symbols of the Levi-Civita connections ∇ and ∇' of (M, g) and (N, h), respectively. We then have the usual tension field

$$\tau(g, \nabla, \nabla') = g^{\alpha\beta}(u^k_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}u^k_\gamma + \Gamma^k_{ij}u^i_\alpha u^j_\beta)\partial_{y^k}.$$

Let

$$\begin{aligned} V :=& g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_{x^\gamma}, \\ T^k_{ij} :=& \tilde{\Gamma}'^k_{ij} - \Gamma^k_{ij}. \end{aligned}$$

Then, we have

$$\tau(g,\tilde{\nabla},\tilde{\nabla}') = \tau(g,\nabla,\nabla') + du(V) + \mathrm{Tr}_g(du,du).$$

Therefore,

$$\tau(g,\tilde{\nabla},\tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \mathrm{Tr}_g T(du,du) = 0.$$

Corollary 2 *Let $(M, g, \tilde{\nabla})$ be a compact affine manifold with nonempty boundary ∂M and $(N, h, \tilde{\nabla}')$ a complete affine manifold with sectional curvature bounded from above by $\kappa \geq 0$, where $\tilde{\nabla}$ is a global flat and torsion-free connection on M and $\tilde{\nabla}'$ is a torsion-free connection on N. Let $u_0 : M \to N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. Denote $T^k_{ij} := \tilde{\Gamma}'^k_{ij} - \Gamma^k_{ij}$, where $\tilde{\Gamma}'^k_{ij}$ and Γ^i_{jk} stand for the Christoffel symbols of $\tilde{\nabla}'$ and ∇', respectively. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max|\nabla T| + \max|T| \leq C_0,$$

then there exists a unique affine harmonic map $u : M \to B_R(p) \subset N$ with $u = u_0$ on ∂M.

5.3 Hermitian harmonic maps (c.f. [11,15])

Let $(M^m, g, \tilde{\nabla})$, $(N^n, h, \tilde{\nabla}')$ are both Hermitian manifolds, where $\tilde{\nabla}$ and $\tilde{\nabla}'$ are holomorphic torsion-free connections on M and N, respectively. Direct calculation gives us

$$\tau(g,\tilde{\nabla},\tilde{\nabla}') = g^{\alpha\bar{\beta}}\left(\frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma'^i_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial \bar{z}^\beta}\right)\frac{\partial}{\partial w^i} + \overline{g^{\alpha\bar{\beta}}\left(\frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma'^i_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial \bar{z}^\beta}\right)}\frac{\partial}{\partial \bar{w}^i},$$

where Γ'^i_{jk} are the Christoffel symbols of $\tilde{\nabla}'$.

Let J be the almost complex structure, and $\{e_A\} = \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$ a local basis of M. Let ∇, ∇' be the Levi-Civita connections on M and N, respectively, and $\Gamma^i_{jk}$the Christoffel symbols of ∇'. Set

$$V := \tilde{\nabla}_{e_A} e_A - \nabla_{e_A} e_A \quad \text{and} \quad T^i_{jk} := \Gamma'^i_{jk} - \Gamma^i_{jk},$$

then we have

$$\tau(g,\tilde{\nabla},\tilde{\nabla}') = \tau(u) + du(V) + \mathrm{Tr}_g T(du,du).$$

Namely,

$$\tau(g,\tilde{\nabla},\tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \mathrm{Tr}_g T(du,du) = 0.$$

Corollary 3 *Let $(M^m, g, \tilde{\nabla})$ be a compact Hermitian manifold with nonempty boundary ∂M and $(N^n, h, \tilde{\nabla}')$ a complete Hermitian manifold with sectional curvature bounded from above by $\kappa \geq 0$, where $\tilde{\nabla}$ and $\tilde{\nabla}'$ are holomorphic torsion-free connections on M and N, respectively. Let $u_0 : M \to N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$. Denote $T^k_{ij} := \Gamma'^k_{ij} - \Gamma^k_{ij}$, where Γ'^k_{ij} and Γ^i_{jk} stand for the Christoffel symbols of $\tilde{\nabla}'$ and ∇', respectively. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max|\nabla T| + \max|T| \leq C_0,$$

then there exists a unique Hermitian harmonic map $u : M \to B_R(p) \subset N$ with $u = u_0$ on ∂M.

5.4 Magnetic harmonic maps

We now consider a case that, in contrast to the previous ones, does not arise from a structure different from the Riemannian, but from on additional structure on a Riemannian manifold. Let (Σ^m, g) be an m-dimensional compact oriented Riemannian manifold with nonempty boundary, $(N, \widetilde{g})$ a Riemannian manifold of dimension n. Let $u : (\Sigma^m, g) \to (N, \widetilde{g})$ be a map and $Z \in \Gamma(\mathrm{Hom}(\Lambda^m TN, TN)) \cong \Gamma(\Lambda^m T^*N \otimes TN)$.

Consider the following system:

$$\tau(u) + Z(du(e_1) \wedge \cdots \wedge du(e_m)) = 0, \tag{5.1}$$

where $\{e_1, \ldots, e_m\}$ is a positively oriented local orthonormal frame of Σ^m. In string theory, it can be interpreted as the motion equation of an $(m-1)$-brane under an extrinsic magnetic force (c.f. [13]). In [13], the author obtained the global existence of the heat flow in one-dimensional case.

Using a similar method as above, in the two-dimensional case, we can obtain the following

Theorem 6 *Let $u_1, u_2 \in C^0(\Sigma^2, N)$ be two magnetic harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max|\nabla Z| + \max|Z| \le C_0,$$

then the function $\Theta : \Sigma^2 \to \mathbb{R}$ defined by

$$\Theta := \frac{q_{\frac{\kappa}{4}}(\rho)}{(q_\kappa((1+\sigma)R) - q_\kappa(\rho_1))^{\frac{1}{2}} \cdot (q_\kappa((1+\sigma)R) - q_\kappa(\rho_2))^{\frac{1}{2}}} \tag{5.2}$$

satisfies the maximum principle, namely

$$\max_{\Sigma^2} \Theta \le \max_{\partial\Sigma^2} \Theta.$$

Here $\rho := d(u_1, u_2)$, $\rho_i := d(p, u_i)$, $i = 1, 2$.

In particular, if $u_1 = u_2$ on the boundary $\partial\Sigma^2$, then $u_1 \equiv u_2$ on Σ^2.

For the heat flow of magnetic harmonic maps, an analogous result holds. For $T > 0$, we set

$$\Sigma_T^2 := \Sigma^2 \times [0, T]$$

and denote the parabolic boundary of Σ_T^2 by

$$\partial_p \Sigma_T^2 := (\Sigma^2 \times \{0\}) \cup (\partial\Sigma^2 \times [0, T]).$$

For the heat flow of magnetic harmonic maps

$$\partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \tag{5.3}$$

we have

Theorem 7 *Let $u_1, u_2 \in C^0(\Sigma^2, N)$ be two solutions of heat flow Eq. (5.3) for magnetic harmonic maps into a geodesic ball $B_R(p)$. For appropriate σ and R, there exists a constant C_0 depending only on κ, σ, R and the geometry of N, such that if*

$$\max|\nabla Z| + \max|Z| \le C_0,$$

then the function $\Theta : \Sigma_T^2 \to \mathbb{R}$ *defined by* (5.2) *with* Σ^2 *replaced by* Σ_T^2 *satisfies the maximum principle:*

$$\max_{\Sigma_T^2} \Theta \le \max_{\partial_p \Sigma_T^2} \Theta.$$

In particular, if $u_1 = u_2$ *on the boundary* $\partial_p \Sigma_T^2$, *then* $u_1 \equiv u_2$ *on* Σ_T^2.

As an application of the above maximum principle, we obtain the existence of magnetic harmonic maps into a geodesic ball.

Theorem 8 *Let* $\Sigma^2, N, Z, B_R(p)$ *be as in Theorem* 7. *Suppose* $u_0 \in H^{2,q}(\Sigma^2, N)(q > 2)$ *with* $u_0(\Sigma^2) \subset B_R(p)$. *For appropriate* σ *and* R, *there exists a constant* C_0 *depending only on* κ, σ, R *and the geometry of* N, *such that if*

$$\max |\nabla Z| + \max |Z| \le C_0,$$

then the initial boundary value problem

$$\begin{cases} \partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \\ u - u_0 \in H_0^{2,q}(\Sigma^2, N), \quad u(0) = u_0, \quad u(\Sigma^2 \times [0, \infty)) \subset B_R(p), \end{cases} \tag{5.4}$$

admits a unique global solution u *which subconverges to a unique solution* $u \in H^{2,q}(\Sigma^2, N)$ *of the Dirichlet problem*

$$\begin{cases} \tau(u) + Z(du(e_1) \wedge du(e_2)) = 0, \\ u - u_0 \in H_0^{2,q}(\Sigma^2, N), \end{cases} \tag{5.5}$$

such that $u(\Sigma^2) \subset B_R(p)$.

Remark 2 In local coordinates $\{x^\alpha\}$ on M and $\{y^j\}$ on N, respectively, the term $\mathrm{Tr}_g T(du, du)$ can be written as $T^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$, where $T := T^i_{jk} \frac{\partial}{\partial y^i} \otimes dy^j \otimes dy^k$. Correspondingly, the term $Z(du(e_1) \wedge du(e_2))$ in (5.3) could be written as $(Z^i_{jk} - Z^i_{kj}) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$. Using $Z^i_{jk} - Z^i_{kj}$ in place of T^i_{jk} in the proof of the results for VT-harmonic maps, we can conclude the above theorems for magnetic harmonic maps.

6 *VT*-harmonic maps from complete manifolds into geodesic balls

In this section, we shall establish the existence of VT-harmonic maps from complete noncompact manifolds into geodesic balls in complete Riemannian manifolds with sectional curvature bounded above by a positive constant.

Before proving the existence theorem, we first give the following Bochner formula:

Lemma 3 *Let* (M^m, g) *and* $(N^n, \tilde{g})$ *be Riemannian manifolds. Let* $\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2} L_V g$, *where* Ric *is the Ricci curvature of* M *and* L_V *is the Lie derivative. Suppose* u *is a* VT-*harmonic map from* M *to* N, *then*

$$
\begin{aligned}
\frac{1}{2}\Delta_V e(u) =& |\nabla du|^2 + \sum_{\alpha=1}^{m}\langle du(\mathrm{Ric}_V(e_\alpha)), du(e_\alpha)\rangle \\
&- \sum_{\alpha,\beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) \\
&- \sum_{\alpha,\beta=1}^{m} \langle(\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)), du(e_\alpha)\rangle \\
&- \sum_{\alpha,\beta=1}^{m} \langle 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha)\rangle,
\end{aligned}
\tag{6.1}
$$

where $\{e_\alpha\}$ is a local orthonormal frame of M.

Proof By the proof of Proposition 1.3.5 in [18] (c.f. also in [1]), we have

$$
\begin{aligned}
\frac{1}{2}\Delta e(u) =& |\nabla du|^2 + \sum_{\alpha=1}^{m}\langle \nabla_{e_\alpha}\tau(u), du(e_\alpha)\rangle + \sum_{\alpha=1}^{m}\langle du(\mathrm{Ric}(e_\alpha)), du(e_\alpha)\rangle \\
&- \sum_{\alpha,\beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)).
\end{aligned}
$$

Let $\{e_\alpha\}$ be a local orthonormal normal frame of M at the considered point. Since

$$
\begin{aligned}
&\sum_{\alpha=1}^{m}\langle \nabla_{e_\alpha} du(V), du(e_\alpha)\rangle = \sum_{\alpha=1}^{m}\langle (\nabla_{e_\alpha} du)(V) + du(\nabla_{e_\alpha} V), du(e_\alpha)\rangle \\
&= \sum_{\alpha=1}^{m}\langle (\nabla_V du)(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha)\rangle = \sum_{\alpha=1}^{m}\langle \nabla_V du(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha)\rangle \\
&= \frac{1}{2}V|du|^2 + \sum_{\alpha,\beta=1}^{m} \langle \nabla_{e_\alpha} V, e_\beta\rangle\langle du(e_\beta), du(e_\alpha)\rangle \\
&= \frac{1}{2}V|du|^2 + \frac{1}{2}\sum_{\alpha,\beta=1}^{m} (L_V g)(e_\alpha, e_\beta)\langle du(e_\alpha, du(e_\beta))\rangle
\end{aligned}
,
$$

and

$$
\begin{aligned}
\nabla_{e_\alpha}(\mathrm{Tr}_g T(du, du)) &= \nabla_{e_\alpha}(T(du(e_\beta), du(e_\beta))) \\
&= (\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\frac{1}{2}\Delta e(u) =& |\nabla du|^2 - \frac{1}{2}V|du|^2 - \frac{1}{2}\sum_{\alpha,\beta=1}^{m} (L_V g)(e_\alpha, e_\beta)\langle du(e_\alpha, du(e_\beta))\rangle \\
&- \sum_{\alpha,\beta=1}^{m} \langle(\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha)\rangle \\
&+ \sum_{\alpha=1}^{m}\langle du(\mathrm{Ric}(e_\alpha)), du(e_\alpha)\rangle - \sum_{\alpha,\beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)),
\end{aligned}
$$

which implies that (6.1) holds. □

Using the above Bochner formula and the estimate of $\Delta_V r$ in [6] (here r denotes the distance function on M), we establish the gradient estimate for VT-harmonic maps.

Theorem 9 *Let (M^m, g) be a complete noncompact Riemannian manifold with*

$$\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2}L_V g \geq -A,$$

where $A \geq 0$ is a constant, Ric *is the Ricci curvature of M and L_V is the Lie derivative. Let $(N^n, \widetilde{g})$ be a complete Riemannian manifold with sectional curvature bounded from above by a positive constant κ. Let $u : M \to N$ be a VT-harmonic map such that $u(M) \subset B_{\widetilde{R}}(p)$, where $B_{\widetilde{R}}(p)$ is a regular ball in N, i.e., disjoint from the cut locus of p and $\widetilde{R} < \frac{\pi}{2\sqrt{\kappa}}$. Suppose $\|V\|_{L^\infty} < +\infty$, $\|T\|_{L^\infty} < +\infty$, $\|\nabla T\|_{L^\infty} < +\infty$ and*

$$\left(1 + (m+1)^2 - \frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty} + \frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa}\widetilde{R})}\|T\|_{L^\infty} < \frac{\kappa}{\min\{m,n\}}. \tag{6.2}$$

Then, we have

$$|\nabla u| \leq C_6(\sqrt{A} + 1),$$

where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$.

Proof Let r, ρ be the respective distance functions on M and N from some fixed points $\tilde{p} \in M$, $p \in N$. Let $B_a(\tilde{p})$ be a geodesic ball of radius a around $\tilde{p}$. Define $\varphi := \cos(\sqrt{\kappa}\rho)$. Then, the Hessian comparison theorem implies

$$\mathrm{Hess}^N(\varphi) \leq -\kappa(\cos(\sqrt{\kappa}\rho))\widetilde{g}. \tag{6.3}$$

Define $f : B_a(\tilde{p}) \to \mathbb{R}$ by

$$f := (a^2 - r^2)\frac{|\nabla u|}{\varphi \circ u}.$$

Denote $\psi := \frac{|\nabla u|}{\varphi \circ u}$. Clearly, f achieves its maximum at some interior point of $B_a(\tilde{p})$, say q. WLOG, we assume that $\nabla u(q) \neq 0$. Then, from

$$\nabla f(q) = 0,$$
$$\Delta_V f(q) \leq 0,$$

we obtain at q:

$$\frac{\nabla r^2}{a^2 - r^2} = \frac{\nabla \psi}{\psi}, \tag{6.4}$$

$$\frac{\Delta_V \psi}{\psi} - \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2\langle \nabla r^2, \nabla \psi\rangle}{(a^2 - r^2)\psi} \leq 0. \tag{6.5}$$

It follows from the above two inequalities that

$$\frac{\Delta_V \psi}{\psi} - \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2|\nabla r^2|^2}{(a^2 - r^2)^2} \leq 0. \tag{6.6}$$

By formula (2.4) in [6] (see also [16]), we have

$$\Delta_V r^2 = 2r\Delta_V r + 2|\nabla r|^2 \leq 2r(A(r - r_0) + \widetilde{C}_0) + 2, \tag{6.7}$$

where $r_0 > 0$ is a sufficiently small constant and $\widetilde{C}_0 := \max_{\partial B_{r_0}(\tilde{p})} \Delta_V r$.

Let $\{e_\alpha\}$ be a local orthonormal frame field of M and s the rank of u at the point. We shall compute in normal coordinates at the considered point of N. By Newton's inequality, we get

$$\sum_{\alpha,\beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) = \sum_{\alpha\neq\beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta))$$
$$\leq 2\kappa \sum_{1\leq\alpha<\beta\leq s} \left(\sum_i (u^i_\alpha)^2\right)\left(\sum_j (u^j_\beta)^2\right) \leq 2\kappa \cdot \frac{s(s-1)}{2} \cdot \frac{1}{s^2}\left(\sum_{\alpha,i} (u^i_\alpha)^2\right)^2$$
$$= \frac{s-1}{s}\kappa|du|^4 \leq \frac{s_0-1}{s_0}\kappa|du|^4,$$

where we have used the fact that $s_0 := \min\{m, n\} \geq s$ in the third " $\leq$ ".

The Cauchy–Schwarz inequality gives us

$$\left|\langle(\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)), du(e_\alpha)\rangle\right| \leq \|\nabla T\|_{L^\infty}|du|^2 \cdot |du|$$
$$\leq \varepsilon_2 e(u) + \frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty} e(u)^2,$$
$$\left|\langle 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha)\rangle\right| \leq 2\|T\|_{L^\infty}|\nabla du||du| \cdot |du| \leq \varepsilon_3|\nabla du|^2$$
$$+\frac{1}{\varepsilon_3}\|T\|^2_{L^\infty} e(u)^2.$$

The formula (3.12) in [2] (see also [3]) implies that for any $\epsilon > 0$

$$|\nabla du|^2 \geq \frac{1-\epsilon}{m-1}|\tau(u)|^2 + \left(\frac{m}{m-1} - \frac{1}{(m-1)\epsilon}\right)|\nabla\sqrt{e(u)}|^2.$$

Choosing $\epsilon = m$, then we have

$$|\nabla du|^2 \geq -|\tau(u)|^2 + \left(1 + \frac{1}{m}\right)|\nabla\sqrt{e(u)}|^2.$$

By the VT-harmonic map equation (1.1), it is easy to see that

$$|\tau(u)|^2 \leq e(u)\|V\|^2_{L^\infty} + e(u)^2\|T\|^2_{L^\infty}.$$

Hence, from the Bochner formula (6.1), we obtain

$$\frac{1}{2}\Delta_V e(u) \geq \left(-(1-\varepsilon_3)\|V\|^2_{L^\infty} - A - \varepsilon_2\right) e(u) + (1-\varepsilon_3)\left(1 + \frac{1}{m}\right)|\nabla\sqrt{e(u)}|^2$$
$$+ \left(-(1-\varepsilon_3)\|T\|^2_{L^\infty} - \kappa\left(1 - \frac{1}{s_0}\right) - \frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty} - \frac{1}{\varepsilon_3}\|T\|^2_{L^\infty}\right) e(u)^2.$$

Since

$$\frac{1}{2}\Delta_V e(u) = \frac{1}{2}\Delta_V|\nabla u|^2 = |\nabla|\nabla u||^2 + |\nabla u|\Delta_V|\nabla u|,$$

therefore

$$\Delta_V|\nabla u| \geq \left[(1-\varepsilon_3)\left(1 + \frac{1}{m}\right) - 1\right]\frac{|\nabla|\nabla u||^2}{|\nabla u|} + \left(-(1-\varepsilon_3)\|V\|^2_{L^\infty} - A - \varepsilon_2\right)|\nabla u|$$
$$+ \left[-(1-\varepsilon_3)\|T\|^2_{L^\infty} - \kappa\left(1 - \frac{1}{s_0}\right) - \frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty} - \frac{1}{\varepsilon_3}\|T\|^2_{L^\infty}\right]|\nabla u|^3.$$

Choose $\varepsilon_3=\frac{1}{(m+1)^2}$, then

$$\begin{aligned}\Delta_V|\nabla u|\geq&\frac{1}{m+1}\frac{|\nabla|\nabla u||^2}{|\nabla u|}+\left[-\left(1-\frac{1}{(m+1)^2}\right)\|V\|^2_{L^\infty}-A-\varepsilon_2\right]|\nabla u|\\&+\left[-\left(1-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}-\kappa\left(1-\frac{1}{s_0}\right)\right.\\&\left.-\frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty}-(m+1)^2\|T\|^2_{L^\infty}\right]|\nabla u|^3.\end{aligned}\tag{6.8}$$

For simplicity in the following computation, we denote

$$\begin{aligned}&C_1:=\frac{1}{m+1},\quad C_2:=-\left(1-\frac{1}{(m+1)^2}\right)\|V\|^2_{L^\infty}-A-\varepsilon_2,\\&C_3:=-\left(1-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}-\kappa\left(1-\frac{1}{s_0}\right)-\frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty}-(m+1)^2\|T\|^2_{L^\infty}.\end{aligned}$$

By direct calculation, we have

$$\begin{aligned}\nabla\psi=&\frac{\nabla|\nabla u|}{\varphi\circ u}-\frac{|\nabla u|\nabla(\varphi\circ u)}{(\varphi\circ u)^2},\\\Delta_V\psi=&\frac{\Delta_V|\nabla u|}{\varphi\circ u}-\frac{|\nabla u|\Delta_V(\varphi\circ u)}{(\varphi\circ u)^2}-\frac{2}{\varphi\circ u}\langle\nabla\psi,\nabla(\varphi\circ u)\rangle\\\geq&C_1\frac{|\nabla|\nabla u||^2}{(\varphi\circ u)|\nabla u|}+C_2\frac{|\nabla u|}{\varphi\circ u}+C_3\frac{|\nabla u|^3}{\varphi\circ u}-\frac{\psi\Delta_V(\varphi\circ u)}{\varphi\circ u}-\frac{2\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}.\end{aligned}$$

The Cauchy–Schwarz implies that

$$\begin{aligned}&-\frac{2\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}\\&=-(2-2C_1)\frac{\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}-2C_1\frac{\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}\\&=-(2-2C_1)\frac{\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}-2C_1\frac{\langle\nabla|\nabla u|,\nabla(\varphi\circ u)\rangle}{(\varphi\circ u)^2}+2C_1\frac{|\nabla(\varphi\circ u)|^2|\nabla u|}{(\varphi\circ u)^3}\\&\geq-(2-2C_1)\frac{\langle\nabla\psi,\nabla(\varphi\circ u)\rangle}{\varphi\circ u}-C_1\frac{|\nabla|\nabla u||^2}{(\varphi\circ u)|\nabla u|}+C_1\frac{|\nabla(\varphi\circ u)|^2|\nabla u|}{(\varphi\circ u)^3}.\end{aligned}$$

From the above two inequalities and (6.4), we get

$$\frac{\Delta_V\psi}{\psi}\geq C_2+C_3|\nabla u|^2-\frac{\Delta_V(\varphi\circ u)}{\varphi\circ u}+C_1\frac{|\nabla(\varphi\circ u)|^2}{(\varphi\circ u)^2}-(2-2C_1)\frac{\langle\nabla r^2,\nabla(\varphi\circ u)\rangle}{(a^2-r^2)(\varphi\circ u)}.\tag{6.9}$$

Since

$$\left|\frac{\langle\nabla r^2,\nabla(\varphi\circ u)\rangle}{(a^2-r^2)(\varphi\circ u)}\right|\leq\frac{2r|\nabla(\varphi\circ u)|\cdot|\nabla r|}{(a^2-r^2)(\varphi\circ u)}\leq\frac{2r|\nabla(\varphi\circ u)|}{(a^2-r^2)(\varphi\circ u)}\leq\frac{2\sqrt{\kappa}r|\nabla u|}{(a^2-r^2)(\varphi\circ u)}\tag{6.10}$$

and from (6.3),

$$\begin{aligned}\Delta_V(\varphi\circ u) =&g^{\alpha\beta}\nabla^2\varphi(\partial_\alpha u,\partial_\beta u)+\langle(\nabla\varphi)\circ u,\tau(u)+du(V)\rangle\\ =&g^{\alpha\beta}u^j_{x^\alpha}u^k_{x^\beta}\mathrm{Hess}^N(\varphi)\left(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\right)+\langle(\nabla\varphi)\circ u,-\mathrm{Tr}_g T(du,du)\rangle\\ \le&\left(-\kappa\cos(\sqrt{\kappa}\rho)+\sqrt{\kappa}\|T\|_{L^\infty}\right)|\nabla u|^2.\end{aligned}\tag{6.11}$$

Therefore, from (6.6), (6.7), (6.9)–(6.11), we obtain

$$\begin{aligned}&C_2+\left(C_3+\kappa-\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\rho\circ u)}\right)|\nabla u|^2-\frac{4(1-C_1)\sqrt{\kappa}r}{(a^2-r^2)(\varphi\circ u)}|\nabla u|\\ &-\frac{2Ar^2-2Ar_0r+2\widetilde{C}_0r+2}{a^2-r^2}-\frac{8r^2}{(a^2-r^2)^2}\le 0.\end{aligned}\tag{6.12}$$

Since the condition (6.2) tells us

$$\left(1+(m+1)^2-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}+\frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa}\widetilde{R})}\|T\|_{L^\infty}<\frac{\kappa}{s_0},$$

there exists a constant $\varepsilon_0>0$, such that

$$\left(1+(m+1)^2-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}+\frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa}\widetilde{R})}\|T\|_{L^\infty}+\varepsilon_0<\frac{\kappa}{s_0}.$$

Choosing $\varepsilon_2=\frac{\|\nabla T\|^2_{L^\infty}+1}{4\varepsilon_0}$, then we have

$$\begin{aligned}C_3+\kappa-\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\rho\circ u)}&=-\left(1-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}-\kappa\left(1-\frac{1}{s_0}\right)\\ &-\frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty}-(m+1)^2\|T\|^2_{L^\infty}+\kappa-\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\rho\circ u)}\\ &\ge\frac{\kappa}{s_0}-\left(1-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}-\frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty}-(m+1)^2\|T\|^2_{L^\infty}-\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\widetilde{R})}\\ &=\frac{\kappa}{s_0}-\left[\left(1+(m+1)^2-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}+\frac{1}{4\varepsilon_2}\|\nabla T\|^2_{L^\infty}+\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\widetilde{R})}\right]\\ &>\frac{\kappa}{s_0}-\left[\left(1+(m+1)^2-\frac{1}{(m+1)^2}\right)\|T\|^2_{L^\infty}+\varepsilon_0+\frac{\sqrt{\kappa}\|T\|_{L^\infty}}{\cos(\sqrt{\kappa}\widetilde{R})}\right]=:C_4>0.\end{aligned}$$

Therefore, it follows from (6.12) that

$$\begin{aligned}&C_4|\nabla u|^2-\frac{4(1-C_1)\sqrt{\kappa}r}{(a^2-r^2)(\varphi\circ u)}|\nabla u|-\left(\frac{2Ar^2-2Ar_0r+2\widetilde{C}_0r+2}{a^2-r^2}+\frac{8r^2}{(a^2-r^2)^2}-C_2\right)\\ &\le 0.\end{aligned}$$

Note the elementary fact that if $ax^2-bx-c\le 0$ with a,b,c all positive, then

$$x\le\frac{b}{a}+\sqrt{\frac{c}{a}}.$$

Hence, at the point q,

$$|\nabla u|\le\frac{4(1-C_1)\sqrt{\kappa}r}{C_4(a^2-r^2)\cos(\sqrt{\kappa}\widetilde{R})}+\sqrt{\frac{1}{C_4}\left(\frac{2Ar^2-2Ar_0r+2\widetilde{C}_0r+2}{a^2-r^2}+\frac{8r^2}{(a^2-r^2)^2}-C_2\right)}.$$

From this, we can derive the upper bound of f, and it is easy to conclude that at every point of $B_{\frac{a}{2}}(\tilde{p})$, we have

$$\begin{aligned}|\nabla u| \leq & C_5\left(\sqrt{A+\frac{\|\nabla T\|_{L^\infty}^2+1}{4\varepsilon_0}+\left(1-\frac{1}{(m+1)^2}\right)\|V\|_{L^\infty}}+\frac{1}{a}+\frac{1}{\sqrt{a}}\right) \\ \leq & C_6\left(\sqrt{A}+1+\frac{1}{a}+\frac{1}{\sqrt{a}}\right),\end{aligned} \tag{6.13}$$

where $C_6>0$ is a constant depending only on $m, n, \kappa, \widetilde{R}, V, T$.

For any fixed $x \in M$, letting $a \to \infty$ in (6.13), we obtain $|\nabla u| \leq C_6(\sqrt{A}+1)$. □

Proof of Theorem 4 We first choose a constant C_0' depending only on κ, σ, R and the geometry of N, such that if

$$\max|\nabla T|+\max|T| \leq C_0',$$

then both the condition (4.3) in Theorem 5 and the condition (6.2) in Theorem 9 hold. Let $\{\Omega_i\}$ be a compact exhaustion of M. By Theorem 5, we have a sequence of maps $\{u_i\}$ which solve the Dirichlet problem

$$\begin{cases}\tau(u_i)+du_i(V)+\operatorname{Tr}_g T(du_i, du_i)=0, \\ u_i|_{\partial\Omega_i}=u_0|_{\partial\Omega_i}, \\ u_i \text{ homotopic to } u_0 \text{ rel. } \partial\Omega_i,\end{cases}$$

where $u_i \in C^\infty(\Omega_i, N) \cap C^0(\overline{\Omega_i}, N)$ such that $u_i(\Omega_i) \subset B_R(p)$.

For any compact set $K \subset M$, there exists an integer $i_0>0$, such that $K \subset \Omega_i$ for $i>i_0$. Then, by (6.8),

$$\begin{aligned}\Delta_V|\nabla u_i| \geq & \frac{1}{m+1}\frac{|\nabla|\nabla u_i||^2}{|\nabla u_i|}+\left[-\left(1-\frac{1}{(m+1)^2}\right)\|V\|_{L^\infty}^2-\widetilde{A}-\frac{\|\nabla T\|_{L^\infty}^2+1}{4\varepsilon_0}\right]|\nabla u_i| \\ & +\left[-\left(1+(m+1)^2-\frac{1}{(m+1)^2}\right)\|T\|_{L^\infty}^2-\kappa\left(1-\frac{1}{s_0}\right)-\varepsilon_0\right]|\nabla u_i|^3,\end{aligned} \tag{6.14}$$

where $\widetilde{A}$ is a positive constant depending only on the bounds for Ricci curvature of K and $\|V\|_{C^1(K)}$.

Since K is compact, there exist finitely many such geodesic balls $\{B_{a_j}(\tilde{p}_j)\}_{j=1}^{k_0} \subset M$, such that $\cup_{j=1}^{k_0} B_{a_j}(\tilde{p}_j) \supset K$. Hence, for any $q \in K$, there is a geodesic ball, say $B_{a_{j_0}}(\tilde{p}_{j_0})$ $(1 \leq j_0 \leq k_0)$, containing q. Then, by Theorem 9, we can conclude that

$$|\nabla u_i|(q) \leq C_{7j_0}(\sqrt{\widetilde{A}}+1).$$

Hence,

$$\sup_K|\nabla u_i| \leq \max_{1\leq j_0\leq k_0}\left\{C_{7j_0}(\sqrt{\widetilde{A}}+1)\right\}=: C_8,$$

where C_8 is a positive constant independent of i. Then, by the standard elliptic theory, u_i subconverges to a VT-harmonic map $u \in C^\infty(M,N)$ with $u(M) \subset B_R(p)$ and u is homotopic to u_0. □

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