ON PROJECTIVE CLASS GROUPS(¹)

BY

DOCK SANG RIM

1. **Introduction.** For any ring A, the notion of the projective class group $C(A)$ was introduced in [12; 13]. It enjoys functorial properties with respect to ring homomorphisms, and can be described as a certain factor group of the Grothendieck group associated with the category of all finitely generated A-projective modules. This group appeared originally to measure the extent to which A-projective modules are not free. However, this group being closely related with the arithmetic of the ring A, it has its own interest. This note is to study the projective class group of certain algebras over a noetherian domain. Let R be a noetherian domain with quotient field K, and A an R-algebra, finitely generated as R-module. In this case, it is more effective to consider the reduced projective class group $C_0(A)$. This is obtained from the category of all finitely generated A-projective modules P such that $K\otimes_R P$ is $K\otimes_R A$-free (see §3). If A is R-projective as an R-module then $A\subset K\otimes_R A$. If, furthermore, A has no nilpotent ideals, then $K\otimes_R A$ is a simple K-algebra and hence we may consider a maximal order Λ of $K\otimes_R A$ containing A (see §4). Since Λ has a relatively simple structure (for example, if R is a Dedekind domain, then Λ is a hereditary ring), a natural question is to ask for a relation between $C_0(A)$ and $C_0(\Lambda)$. One main objective of this note is to prove Theorem 10, which states that $C_0(A)\to C_0(\Lambda)\to 0$ is exact when $\Lambda/\mathfrak{c}$ is Artinian and the Cartan matrix of $\Lambda/\mathfrak{c}$ is nonsingular, where $\mathfrak{c}$ is the conductor of Λ in A. To obtain this we generalize in §1 a theorem of J.-P. Serre to noncommutative algebras. The study of $C_0(\Lambda)$ is trivially reduced to the case when Λ is a maximal order in a simple algebra. A maximal order in a simple algebra is closely related with that of a division algebra. Namely, if R is a Dedekind domain and Λ is a maximal order of a simple algebra over R, and Γ is a maximal order of a division algebra associated with Λ, then $\mathfrak{M}(\Lambda)\approx\mathfrak{M}(\Gamma)$ as categories, where $\mathfrak{M}(\Lambda)$ (or $\mathfrak{M}(\Gamma)$) denotes the category of all finitely generated left Λ-modules (or Γ-modules). This categorial isomorphism preserves projective modules in both directions but does not preserve free modules in general. This causes a difficulty in deducing a relation on the projective class groups. This is studied in §3 where we prove that $C_0(\Lambda)\approx C(\Gamma)$. Another interesting result is Theorem 13 concerning the finiteness of the projective class group. Most of our theorems are valid for integral group algebra of finite groups and a result of R. G. Swan is deduced as a corollary.

Received by the editors April 21, 1960.

(¹) This work has been supported by the Office of Naval Research under Contract Nonr. 266(57).

For the sake of simplicity, we make some conventions on terminology and notation. Throughout this note, unless otherwise stated, we mean: (1) ring $=$ring with unit element; (2) module$=$finitely generated unitary left module; (3) $R=$commutative noetherian domain and $K=$quotient field of R; (4) R-algebra$=R$-algebra with unit element, which is finitely generated as R-module; and (5) $\otimes=\otimes_R$. The reader should keep in his mind these agreements throughout this note. (For example, when we say "all Λ-modules," then we mean "all finitely generated left Λ-modules".)

The author wishes to thank J.-P. Serre for his private letter [14], from which the present investigation originates. Thanks also goes to H. Bass for useful conversations.

2. **Modulo the conductor.** Let $A\subset\Lambda$ be R-algebras. Then Λ, being finitely generated as an R-module, is so as a module over A. Now $\mathfrak{q}=\{r\mid r\in R, r\Lambda\subset A\}$ is an ideal in R and, if we put $\mathfrak{c}=\mathfrak{q}\Lambda=\Lambda\mathfrak{q}$, then $\mathfrak{c}$ is a two-sided ideal in A as well as in Λ. We shall call $\mathfrak{c}$ the conductor of Λ in A in analogy with number theory. For the sake of simplicity, we shall use throughout this section the notations: $R/\mathfrak{q}=\bar{R}$, $A/\mathfrak{c}=\bar{A}$, $\Lambda/\mathfrak{c}=\bar{\Lambda}$. Then $\bar{A}\subset\bar{\Lambda}$ and they are $\bar{R}$-algebras. Furthermore, for any A-module (or Λ-module) M, we put $\bar{M}=\bar{A}\otimes_A M$ (or $\bar{M}=\bar{\Lambda}\otimes_\Lambda M$).

Let P be an A-module and P' a Λ-module such that $P\subset P'$. Adopting J.-P. Serre's terminology we say that P is an *A-form* of P' if (i) P is A-projective and P' is Λ-projective and (ii) the map $\Lambda\otimes_A P\rightarrow P'$ induced by $P\subset P'$ is an isomorphism. If P is A-projective, then $P'=\Lambda\otimes_A P$ is Λ-projective and thus, identifying P with its image under the canonical imbedding $P\rightarrow\Lambda\otimes_A P=P'$, we see that P is an A-form of P'. Conversely, given a Λ-projective module P', what are the sub-A-modules P of P' which are A-forms of P'? A milder question asks whether every Λ-projective module comes from an A-projective module by tensoring with Λ. Theorem 1 below reduces these questions modulo the conductor $\mathfrak{c}$. This theorem was proved by J.-P. Serre when A, Λ are both commutative noetherian rings [13; 14]. Our theorem is a generalization of it to noncommutative algebras.

THEOREM 1. *With the notations as above, let P' be a Λ-projective module, and M an A-submodule contained in P'. Then the following statements are equivalent*:

(i) *M is an A-form of P'.*

(ii) *$M\supset\mathfrak{c}P'$ and $\bar{M}$ is an $\bar{A}$-form of $\bar{P}'$.*

Thus, if $\phi: P'\rightarrow\bar{P}'$ is the natural map, ϕ^{-1} puts the $\bar{A}$-forms of $\bar{P}'$ in bijective correspondence with the A-forms of P'. In particular P' has an A-form if and only if $\bar{P}'$ has an $\bar{A}$-form.

We shall reduce the proof of the above theorem to the case when R is a complete local ring. This procedure is accomplished by Proposition 3 below. The following lemma is known [2] and will be used in the proof of Proposition 3. We quote it below without proof.

LEMMA 2. *Let R be a noetherian ring and Γ an R-algebra which is left noetherian. Let S be an R-algebra (not necessarily finitely generated) which is a flat R-module. If C is a finitely generated Γ-module, then* $S\otimes_R \operatorname{Ext}^n_\Gamma(C, D) \approx \operatorname{Ext}^n_S \otimes_R \Gamma(S\otimes_R C, S\otimes_R D)$ *for any Γ-module D.*

Now for each maximal ideal $\mathfrak{p}$ of R, let $\hat{R}_\mathfrak{p}=$ completion of R with respect to the maximal ideal $\mathfrak{p}$ of R. For any R-module M, we write $\hat{M}_\mathfrak{p}=\hat{R}_\mathfrak{p}\otimes_R M$.

PROPOSITION 3. *Let P' be a Λ-module and P an A-module with $P\subset P'$. Then P is an A-form of P' if and only if $\hat{P}_\mathfrak{p}$ is an $\hat{A}_\mathfrak{p}$-form of $\hat{P}'_\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of R.*

Proof. $P\subset P'$ induces $\Lambda\otimes_A P\rightarrow P'$. Taking its kernel X and cokernel Y, $0\rightarrow X\rightarrow\Lambda\otimes_A P\rightarrow P'\rightarrow Y\rightarrow 0$ is exact. We observe that every module in this exact sequence is finitely generated as an R-module. Regarding this as an exact sequence of R-modules, we get an exact sequence $0\rightarrow\hat{X}_\mathfrak{p}\rightarrow\Lambda_\mathfrak{p}\otimes_{\hat{A}_\mathfrak{p}}\hat{P}_\mathfrak{p}\rightarrow\hat{P}'_\mathfrak{p}\rightarrow\hat{Y}_\mathfrak{p}\rightarrow 0$ for each maximal ideal $\mathfrak{p}$ of R. Now for any $R_\mathfrak{p}$-module M, $\hat{M}_\mathfrak{p}=0$ if and only if $M=0$. Therefore $\Lambda\otimes_A P\rightarrow P'$ is an isomorphism $\Leftrightarrow X=0=Y\Leftrightarrow X_\mathfrak{p}=0=Y_\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of $R\Leftrightarrow\hat{X}_\mathfrak{p}=0=\hat{Y}_\mathfrak{p}$ for all maximal ideals $\mathfrak{p}$ of $R\Leftrightarrow\hat{\Lambda}_\mathfrak{p}\otimes^n_{\hat{A}_\mathfrak{p}}\hat{P}_\mathfrak{p}\rightarrow\hat{P}'_\mathfrak{p}$ is an isomorphism for all maximal ideals $\mathfrak{p}$ of R. Furthermore, A being noetherian and $\hat{R}_\mathfrak{p}$ being R-flat, we get from Lemma 2 that $\operatorname{Ext}^n_{\hat{A}_\mathfrak{p}}(\hat{C}_\mathfrak{p}, \hat{D}_\mathfrak{p})\approx\hat{R}_\mathfrak{p}\otimes_R \operatorname{Ext}^n_A(C, D)$ for all finitely generated A-modules C and every A-module D. If C and D are both finitely generated as A-modules, then $\operatorname{Ext}^n_A(C, D)$ is finitely generated as an R-module for all n. Therefore, if D is a finitely generated A-module, then $\operatorname{Ext}^n_A(P, D)=0$ for all $n>0\Leftrightarrow R_\mathfrak{p}\otimes_R \operatorname{Ext}^n_A(P, D)\approx\operatorname{Ext}^n_{A_\mathfrak{p}}(P_\mathfrak{p}, D_\mathfrak{p})=0$ for all $n>0$ and for all maximal ideals $\mathfrak{p}$ of $R\Leftrightarrow\hat{R}_\mathfrak{p}\otimes_R \operatorname{Ext}^n_A(P, D)\approx\operatorname{Ext}^n_{\hat{A}_\mathfrak{p}}(\hat{P}_\mathfrak{p}, \hat{D}_\mathfrak{p})=0$ for all $n>0$ and for all maximal ideals $\mathfrak{p}$ of R. Therefore using the fact that A is noetherian, we see immediately that P is A-projective if and only if $\hat{P}_\mathfrak{p}$ is $\hat{A}_\mathfrak{p}$-projective for all maximal ideals $\mathfrak{p}$ of R. The same thing is true for Λ-modules P'.

Proof of Theorem 1. (i)$\Rightarrow$(ii): $\overline{M}=\overline{A}\otimes_A M$ is $\overline{A}$-projective and $\Lambda\otimes_A M=P'$ entails that $\Lambda M=P'$ so that $M\supset\mathfrak{c}M=\mathfrak{c}\Lambda M=\mathfrak{c}P'$. Furthermore, $\overline{\Lambda}\otimes_{\overline{A}}\overline{M}=\overline{\Lambda}\otimes_{\overline{A}}(\overline{A}\otimes_A M)=\overline{\Lambda}\otimes_A M=(\overline{\Lambda}\otimes_\Lambda\Lambda)\otimes_A M=\overline{\Lambda}\otimes_\Lambda P'=\overline{P}'$.

(ii)$\Leftrightarrow$(i): By Proposition 3, we may assume that R is a complete local ring with the maximal ideal $\mathfrak{m}$. Therefore A is a semi-perfect ring [3]. If $\mathfrak{q}\not\subset\mathfrak{m}$, then the theorem is trivial since $\mathfrak{q}=R$ entails that $A=\Lambda$. Therefore we may assume that $\mathfrak{q}\subset\mathfrak{m}$. In this case, we contend that $\mathfrak{c}$ is contained in the J-radical of Λ as well as of A. For, since A and Λ are finitely generated as R-modules, it is clear that $\mathfrak{c}=\mathfrak{q}\Lambda$ is contained in the J-radical of Λ and $\mathfrak{q}A$ is contained in the J-radical of A. However, $\mathfrak{c}^2=\mathfrak{q}^2\Lambda\subset\mathfrak{q}A$ and hence $\mathfrak{c}$ is also contained in the J-radical of A. Now since $\overline{M}$ is $\overline{A}$-projective, there exists an $\overline{A}$-projective module P_1^* such that $\overline{M}\oplus P_1^*$ is $\overline{A}$-free. Since $\mathfrak{c}$ is contained in the J-radical of a semi-perfect ring A, there exists an A-projective cover of P_1^*, i.e., there exists an A-projective module P_1 such that $\overline{P}_1=P_1^*$ (see [3; 9]). Put $\Lambda\otimes_A P_1$

$=P_1'$. Then P_1 is an A-form of P_1' and hence, if we set $X=M\oplus P$ and $Y=P'\oplus P_1'$, then $X\subset Y$ and Y is Λ-projective and the condition (ii) is valid for X and Y. $\overline{X}$ is $\overline{A}$-free, so let $x_1, x_2, \cdots, x_n$ be the elements of X such that their canonical images in $\overline{X}$ form an $\overline{A}$-free basis of $\overline{X}$. Then they are $\overline{\Lambda}$-free basis of $\overline{Y}$ since $\overline{\Lambda}\otimes_{\overline{A}}\overline{X}=\overline{Y}$. Therefore Y is Λ-free with the basis $x_1, x_2, \cdots, x_n$ since $Y\to\overline{Y}$ is a Λ-projective cover. Therefore $x_1, x_2, \cdots, x_n$ are, a fortiori, linearly independent over A. However, they generate X by Nakayama's lemma, and hence X is A-free and $\Lambda\otimes_A X=Y$. Therefore, we see that the direct summand M of X is A-projective and $\Lambda\otimes_A M=P'$.

Corollary 4. *Let A, Λ, $\mathfrak{c}$ be as above. If P' is a Λ-projective module such that $\overline{P}'$ is $\overline{\Lambda}$-free, then it admits an A-form, i.e., there exists an A-projective module P such that $\Lambda\otimes_A P=P'$.*

Proof. $\overline{P}'$, being $\overline{\Lambda}$-free, clearly admits an $\overline{A}$-form, so P' has an A-form by Theorem 1.

Corollary 5. *Let A, Λ, $\mathfrak{c}$ be as above, and P' a Λ-projective module such that $\overline{P}'$ is $\overline{\Lambda}$-free (of rank n). Then the number of isomorphism classes of A-forms M of P' such that $\overline{M}$ is $\overline{A}$-free is the number of the double cosets of $GL(n, \overline{\Lambda})$ modulo $GL(n, \overline{A})$ and the subgroup T of the automorphisms of $\overline{P}'$ induced by the automorphisms of P'.*

Proof. It follows from Theorem I that A-forms M of P' such that $\overline{M}$ is $\overline{A}$-free, are in bijective correspondence with the points of $GL(n, \overline{\Lambda})/GL(n, \overline{A})$. Now let P_1, P_2 be A-forms of P' such that $\theta\colon P_1\approx P_2$. Then θ extends uniquely to an automorphism θ of P' such that $\bar{\theta}\colon \overline{P}_1\approx\overline{P}_2$. Conversely if there exists an automorphism θ of P' such that $\bar{\theta}\colon \overline{P}_1\approx\overline{P}_2$, then $\phi\theta(P_1)=\overline{P}_2$ and hence $\theta(P_1)=P_2$, i.e., $P_1\approx P_2$. Therefore the isomorphism classes of A-forms M of P' such that $\overline{M}$ is $\overline{A}$-free is in bijective correspondence with the double cosets of $GL(n, \overline{\Lambda})$ modulo $GL(n, \overline{A})$ and the subgroup T induced by Aut (P').

Corollary 6. *Let R be either a ring of integers in a number field, or function field in one-variable over a finite field. Let A, Λ, $\mathfrak{c}$ be as above and assume that $\mathfrak{c}\neq 0$. Then, given a Λ-projective module P' the number of A-forms of P' is finite.*

Proof. Let $\mathfrak{c}=\mathfrak{q}\Lambda$. Then by hypothesis $\mathfrak{q}\neq 0$. By virtue of the hypothesis imposed on R, $R/\mathfrak{q}$ has only a finite number of elements. Consequently, the $R/\mathfrak{q}$-algebra $\overline{\Lambda}$ which is finitely generated as $R/\mathfrak{q}$-module has only a finite number of elements. Therefore the assertion follows from Theorem 1.

We know now that every Λ-projective module P' such that $\overline{P}'$ is $\overline{\Lambda}$-free comes from an A-projective module. In view of this fact, we may ask when a Λ-projective module P' has the property that $\overline{P}'$ is $\overline{\Lambda}$-free. An answer to this question is provided by the following theorem, which is a trivial consequence of Theorem 2 in [4]. We simply quote it below for the benefit of the reader.

THEOREM 7. *Let R be a commutative ring with K its full ring of quotients, $\mathfrak{q}$ an ideal of R such that $R/\mathfrak{q}$ is Artinian. Let Λ be an R-projective R-algebra and assume that the Cartan matrix of $\Lambda/\mathfrak{q}\Lambda$ is nonsingular. Then, given any two Λ-projective modules P and P', $K\otimes_R P\approx K\otimes_R P'$ implies $P/\mathfrak{q}P\approx P'/\mathfrak{q}P'$. In particular, if $K\otimes P$ is $K\otimes\Lambda$-free, then $P/\mathfrak{q}P$ is $\Lambda/\mathfrak{q}\Lambda$-free.*

3. **Projective class group and categorical isomorphism.** First let us recall the definition of Grothendieck groups [13; 16]. Let $\mathcal{C}$ be an additive category. The Grothendieck group $G(\mathcal{C})$ associated with the category $\mathcal{C}$ is an abelian group which is described by giving generators and relations. The generators are symbols $[A]$, one for each object A of $\mathcal{C}$, and relations are $[A]=[A']+[A'']$ for each exact sequence $0\to A'\to A\to A''\to 0$ in the category $\mathcal{C}$. (In what follows, if $A\in\mathcal{C}$, then $[A]$ will denote the element in $G(\mathcal{C})$ represented by the object A.) If $\mathcal{C}_1$, $\mathcal{C}_2$ are additive categories and F: $\mathcal{C}_1\to\mathcal{C}_2$ is an exact functor, then F naturally induces a homomorphism $G(F): G(\mathcal{C}_1)\to G(\mathcal{C}_2)$, since relations are preserved. For any R-algebra Λ (where R is a noetherian domain with K its quotient field), we shall denote by $\mathcal{P}(\Lambda)$ the category of all (finitely generated) Λ-projective modules and by $\mathcal{P}_0(\Lambda)$ the subcategory of $\mathcal{P}(\Lambda)$ consisting of all Λ-projective modules P such that $K\otimes_R P$ is $K\otimes_R\Lambda$-free. In these categories, maps are of course Λ-homomorphisms. The associated Grothendieck groups will be denoted by $P(\Lambda)$ and $P_0(\Lambda)$ respectively. The factor groups $C(\Lambda)=P(\Lambda)/\{[\Lambda]\}$, $C_0(\Lambda)=P_0(\Lambda)/\{[\Lambda]\}_0$ are called the projective class group of Λ, and the reduced projective class group of Λ respectively, where $\{[\Lambda]\}$ or $\{[\Lambda]\}_0$ indicates the cyclic subgroup generated by $[\Lambda]$ in $P(\Lambda)$ or $P_0(\Lambda)$ respectively. The reader will find immediately that this definition coincides with the original one given in [12; 13]. If P is in $\mathcal{P}(\Lambda)$ or $\mathcal{P}_0(\Lambda)$ then we shall indicate by $[[P]]$ the element in $C(\Lambda)$ or $C_0(\Lambda)$ represented by P.

Let Λ be a ring and E a right Λ-module. Then E gives rise to the endomorphism ring $\Gamma=\operatorname{Hom}_\Lambda(E, E)$ and E becomes a right Λ, left Γ, bimodule. The dual $E^*=\operatorname{Hom}_\Lambda(E,\Lambda)$ is in an obvious way a left Λ, right Γ, bimodule. We call Λ-module E *regular* if (i) E is (finitely generated) Λ-projective and (ii) $E\otimes E^*\to\Lambda$ given by $(x, f)\to f(x)$ is an epimorphism. If E is a regular module, then there exists a categorical isomorphism between the category of (left) Λ-modules and the category of (left) Γ-modules. These facts were fully explored by Auslander-Goldman [2], Curtis [8] and Morita [11] and also by S. Chase. We state below the known results in the form convenient to our purpose.

THEOREM 8. *Let Λ be a ring, E a regular right Λ-module, and set $E^*=\operatorname{Hom}_\Lambda(E,\Lambda)$ and $\Gamma=\operatorname{Hom}_\Lambda(E, E)$. $\mathfrak{M}(\Lambda)$ and $\mathfrak{M}(\Gamma)$ denote the category of (left) Λ-modules and (left) Γ-modules respectively. Then*

(a) *E and E^* are both (finitely generated) Λ-projective as well as Γ-projective;*

(b) *$\Lambda\approx\operatorname{Hom}_\Gamma(E, E)$ as rings (of course $\Gamma=\operatorname{Hom}_\Lambda(E, E)$ by definition);*

(c) $E^*\otimes_\Gamma E\approx\Lambda$ *as two-sided* Λ-*modules and* $E\otimes_\Lambda E^*\approx\Gamma$ *as two-sided* Γ-*modules*;

(d) $S\colon \mathfrak{M}(\Lambda)\rightarrow\mathfrak{M}(\Gamma)$ *given by* $S(M)=E\otimes_\Lambda M$ *and* $T\colon\mathfrak{M}(\Gamma)\rightarrow\mathfrak{M}(\Lambda)$ *given by* $T(N)=E^*\otimes_\Gamma N$ *are categorical isomorphisms* $\mathfrak{M}(\Lambda)\approx\mathfrak{M}(\Gamma)$. *In fact,* $TS(M)\approx M$ *for all* $M\in\mathfrak{M}(\Lambda)$ *and* $ST(N)\approx N$ *for all* $N\in\mathfrak{M}(\Gamma)$.

Now E being Γ-projective, if M is Λ-projective then $S(M)=E\otimes_\Lambda M$ is Γ-projective. Likewise, if N is Γ-projective then $T(N)=E^*\otimes_\Gamma N$ is Λ-projective. Therefore $S\colon\mathcal{P}(\Lambda)\rightarrow\mathcal{P}(\Gamma)$ and $T\colon\mathcal{P}(\Gamma)\rightarrow\mathcal{P}(\Lambda)$ provides a categorical isomorphism $\mathcal{P}(\Lambda)\approx\mathcal{P}(\Gamma)$. If E is Γ-free, then the functor S will send Λ-free modules to Γ-free modules. Likewise, if E^* is Λ-free, then T will send Γ-free modules to Λ-free modules. Thus, if E is Γ-free and E^* is Λ-free, then we could conclude $C(\Lambda)\approx C(\Gamma)$. However, E and E^* need not be free in general. What conclusion can we draw if E and E^* are not known to be free? This is the question we are concerned with in this section (for a particular case).

Now let Λ be an R-algebra, E a regular right Λ-module and let E^* and Γ be as above. Then Γ is also an R-algebra. If $P\in\mathcal{P}_0(\Lambda)$, then $K\otimes P$ is $K\otimes\Lambda$-free by definition, and hence the rank of $K\otimes P$ over $K\otimes\Lambda$ is defined (recall that $\otimes=\otimes_R$). We put $\rho(P)=$ rank of $K\otimes P$ over $K\otimes\Lambda$. This function ρ, being additive, induces $\rho\colon P_0(\Lambda)\rightarrow Z$ which is clearly an epimorphism. With these hypotheses and notations, we have:

THEOREM 9. $C_0(\Lambda)\approx\operatorname{Ker}(\rho\colon P_0(\Lambda)\rightarrow Z)$. *If* $K\otimes E$ *is a simple* $K\otimes\Lambda$-*module, then* $K\otimes\Gamma$ *is a division algebra and* $C(\Gamma)\approx\operatorname{Ker}(\rho\colon P_0(\Lambda)\rightarrow Z)\approx C_0(\Lambda)$.

Proof. The first statement follows trivially from the splitting exact sequence $0\rightarrow\{[\Lambda]\}_0\rightarrow P_0(\Lambda)\rightarrow C_0(\Lambda)\rightarrow 0$. We now prove the second statement. Firstly, if $K\otimes E$ is a simple $K\otimes\Lambda$-module, then $K\otimes\Gamma=\operatorname{Hom}_{K\otimes\Lambda}(K\otimes E, K\otimes E)$ is a division algebra by Schur's lemma. Furthermore, $K\otimes\Lambda\approx K\otimes\operatorname{Hom}_\Gamma(E,E)$ $=\operatorname{Hom}_{K\otimes\Gamma}(K\otimes E,\ K\otimes E)$ is a total matrix algebra over the division algebra $K\otimes\Gamma$. $K\otimes E$ being a simple $K\otimes\Lambda$-module, so is $K\otimes E^*$ and hence $K\otimes\Lambda$ $\approx r(K\otimes E^*)$ for some integer r as left $K\otimes\Lambda$-modules, where $r(K\otimes E^*)$ $=K\otimes E^*+\cdots+K\otimes E^*$ (r times). Now the exact functor $S\colon\mathcal{P}_0(\Lambda)\rightarrow\mathcal{P}(\Gamma)$ induces $\overline{S}\colon P_0(\Lambda)\rightarrow C(\Gamma)$ which is given by $[P]\rightarrow[[S(P)]]=[[E\otimes_\Lambda P]]$. $\overline{S}$ is an epimorphism. For, let Q be a Γ-projective module. Consider the Λ-projective module $T(Q)=E^*\otimes_\Gamma Q$. $T(Q)$ need not be in $\mathcal{P}_0(\Lambda)$ in general i.e., $K\otimes T(Q)$ need not be $K\otimes\Lambda$-free. However, since $K\otimes\Lambda\approx r(K\otimes E^*)$, it is clear that $T(Q)\oplus mE^*$ will lie in $\mathcal{P}_0(\Lambda)$ for some integer m. Then $\overline{S}(T(Q)\oplus mE^*)=[[ST(Q)\oplus mS(E^*)]]=[[Q\oplus m(E\otimes_\Lambda E^*)]]=[[Q\oplus m\Gamma]]$ $=[[Q]]$. This proves that $\overline{S}$ is an epimorphism. Let us determine the kernel of $\overline{S}$. Suppose that $\overline{S}(P_1)=\overline{S}(P_2)$, i.e., $[[S(P_1)]]=[[S(P_2)]]$, i.e., $S(P_1)$ $\oplus m_1\Gamma\approx S(P_2)\oplus m_2\Gamma$. Then $TS(P_1)\oplus m_1T(\Gamma)\approx TS(P_2)\oplus m_2T(\Gamma)$, i.e., P_1 $\oplus m_1E^*\approx P_2\oplus m_2E^*$. Tensoring with K, we find that $m_1\equiv m_2 \pmod r$ since $P_i\in\mathcal{P}_0(\Lambda)$. Conversely, if $mE^*\in\mathcal{P}_0(\Lambda)$, i.e., if $m\equiv 0 \pmod r$, then $\overline{S}(mE^*)$ $=[[mS(E^*)]]=[[m\Gamma]]=0$. Therefore $\operatorname{Ker}\overline{S}=$ cyclic subgroup of $P_0(\Lambda)$

generated by $[rE^*]$. Therefore $0 \to Z \xrightarrow{i} P_0(\Lambda) \xrightarrow{s} C(\Gamma) \to 0$ is exact where $i(m) = m[rE^*]$. Then it is clear that $pi =$ identity and hence the above exact sequence splits so that we have $C(\Gamma) \approx \mathrm{Ker}(p\colon P_0(\Lambda) \to Z)$.

4. **Applications.** Let A be an R-projective R-algebra without nilpotent ideals. Then $\Sigma = K \otimes A$ is a semi-simple K-algebra. An R-algebra Λ contained in Σ is called an *order* of Σ over R if $K \otimes \Lambda = \Sigma$. A *maximal order* is an order which is not properly contained in another order. It is well known that every order is imbedded in a maximal order [1; 2]. (The reader should observe that in the above remark, we used our convention that algebra = algebra which is finitely generated as a module.)

THEOREM 10. *Let A be an R-projective R-algebra without nilpotent ideals, Λ a maximal order of $\Sigma = K \otimes A$ containing A, and let $\mathfrak{c}$ be the conductor of Λ in A. If $\Lambda/\mathfrak{c}$ is Artinian and its Cartan matrix is nonsingular, then $C_0(A) \to C_0(\Lambda) \to 0$ is exact.*

Proof. Let $\mathcal{P}_0(A)$ (or $\mathcal{P}_0(\Lambda)$) be the category of all A-projective (or Λ-projective) modules P such that $K \otimes P$ is Σ-free. Then $\mathcal{P}_0(A) \to \mathcal{P}_0(\Lambda)$ given by $P \to \Lambda \otimes_A P$ is an exact functor and hence induces a homomorphism $j\colon P_0(A) \to P_0(\Lambda)$. Since $j[A] = [\Lambda]$, it induces $C_0(A) \to C_0(\Lambda)$. To see that this is an epimorphism it suffices to see that $j\colon P_0(A) \to P_0(\Lambda)$ is an epimorphism. However this follows immediately from Corollary 4, Theorem 7 and the hypothesis.

COROLLARY 11. *Let R be a Dedekind domain with quotient field K, and A a torsion-free R-algebra without nilpotent ideals. Let Λ be a maximal order of $K \otimes A$ containing A. Then $C_0(A) \to C_0(\Lambda) \to 0$ is exact. If $K \otimes \Lambda$ splits completely, i.e., is a ring direct sum of total matrix algebras, then $C_0(\Lambda) \approx C(\Lambda) \approx$ ideal class group of the center of Λ.*

Proof. As for the first statement, it suffices to check that if $\mathfrak{c}$ is the conductor of Λ in A, then $\Lambda/\mathfrak{c}$ is Artinian and its Cartan matrix is nonsingular. Let $\mathfrak{q} = \{r \in R \mid r\Lambda \subset A\}$. Then, by definition, $\mathfrak{c} = \mathfrak{q}\Lambda$. Since Λ is finitely generated as an R-module and $K \otimes \Lambda = K \otimes A$, we see that $\mathfrak{q} \neq 0$, and hence $\Lambda/\mathfrak{c} = R/\mathfrak{q} \otimes_R \Lambda$ is Artinian. Now Λ being a maximal order in a semi-simple algebra $K \otimes \Lambda$, it is a ring direct sum of maximal orders in simple algebras, i.e., $\Lambda = \Lambda_1 + \cdots + \Lambda_s$ and Λ_i is a maximal order in the simple algebra $K \otimes \Lambda_i$. Furthermore, we know that $\Lambda_i/\mathfrak{p}\Lambda_i$ is a primary algebra for any nonzero prime ideal $\mathfrak{p}$ of R [5]. Thus, for any nonzero ideal $\mathfrak{q}$ of R, $\Lambda_i/\mathfrak{q}\Lambda_i$ is a ring direct sum of primary algebras and consequently the Cartan matrix of $\Lambda/\mathfrak{q}\Lambda$ is nonsingular. This proves the first statement. As for the second statement, we may assume that Λ is a maximal order in a simple algebra since $C_0(\Lambda) = \coprod_{i=1}^{s} C_0(\Lambda_i)$. If Λ is a maximal order in a simple algebra, it is known [2] that Λ admits a regular right Λ-module E such that $K \otimes E$ is $K \otimes \Lambda$-simple. Therefore by Theorem 9, we have $C_0(\Lambda) \approx C(\Gamma)$ where $\Gamma = \mathrm{Hom}_\Lambda(E, E)$ and

$K\otimes\Gamma$ is a division algebra associated with the simple algebra $K\otimes\Lambda$. Furthermore, Γ is a maximal order in $K\otimes\Gamma$ [2]. If $K\otimes\Lambda$ is a total matrix algebra over a field, then $K\otimes\Gamma$ is the center of $K\otimes\Lambda$ and hence Γ is the center of Λ. R being a Dedekind domain, Γ is also a Dedekind domain and hence $C(\Gamma)$ $\approx$ideal class group of Γ. Therefore $C_0(\Lambda)\approx$ideal class group of the center of Λ. This completes the proof.

COROLLARY 12. *Let π be a finite group of order n and R the ring of integers in a number field K. Consider the group algebras $R\pi$ and $K\pi$. Let Λ be a maximal order of $K\pi$ containing $R\pi$. Then $C(R\pi)\rightarrow C_0(\Lambda)\rightarrow 0$ is exact. If K contains a primitive nth root of unity, then $C(R\pi)\rightarrow$(ideal class group of the center of $K\pi$) $\rightarrow 0$ is exact.*

Proof. From a theorem of Swan [15; 16], we know that every R-projective module P has the property that $K\otimes P$ is $K\pi$-free. Therefore $C(R\pi)=C_0(R\pi)$ and hence the first statement follows from Corollary 11. If K contains a primitive nth root of unity, then it follows from a theorem of Brauer [6] that $K\pi$ splits completely, i.e., a ring direct sum of total matrix algebra. Therefore $C_0(\Lambda)=$ideal class group of the center of Λ. However, the center of $K\pi$ is a direct sum of number fields and hence the center of Λ is the unique maximal order of the center of $K\pi$, i.e., the ring of integers in the center of $K\pi$. Therefore by the definition of the ideal class group in a number field, we get the result.

REMARK. The above Corollary 12 was also obtained by R. G. Swan. His method depends on the nature of the group ring and uses the author's characterization of projective modules over $R\pi$.

As another application of Theorem 1, we give below a theorem concerning the finiteness of class numbers. Namely:

THEOREM 13. *Let R be the ring of integers in a number field or a function field in one variable over a finite field. Let K be the quotient field of R and let A be a torsion-free R-algebra without nilpotent ideals. If the Cartan matrix of $R/\mathfrak{p}\otimes A$ is nonsingular for all nonzero prime ideals $\mathfrak{p}$ of R, then $C_0(A)$ is a finite group.*

Proof. Let Λ be a maximal order of $\Sigma=K\otimes A$ containing A. Then Λ is hereditary [10] and every Λ-projective module can be written as a direct sum of free modules and an ideal. Therefore every element in $C_0(\Lambda)$ can be represented by an ideal in Λ. However, it is a theorem of Artin [1] that the number of isomorphism classes of left ideals in Λ is finite. (Artin proves it only for the number field case but the same argument applies to the other case.) Therefore it follows that $C_0(\Lambda)$ is a finite group. There remains to see that $\mathrm{Ker}(C_0(A)\rightarrow C_0(\Lambda))$ is a finite group. However, under the hypothesis on Cartan-matrix of A, every A-projective module can be written as a direct sum of an A-free module and an ideal of A, according to a theorem of H. Bass [4]. Therefore the assertion follows immediately from Corollary 6.

COROLLARY 14 (R. G. Swan). *Let R be as above and π a finite group. Then $C_0(R\pi)$ is a finite group, where $R\pi$ stands for the group algebra of π over R.*

BIBLIOGRAPHY

1. E. Artin, *Zur Arithmetik hyperkomplexes Zahlen*, Abh. Math. Sem. Univ. Hamburg vol. 5 (1927) pp. 261–289.

2. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. vol. 97 (1960) pp. 1–24.

3. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. vol. 95 (1960) pp. 466–488.

4. ———, *Projective modules over algebras*, to appear.

5. H. Brandt, *Zur Idealtheorie Dedekindscher Algebren*, Comment. Math. Helv. vol. 2 (1930) pp. 13–17.

6. R. Brauer and J. Tate, *On the characters of finite groups*, Ann. of Math. vol. 62 (1955) pp. 1–7.

7. Cartan-Eilenberg, *Homological algebra*, Princeton, Princeton University Press, 1956.

8. C. W. Curtis, *On commuting rings of endomorphisms*, Canad. J. Math. vol. 8 (1956) pp. 271–291.

9. S. Eilenberg, *Homological dimension and syzygies*, Ann. of Math. vol. 64 (1956) pp. 328–336.

10. M. Deuring, *Algebren*, Berlin, Julius Springer, 1935.

11. K. Morita, *Duality for modules*, Sci. Rep. Tokyo Kyoiku Daigaku, Sec. A. vol. 6 (1958) pp. 83–142.

12. D. S. Rim, *Modules over finite groups*, Ann. of Math. vol. 69 (1959) pp. 700–712.

13. J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, Séminaire Dubreil, Algèbre et théorie des nombres, Paris, 1958.

14. ———, *Construction of projective modules*, (unpublished).

15. R. G. Swan, *Induced representations and projective modules*, Ann. of Math. vol. 71 (1960) pp. 552–578.

16. ———, *Induced representations and projective modules*, mimeographed notes, University of Chicago, 1959.

COLUMBIA UNIVERSITY,
NEW YORK, NEW YORK