ON A SCHEDULING PROBLEM

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In this paper we consider the problem of deciding the order of the two items which should be processed by n machines in order to minimize the time required to complete all the operations.

Up to the present, many studies on the scheduling problem have been published. However, those studies only attempted to provide some results concerning the system with constant handling time, —the time required for processing.

So, in this paper we are going to obtain an optimal processing order in the case where the handling time is treated as random variables.

1. GENERAL CONSIDERATIONS

At first, we shall treat the system with the structure shown in the figure (1).

in put⇒ | Machine; M_1 | → | Machine; M_2 | →···→ | Machine; M_n | ⇒out put

(Fig. 1)

Two items A and B should be processsed by n machines, that is, $M_1, M_2, \cdots, M_n$.

Let T_{a_i} be the handling time of the item A on the machine M_i, and let T_{b_i} be the handling time of the item B. $(i=1, 2, \cdots, n)$.

Then these times are mutually independent random variables.

Suppose that the probability density function of the distribution of T_{a_i} and T_{b_i} are $f_i(t)$ and $g_i(t)$, respectively.

If we choose

$$A \rightarrow B$$

for the order of process, then the total time, —the time required to complete all the operations, $T(n)$ becomes

$$T(n) = T_{a_1} + U_n + T_{b_n}. \qquad (1)$$

Where,

$$U_1 \equiv 0$$

$$U_{n+1} = \{U_n + T_{b_n}\} \vee \{T_{a_2} + T_{a_3} + \cdots + T_{a_{n+1}}\}. \quad (\text{for } n \geq 1) \qquad (2)$$

(Note) The symbol $x \vee y$ denotes the maximum of x and y.

In the following paragraph, we shall discussprecisely in the caseswhere $n=2$ and 3.

2. TWO MACHINES

2. 1, General Exprerrions

As regards to the formulas (1)~(2), we get

$$T(2) = T_{a_1} + (T_{b_1} \vee T_{a_2}) + T_{b_2}. \qquad (3)$$

We must introduce the density function $u_2(t)$ of the distribution of

$$U_2 = (T_{b_1} \vee T_{a_2})$$

in order to obtain the moment generating function of the distribution of $T(2)$.

But we can easily find the density function of U_2, as follows.

$$u_2(t) = g_1(t) \cdot \int_0^t f_2(x)dx + f_2(t) \cdot \int_0^t g_1(x)dx \qquad (4)$$

since .

$$P_r\{U_2 < x\} = P_r\{T_{b_1} < x,\ T_{a_2} < x\} = P_r\{T_{b_1} < x\} \cdot P_r\{T_{a_2} < x\}.$$

Let $M_1(\theta)$, $M_2(\theta)$, $M_u(\theta)$ be the moment generating functions of the distribution of random variables T_{a_1}, T_{b_2}, U_2, respectively.

Then $M(\theta)$, the moment generating function of the distribution of $T(2)$, can be obtained by

$$M(\theta) = M_1(\theta) \cdot M_2(\theta) \cdot M_u(\theta) \qquad (5)$$

2. 2. In the case of the Exponential Handling Times

Let

$$f_i(t)=\mu_i e^{-\mu_i t}, \qquad g_i(t)=\nu_i e^{-\nu_i t}. \qquad \text{(for } t>0 \text{ and } i=1,2) \qquad (6)$$

Then,

$$M_1(\theta)=\int_0^\infty e^{\theta t}\cdot\mu_1 e^{-\mu_1 t}dt=\frac{\mu_1}{\mu_1-\theta}$$

$$M_2(\theta)=\int_0^\infty e^{\theta t}\cdot\nu_2 e^{-\nu_2 t}dt=\frac{\nu_2}{\nu_2-\theta},$$

and

$$M_u(\theta)=\int_0^\infty e^{\theta t}\cdot u_2(t)dt=\int_0^\infty e^{\theta t}\left\{\nu_1 e^{-\nu_2 t}\int_0^t \mu_2 e^{-\mu_2 x}dx+\mu_2 e^{-\mu_2 t}\int_0^t \nu_1 e^{-\nu_1 x}dx\right\}dt$$

$$=\frac{\mu_2}{\mu_2-\theta}+\frac{\nu_1}{\nu_1-\theta}-\frac{\mu_2+\nu_1}{\mu_2+\nu_1-\theta}.$$

Therefore, the moment generating function $M(\theta)$ of the distribution of total time $T(2)$ becomes

$$M(\theta)=\left(\frac{\mu_1}{\mu_1-\theta}\right)\left(\frac{\nu_2}{\nu_2-\theta}\right)\left(\frac{\mu_2}{\mu_2-\theta}+\frac{\nu_1}{\nu_1-\theta}-\frac{\mu_2+\nu_1}{\mu_2+\nu_1-\theta}\right). \qquad (7)$$

Using the expression (7), we have the expectation $E_{AB}(T)$ and variance $V_{AB}(T)$ of total time $T(2)$.

(Note) Here, the subscript AB is used to specify the order of jobs.

$$E_{AB}(T)=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\nu_1}+\frac{1}{\nu_2}-\frac{1}{\mu_2+\nu_1}, \qquad (8)$$

$$V_{AB}(T)=\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}+\frac{1}{\nu_1^2}+\frac{1}{\nu_2^2}-\frac{3}{(\mu_2+\nu_1)^2}. \qquad (9)$$

On the otherhand, if we assume the order of jobs as,

$$B\rightarrow A,$$

the expectation and variance are merely obtained by replacing μ_i and ν_i in (8) and (9).

That is,

$$E_{BA}(T)=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\nu_1}+\frac{1}{\nu_2}-\frac{1}{\mu_1+\nu_2}$$

$$V_{BA}(T)=\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}+\frac{1}{\nu_1^2}+\frac{1}{\nu_2^2}-\frac{3}{(\mu_1+\nu_2)^2}.$$

Since

$$E_{AB}(T)-E_{BA}(T)=\frac{1}{\mu_1+\nu_2}-\frac{1}{\mu_2+\nu_1},$$

we can see that, if

$$\mu_2+\nu_1<\mu_1+\nu_2 \tag{10}$$

then

$$E_{AB}(T)<E_{BA}(T)$$

holds.

Moreover, as regards to the variance, noting

$$V_{AB}(T)-V_{BA}(T)=\frac{3}{(\mu_1+\nu_2)^2}-\frac{3}{(\mu_2+\nu_1)^2}$$

we can see that, if

$$E_{AB}(T)<E_{BA}(T)$$

then

$$V_{AB}(T)<V_{BA}(T).$$

2. 3. Comparison to the Results of the Usual Model

We are going to compare the ordering in our model which minimizes the expectation $E(T)$ with an optimal ordering in usual model (—that is, the model with constant time to process), by some simple examples.

(Example 1)

Handling time (min.)

Item \ Machine	M_1	M_2
A	5	10
B	20	40

(Table 1)

An optimal ordering for usual system is determined by Johnson's criterion, which is $A \to B$.

And the total time is 65 (min.).

In our consideration, numerical value 5, 10, 20, 40 in Table (1) will be replaced by $\mu_1=1/5$, $\mu_2=1/10$, $\nu_1=1/20$, $\nu_2=1/40$, since we assumed the system with exponeptial probability density functions $f_i(t)$ and $g_i(t)$ for the handling time, and these distributions have the expectations $1/\mu_i$ and $1/\nu_i$, respectively.

Tnus we have

$$\mu_2+\nu_1<\mu_1+\nu_2 .$$

Hence an optimal ordering which minimizes $E(T)$ is

$$A \to B .$$

That is, we find the same result as that of usual system, in this case and we have $68\frac{1}{3}$ (min.) for the value of $E_{AB}(T)$.

However, in the following example, we can see that the solutions are not always similar.

(Example 2)

Handling Time (min.)

Item \ Machine	M_1	M_2
A	3 ($\mu_1=1/3$)	6 ($\mu_2=1/6$)
B	5 ($\nu_1=1/5$)	60 ($\nu_2=1/60$)

(Table 2)

An optimal ordering for usual system is

$A \to B$. (Total time=68 min.)

But we have

$$E_{AB}(T)>E_{BA}(T) ,$$

since

$$\mu_2+\nu_1=\frac{11}{30}, \quad \mu_1+\nu_2=\frac{7}{20}.$$

Therefore we assert an optimal ordering as

$B \rightarrow A$.

(The value of $E_{BA}(T)$ become $71\frac{1}{7}$ min.)

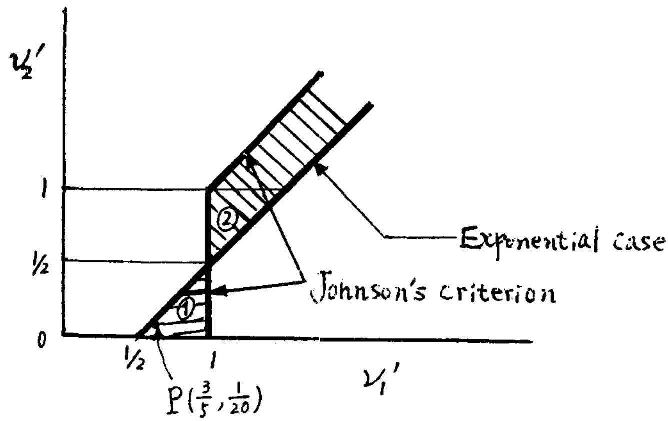
Now, if we put

$$\frac{\mu_2}{\mu_1}=\mu_2', \quad \frac{\nu_1}{\mu_1}=\nu_1', \quad \frac{\nu_2}{\mu_1}=\nu_2',$$

then we see that the values of μ_2' in the above examples are epual to $\frac{1}{2}$.

Whereupon we can compare the optimal ordeaing by Johnson's criterion with our ordering, using the figure (2).

That is, in the case where the points (ν_1', ν_2') are plotted inside the regions ① or ②, the optimal ordering by Johnson's criterion are not identical with our ordering.



(Fig. 2)

In fact, the point

$$P\left(\nu_1'=\frac{3}{5},\quad \nu_2'=\frac{1}{20}\right)$$

in the case of the example (2), is plotted inside the region ①.

Therefore we can see the both ordering are not identical.

2. 4. In the case of the Erlang Handling Times

Similarly to the paragraph (2. 2), we assume that

$$\left.\begin{aligned} f_i(t)&=\frac{(\mu_i k_i)^{k_i}\cdot t^{k_i-1}}{(k_i-1)!}e^{-\mu_i k_i t}, \quad (\text{for } i=1, 2) \\ g_i(t)&=\frac{(\nu_i S_i)^{s_i}\cdot t^{S_i-1}}{(S_i-1)!}e^{-\nu_i S_i t}. \end{aligned}\right\} \tag{11}$$

At first, we have to obtain the moment generating function $M_u(\theta)$ of the distribution of random variable

$$U_2=(T_{b_1}\vee T_{a_2}),$$

noting the total time $T(2)$ is

$$T(2)=T_{a_1}+(T_{b_1}\vee T_{a_2})+T_{b_2}.$$

Now, we get

$$\begin{aligned} M_u(\theta) &=\int_0^\infty \frac{(\mu_2 k_2)^{k_2}\cdot t^{k_2-1}}{(k_2-1)!}e^{-\mu_2 k_2 t}\cdot\left\{1-e^{-\nu_1 S_1 t}\cdot\sum_{r=0}^{S_1-1}\frac{(\nu_1 S_1 t)^r}{r!}\right\}e^{\theta t}dt \\ &+\int_0^\infty \frac{(\nu_1 S_1)^{S_1}\cdot t^{S_1-1}}{(S_1-1)!}e^{-\nu_1 S_1 t}\cdot\left\{1-e^{-\mu_2 k_2 t}\cdot\sum_{r=0}^{k_2-1}\frac{(\mu_2 k_2 t)^r}{r!}\right\}\cdot e^{\theta t}dt \\ &=\left(\frac{k_2\mu_2}{k_2\mu_2-\theta}\right)^{k_2}-\sum_{r=0}^{S_1-1}\left[\frac{(k_2\mu_2)^{k_2}}{(k_2-1)!}\cdot\frac{(S_1\nu_1)^r}{r!}\cdot\left(\frac{1}{k_2\mu_2+S_1\nu_1-\theta}\right)^{k_2+r}\right. \\ &\left.\times\{(k_2+r-1)!\}\right]+\left(\frac{S_1\nu_1}{S_1\nu_1-\theta}\right)^{S_1}-\sum_{r=0}^{k_2-1}\left[\frac{(S_1\nu_1)^{S_1}}{(S_1-1)!}\cdot\frac{(k_2\mu_2)^r}{r!}\right. \\ &\left.\times\left(\frac{1}{k_2\mu_2+S_1\nu_1-\theta}\right)^{S_1+r}\cdot\{(S_1+r-1)!\}\right]. \end{aligned} \tag{12}$$

Therefore, the moment generating function $M(\theta)$ of total time $T(2)$ is obtained by

$$M(\theta)=M_1(\theta)\cdot M_2(\theta)\cdot M_u(\theta)\,.$$

Where,

$$M_1(\theta)=\left(\frac{k_1\mu_1}{k_1\mu_1-\theta}\right)^{k_1},$$

$$M_2(\theta)=\left(\frac{S_2\nu_2}{S_2\nu_2-\theta}\right)^{S_2}.$$

Special Case

At first, let us obtain an optimal processing order in the case where $k_i=S_i=2\quad(i=1,2)$.

In this case, we get

$$M_1(\theta)=\left(\frac{2\mu_1}{2\mu_1-\theta}\right)^2,\qquad M_2(\theta)=\left(\frac{2\nu_2}{2\nu_2-\theta}\right)^2,$$

$$M_u(\theta)=\frac{4\mu_2^2}{(2\mu_2-\theta)^2}-\frac{4\mu_2^2}{(2\mu_2+2\nu_1-\theta)^2}-\frac{16\mu_2^2\nu_1}{(2\mu_2+2\nu_1-\theta)^3}$$
$$+\frac{4\nu_1^2}{(2\nu_1-\theta)^2}-\frac{4\nu_1^2}{(2\mu_2+2\nu_1-\theta)^2}-\frac{16\mu_2\nu_1^2}{(2\mu_2+2\nu_1-\theta)^3}\,.$$

Thus we have

$$M(\theta)=\left(\frac{2\mu_1}{2\mu_1-\theta}\right)^2\cdot\left(\frac{2\nu_2}{2\nu_2-\theta}\right)^2\cdot\left\{\frac{4\mu_2^2}{(2\mu_2-\theta)^2}-\frac{4\mu_2^2}{(2\mu_2+2\nu_1-\theta)^2}\right.$$
$$\left.-\frac{16\mu_2^2\nu_1}{(2\mu_2+2\nu_1-\theta)^3}+\frac{4\nu_1^2}{(2\nu_1-\theta)^2}-\frac{4\nu_1^2}{(2\mu_2+2\nu_1-\theta)^2}-\frac{16\mu_2\nu_1^2}{(2\mu_2+2\nu_1-\theta)^3}\right\}.$$

Hence

$$E_{AB}(T)=\frac{1}{\mu_1}+\frac{1}{\nu_2}+\frac{1}{\mu_2}+\frac{1}{\nu_1}-\frac{\mu_2^2+3\mu_2\nu_1+\nu_1^2}{(\mu_2+\nu_1)^3}\,, \tag{13}$$

and

$$E_{AB}(T)-E_{BA}(T)=\frac{\mu_1^2+3\mu_1\nu_2+\nu_2^2}{(\mu_1+\nu_2)^3}-\frac{\mu_2^2+3\mu_2\nu_1+\nu_1^2}{(\mu_2+\nu_1)^3}\,. \tag{14}$$

Using the expression (14), we can determine an optimal ordering.

Now, we can obtain the ordering in the case where $k_i=1,\ 2\,;s_i=1,\ 2$ (for $i=1,\ 2$), by similar calcurations.

Those results are shown in the following table (3).

Distributions of the Handling Time	$M(\theta)$	$E_{AB}(T)$	Optimal Ordering
Type [1]; $k_i=1,\ S_i=2,$ (for $i=1, 2$)	$M_{\mu_1}(\theta)\cdot M_{2\nu_2}(\theta)\cdot M_{\mu_2,2\nu_1}(\theta)$	T_0-T_1	if $T_1>T_1'$ then $A\to B$ is an optimal ordering
Type [2]; $k_1=S_1=1,$ $k_2=S_2=2$	$M_{\mu_1}(\theta)\cdot M_{2\nu_2}(\theta)\cdot M_{2\mu_2,\nu_1}(\theta)$	T_0-T_2	if $T_2>T_2'$ then $A\to B$
Type [3]; $k_i=2,\ S_i=1,$ (for $i=1, 2$)	$M_{2\mu_1}(\theta)\cdot M_{\nu_2}(\theta)\cdot M_{2\mu_2,\nu_1}(\theta)$	T_0-T_2	if $T_2>T_2'$ then $A\to B$
Type [4]; $k_1=S_1=2,$ $k_2=S_2=1.$	$M_{2\mu_1}(\theta)\cdot M_{\nu_2}(\theta)\cdot M_{\mu_2,2\nu_1}(\theta)$	T_0-T_1	if $T_1>T_1'$ then $A\to B$

Table (3)

The symbols in the table (3) are defined as follows;

$$M_{\mu_1}(\theta)=\frac{\mu_1}{\mu_1-\theta}, \qquad M_{\nu_2}(\theta)=\frac{\nu_2}{\nu_2-\theta},$$

$$M_{2\mu_1}(\theta)=\left(\frac{2\mu_1}{2\mu_1-\theta}\right)^2, \qquad M_{2\nu_2}(\theta)=\left(\frac{2\nu_2}{2\nu_2-\theta}\right)^2,$$

$$M_{\mu_2,2\nu_1}(\theta)=\left\{\frac{\mu_2}{\mu_2-\theta}-\frac{\mu_2}{\mu_2+2\nu_1-\theta}-\frac{2\mu_2\nu_1}{(\mu_2+2\nu_1-\theta)^2}\right.$$
$$\left.+\frac{4\nu_1^2}{(2\nu_1-\theta)^2}-\frac{4\nu_1^2}{(\mu_2+2\nu_1-\theta)^2}\right\},$$

$$M_{2\mu_2\cdot\nu_1}(\theta)=\left\{\frac{\nu_1}{\nu_1-\theta}-\frac{\nu_1}{2\mu_2+\nu_1-\theta}-\frac{2\mu_2\nu_1}{(2\mu_2+\nu_1-\theta)^2}\right.$$
$$\left.+\frac{4\mu_2{}^2}{(2\mu_2-\theta)^2}-\frac{4\mu_2{}^2}{(2\mu_2+\nu_1-\theta)^2}\right\},$$

$$T_0=\frac{1}{\mu_1}+\frac{1}{\nu_1}+\frac{1}{\mu_2}+\frac{1}{\nu_2},$$

$$T_1=\frac{\mu_2+4\nu_1}{(\mu_2+2\nu_1)^2},\qquad T_1'=\frac{\mu_1+4\nu_2}{(\mu_1+2\nu_2)^2},$$

$$T_2=\frac{4\mu_2+\nu_1}{(2\mu_2+\nu_1)^2},\qquad T_2'=\frac{4\mu_1+\nu_2}{(2\mu_1+\nu_2)^2}.$$

(Note) In the case where

$$k_i=1,\quad S_i=1,\qquad (\text{for}\quad i=1,\ 2),$$

of course, we have the same results with the exponential case. (Paragraph 2. 2)

In the case where

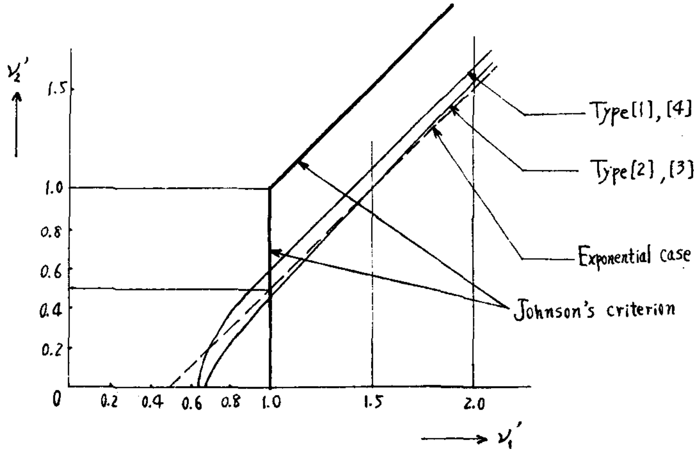
$$k_i\to\infty,\quad S_i\to\infty,\qquad (i=1,\ 2)$$

we have the same results as an optimal ordering by Johnson's criterion.

Now, let us show the figure (3) after the model of figure (2), assuming the same values of μ_1, μ_2, ν_1 and ν_2, that is,

$$\frac{\mu_2}{\mu_1}=\mu_2'=\frac{1}{2},\quad \frac{\nu_1}{\mu_1}=\nu_1',\quad \frac{\nu_2}{\mu_1}=\nu_2'.$$

Then we can said if the point (ν_1', ν_2') are plotted upside the curve, then an optimal ordering is $A\to B$, using the figure (3).



(Fig. 3)
(Type No. ··· Refer to the Table 3.)

3. THREE MACHINES

3. 1. General Expressions

As regards to the formulas (1) (2), we have

$$T(3)=T_{a_1}+\{(T_{b_1}\vee T_{a_2})+T_{b_2}\}\vee\{T_{b_2}+T_{a_3}\}+T_{b_3}\,.$$

At first, let us obtain the moment generating function $M_u(\theta)$ of

$$U_3=\{(T_{b_1}\vee T_{a_2})+T_{b_2}\}\vee\{T_{a_2}+T_{a_3}\}\,.$$

Now,

$$P_r\{U_3<x\}=P_r\{T_{b_1}+T_{b_2}<x,\ T_{a_2}+T_{b_2}<x,\ T_{a_2}+T_{a_3}<x\}\,.$$

Here we consider two cases of partitioned

Case [1] ··· $T_{b_1}+T_{b_2}>T_{a_2}+T_{a_3}$,
Case [2] ··· $T_{b_1}+T_{b_2}<T_{a_2}+T_{a_3}$.

For Case [1], we have

$$P_r\{U_3<x\}=P_r\{(T_{b_1}\vee T_{a_2})+T_{b_2}<x\}\ .$$

On the otherhand, for Case [2], we have

$$P_r\{U_3<x\}=P_r\{(T_{b_2}\vee T_{a_3})+T_{a_2}<x\}\ .$$

Let

the probability that Case [1] ocurs be $P_{[1]}$,

the probability that Case [2] occurs be $P_{[2]}$,

the moment generating function of $(T_{b_1}\vee T_{a_2})+T_{b_2}$ be $M_{[1]}(\theta)$,

the moment generating function of $(T_{b_2}\vee T_{a_3})+T_{a_2}$ be $M_{[2]}(\theta)$.

Then we get

$$M_u(\theta)=P_{[1]}\cdot M_{[1]}(\theta)+P_{[2]}\cdot M_{[2]}(\theta).$$

Using above expressions, we can find the moment generating function $M(\theta)$ of total time.

3. 2. In the case of the Exponential Handling Time

The probability density functions $g_{12}(t)$ and $f_{23}(t)$ of the distributions of random variables $(T_{b_1}+T_{b_2})$ and $(T_{a_2}+T_{a_3})$ are

$$g_{12}(t)=\frac{\nu_1\nu_2}{\nu_2-\nu_1}\cdot(e^{-\nu_1 t}-e^{-\nu_2 t})$$

$$f_{23}(t)=\frac{\mu_2\mu_3}{\mu_3-\mu_2}\cdot(e^{-\mu_2 t}-e^{-\mu_3 t})\ ,$$

respectively.

Accordingly, the probability $P_{[1]}$ of $T_{b_1}+T_{b_2}>T_{a_2}+T_{a_3}$ becomes

$$\begin{aligned}P_{[1]}&=\iint_{t_1>t_2} g_{12}(t)\cdot f_{23}(t_2)dt_1dt_2\\&=\frac{\mu_2\mu_3\cdot\{\nu_1^2+\nu_1\nu_2+\nu_2^2+\mu_2\mu_3+(\nu_1+\nu_2)(\mu_2+\mu_3)\}}{(\mu_2+\nu_1)(\mu_3+\nu_1)(\mu_2+\nu_2)(\mu_3+\nu_2)}\ .\end{aligned}$$

And the probability $P_{[2]}$ of $T_{b_1}+T_{b_2}<T_{a_2}+T_{a_3}$ becomes

$$P_{[2]}=\frac{\nu_1\nu_2\cdot\{\mu_2^2+\mu_2\mu_3+\mu_3^2+\nu_1\nu_2+(\nu_1+\nu_2)(\mu_2+\mu_3)\}}{(\mu_2+\nu_1)(\mu_3+\nu_1)(\mu_2+\nu_2)(\mu_3+\nu_2)}\ .$$

The moment generating functions $M_{[1]}(\theta)$ and $M_{[2]}(\theta)$ of the distributions of $(T_{b_1} \vee T_{a_2})+T_{b_2}$ and $(T_{b_2} \vee T_{a_3})+T_{a_2}$ are

$$M_{[1]}(\theta)=\left(\frac{\mu_2}{\mu_2-\theta}+\frac{\nu_1}{\nu_1-\theta}-\frac{\mu_2+\nu_1}{\mu_2+\nu_1-\theta}\right)\cdot\left(\frac{\nu_2}{\nu_2-\theta}\right),$$

$$M_{[2]}(\theta)=\left(\frac{\mu_3}{\mu_3-\theta}+\frac{\nu_2}{\nu_2-\theta}-\frac{\mu_3+\nu_2}{\mu_3+\nu_2-\theta}\right)\cdot\left(\frac{\mu_2}{\mu_2-\theta}\right).$$

Therefore we can obtain the moment generating function $M(\theta)$ of the total time $T(3)$, using above expressions

$$M(\nu)=\left(\frac{\mu_1}{\mu_1-\nu}\right)\cdot\left(\frac{\nu_3}{\nu_3-\nu}\right)\cdot\{P_{[1]}\cdot M_{[1]}(\nu)+P_{[2]}\cdot M_{[2]}(\nu)\}. \qquad (15)$$

Thus we get the expectation $E_{AB}(T)$ of the total time.
That is,

$$E_{AB}(T)=\frac{1}{\mu_1}+\frac{1}{\nu_3}+\left\{P_{[1]}\left(\frac{1}{\mu_2}+\frac{1}{\nu_1}-\frac{1}{\mu_2+\nu_1}+\frac{1}{\nu_2}\right)\right.$$
$$\left.+P_{[2]}\left(\frac{1}{\mu_3}+\frac{1}{\nu_2}-\frac{1}{\mu_3+\nu_2}+\frac{1}{\mu_2}\right)\right\}.$$
$$=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\nu_2}+\frac{1}{\nu_3}+\frac{\mu_2}{\nu_1(\mu_2+\nu_1)}\cdot P_{[1]}+\frac{\nu_2}{\mu_3(\mu_3+\nu_2)}\cdot P_{[2]}. \qquad (16)$$

On the otherhand, the expectation $E_{BA}(T)$ is obtained simply by replacing the symbols μ_i and ν_i.

Using these values, we can determine an optimal ordering.

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