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On a Decidable Generalized Quantifier Logic Corresponding to a Decidable Fragment of First-Order Logic

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Abstract. Van Lambalgen (1990) proposed a translation from a language containing a generalized quantifier Q into a first-order language enriched with a family of predicates R_i, for every arity i (or an infinitary predicate R) which takes $Qx\phi(x, y_1, \ldots, y_n)$ to $\forall x(R(x, y_1, \ldots, y_n) \rightarrow \phi(x, y_1, \ldots, y_n))$ ($y_1, \ldots, y_n$ are precisely the free variables of $Qx\phi$). The logic of Q (without ordinary quantifiers) corresponds therefore to the fragment of first-order logic which contains only specially restricted quantification. We prove that it is decidable using the method of analytic tableaux. Related results were obtained by Andréka and Németi (1994) using the methods of algebraic logic.

Key words: generalized quantifiers, restricted quantification, analytic tableaux, decidability

1. Introduction

Roughly speaking, in the history of modal logic modalities and their axioms came first and relational semantics giving them their precise meaning came second, bringing with it among other things the standard translation of modal logic into classical first-order logic. Although the translation from a generalized quantifier language into a first-order language considered in this paper reminds of the one for modal logic, the rest of the story is completely different. For generalized quantifiers, the meaning (defined in set theory) comes first, the axioms come (if at all) second, and the 'standard translation' appears as a technical trick used to devise a Gentzen-style proof theory for some generalized quantifiers (see van Lambalgen (1991)). However as argued in van Benthem and Alechina (1993), the standard translation may have independent interest. We now explain these points in more detail.

A. Mostowski defined a generalized quantifier **Q** as a class of subsets of the domain, so that a model M satisfies $Qx\varphi(x, \bar{d})$ if the set of elements $\{e : M \models \varphi[e, \bar{d}]\}$ is in **Q**. For example, **Q** may be the set of all uncountable subsets of the domain. Generalized quantifiers were studied for the most part from a model-theoretic point of view.

It is clear that the language containing generalized quantifiers is much more expressive than the ordinary first-order language. However, in some sense it can be reduced to a fragment of first-order language. Van Lambalgen (1991) proposed natural deduction systems for several generalized quantifiers (the filter quantifier and some of its extensions), where introduction and elimination rules for the quantifier have side conditions involving a dependency relation (between free variables). This dependency relation R comes from the following satisfiability preserving standard translation from a language containing a generalized quantifier Q into a first-order language containing R (I shall denote this language as $\mathcal{L}(R)$):

$(P(x_1, \ldots, x_n))^* = P(x_1, \ldots, x_n)$;

$(\neg\varphi)^* = \neg\varphi^*$;

$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$;

$(\forall x\varphi)^* = \forall x\varphi^*$;

$(Qx\varphi(x, \bar{y}))^* = \forall x(R(x, \bar{y}) \rightarrow \varphi^*(x, \bar{y}))$, where $\bar{y}$ are all the free variables of $Qx\varphi$.

To determine the side conditions, van Lambalgen shows that some quantifier axioms correspond to first-order conditions on R. This correspondence is studied systematically in van Benthem and Alechina (1993) and Alechina and van Lambalgen (1995a), (1995b). The standard translation seems to be a useful technical tool for studying generalized quantifiers.

Van Benthem proposed to look at the standard translation above as corresponding to an alternative, 'modal' semantics for generalized quantifiers. This semantics is described in detail in the following section. The idea behind it (and it is this which makes the corresponding restricted fragment of the first order logic interesting) is as follows. For ordinary quantifiers $\forall$ and $\exists$ the domain of quantification does not depend on an assignment to individual variables or on the variables which are free in the quantifier formula. Classical quantifiers have what Blackburn and Seligman in this volume call an 'external' view of the domain, i.e. they always can 'see' all objects which live there. On the contrary, 'local' quantification depends on the view which one can have from a certain point inside the model. A good example of such local quantification is a modal operator: it quantifies over the worlds which are seen from a given world. To make quantifiers behave in the same way, i.e. to make first-order quantification local, one can define an accessibility relation between assignments (as is done in van Benthem (1994)) or between finite sequences of elements and elements (as the standard translation suggests). In the latter case, the domain of quantification is determined by the parameters of the formula: $\diamond_x\varphi(x, \bar{d})$ is true if there is an object d accessible from $\bar{d}$ such that $\varphi[d, \bar{d}]$ is true. The theory of such quantification has a lot in common with modal logic: for example, we shall see that it is decidable. That is why the present paper gives one more answer to the question treated in Németi (1992): how to weaken standard predicate logic to make it decidable?

2. Language, Models, Analytic Tableaux

Now we define our generalized quantifier logic formally.

$\mathcal{L}(\Diamond)$ is a language which contains predicate symbols, individual variables and a generalized quantifier $\Diamond$ (no ordinary quantifiers and no equality are present). If φ is a w.f.f, so is $\Diamond_x\varphi$. $\Box_x$ (the universal-type generalized quantifier denoted by Q above) is defined as a dual of $\Diamond_x$; $\vee$, $\rightarrow$, $\equiv$ are defined as usual.

DEFINITION 1. A *model* for $\mathcal{L}(\Diamond)$ is a structure of the form $M = \langle D, R, V\rangle$ where D is a domain, V a valuation (a function assigning n-ary predicate symbols subsets of D^n), and R is a relation between elements and finite sets of elements of D, called the dependence relation. We shall write $R(a, b_1, \ldots, b_n)$ or $R(a, \bar{b})$ for $R(a, \{b_1, \ldots, b_n\})$, tacitly assuming that the second argument of R is invariant under permutations and repetitions.

The truth conditions for atomic formulas, negation and conjunction are standard. The truth definition for $\Diamond$ reads as follows. Let $y_1, \ldots, y_n$ be precisely all the free variables of $\Diamond_x\varphi$. Then

$$M \models^{\alpha} \Diamond_x\varphi \Leftrightarrow \exists\alpha' =_x \alpha(R(\alpha'(x), \alpha'(y_1), \ldots, \alpha'(y_n)) \wedge M \models^{\alpha'} \varphi).$$

A formula is satisfiable, if there is a model and a variable assignment under which it is true. A formula is valid if its negation is not satisfiable.

Note that a formula of $\mathcal{L}(\Diamond)$, φ, is valid (satisfiable) if its translation into first-order logic φ^* is valid (satisfiable), with R satisfying, for all permutations π,

$$R(a, \bar{b}) \Leftrightarrow R(a, \pi(\bar{b}))$$

$$R(a, b_1 b_1 \bar{b}) \Leftrightarrow R(a, b_1 \bar{b}).$$

Let us call the logic resulting from this notion of validity L^-_{min}. (The minus sign stands for the absence of ordinary quantifiers).

To check effectively whether a formula of $\mathcal{L}(\Diamond)$ is satisfiable, we introduce *analytic tableaux* for L^-_{min}. A very clear description of the method can be found in Smullyan (1968). The calculus described here is slightly different in form but the same in all essential aspects.

Every formula is *signed* by T or F (intuitively standing for truth and falsity). For every connective, there is a pair of rules for decomposing a signed formula with this connective as a principal connective: one rule for a formula signed with T and one rule for a formula signed with F. They correspond to the truth conditions for this connective.

DEFINITION 2. A tableau for $T(F)\chi$ is a tree where the origin is $T(F)\chi$ and the successors of a node are generated by applying one of the following decompositions rules to this node (where $\dfrac{\Delta, A_1}{\Delta, A_2}$ means: given a node Δ, A_1, create a successor node Δ, A_2, and $\dfrac{\Delta, B}{\Delta, B_1 \mid \Delta, B_2}$ means: given a node Δ, B, create two successors, Δ, B_1 and Δ, B_2):

$$T_\neg \quad \frac{\Delta, T\neg\varphi}{\Delta, F\varphi} \qquad F_\neg \quad \frac{\Delta, F\neg\varphi}{\Delta, T\varphi}$$

$$T_\wedge \quad \frac{\Delta, T(\varphi \wedge \psi)}{\Delta, T\varphi, T\psi} \qquad F_\wedge \quad \frac{\Delta, F(\varphi \wedge \psi)}{\Delta, F\varphi \mid \Delta, F\psi}$$

To give rules for $\Diamond$ we need to introduce, in addition to signed formulas, atomic formulas keeping track of the dependency relation, of the form $R(x, \bar{y})$. Now we assume that Δ (also in the rules above) is a union of a set of signed formulas Σ and a set of R-formulas Γ.

$$T_\Diamond \quad \frac{\Delta, T\Diamond_x\varphi(x, \bar{y})}{\Delta, R(z, \bar{y}), T\varphi(z, \bar{y})}$$

where z is a new variable;

$$F_\Diamond \quad \frac{\Delta, F\Diamond_x\varphi(x, \bar{y})}{\Delta, F\varphi(z, \bar{y}), F\Diamond_x\varphi(x, \bar{y})}$$

for every variable z such that $R(z, \bar{y}) \in \Delta$.

Note that the rule $F_\Diamond$ can be used only if previously some variable z with $R(z, \bar{y})$ was introduced by $T_\Diamond$. $F\Diamond_x\varphi(x, \bar{y})$ in the conclusion of the rule means that the application of the rule should be repeated if a new variable u with $R(u, \bar{y})$ is introduced.

A branch is called *closed* if the same formula occurs both under T and under F. A branch is open if this is not the case and no rule can be applied any more. A tableau is closed if all its branches are closed.

THEOREM 1. The tableau calculus described above is sound and complete for $L^-_{\min}$: for every formula χ of $\mathcal{L}(\Diamond)$, χ is valid if, and only if, there is a closed tableau for $F\chi$.

Proof. The proof is analogous to the proofs in Smullyan (1968). To demonstrate consistency, we must show that for every rule, if a set in the premise is satisfiable (i.e. the formulas signed with T are true and the formulas signed with F are false in some model M under some assignment α), then at least one of the successors is satisfiable. For the $T_\Diamond$ rule, notice that if $\Delta \cup \{\Diamond_x\varphi(x, \bar{y})\}$ is satisfiable in M, then there is an element a with $R(a, \alpha(\bar{y}))$ such that for $\alpha' = \alpha[x/a]$ $M \models^{\alpha'} \varphi(x, \bar{y})$. Therefore there is a model and an assignment α' such that the conclusion of the rule is true.

This shows that if the origin is satisfiable, then there is at least one branch, such that the set of all formulas on this branch is satisfiable. But that cannot be a closed branch. Therefore, if a tableau for $F\chi$ is closed, then there is no model and assignment under which χ is false, which means that χ is valid.

To prove completeness, we define a Hintikka set for (a set of variables) D and a dependence relation R between the variables as a set of signed formulas Σ such that

for no formula φ both $T\varphi$ and $F\varphi$ are in Σ;

if $T\neg\varphi \in \Sigma$, then $F\varphi \in \Sigma$, dually for $F\neg\varphi$;

if $T\varphi\wedge\psi \in \Sigma$, then $T\varphi, T\psi \in \Sigma$; if $F\varphi\wedge\psi \in \Sigma$, then either $F\varphi \in \Sigma$ or $F\psi \in \Sigma$;

if $T\Diamond_x\varphi(x,\bar{y}) \in \Sigma$, then $T\varphi(z,\bar{y}) \in \Sigma$ for at least one $z \in D$ with $R(z,\bar{y})$;

if $F\Diamond_x\varphi(x,\bar{y}) \in \Sigma$, then $F\varphi(z,\bar{y}) \in \Sigma$ for all z such that $R(z,\bar{y})$ (may be none such z).

It is easy to check that every Hintikka set for D and R is satisfiable. A satisfying model has domain D, dependence relation R, the assignment is defined by $\alpha(x) = x$ and the valuation is any valuation satifying

$$TP(x_1,\ldots,x_n) \in \Sigma \Rightarrow \langle x_1,\ldots,x_n\rangle \in V(P)$$

and

$$FP(x_1,\ldots,x_n) \in \Sigma \Rightarrow \langle x_1,\ldots,x_n\rangle \notin V(P).$$

Assume that there is no closed tableau for $F\chi$. We are going to show that then there is a tableau for $F\chi$ which has an open branch which constitutes a Hintikka set and therefore is satisfiable. This will imply that χ is not valid.

Let us call a tableau *systematic* if it is constructed in accordance with the following procedure. Given a node Δ, create first all nodes which can be obtained by propositional rules; then all nodes which can be obtained by the $T\Diamond$-rule; and then all nodes which can be obtained by the $F\Diamond$-rule. Repeat this procedure as long as the nodes contain signed formulas which can be used to generate new nodes. When there are no such formulas any more, a systematic tableau is finished (note that it can be infinite). It is easy to check that an open branch of a systematic tableau forms a Hintikka set. Therefore, if a systematic tableau for $F\chi$ is open, then χ is not valid. □

3. Decidability

We show that a tableau construction for formulas in *normal form* always stops. Then we prove that every formula has an equivalent in normal form.

Let us say that a subformula φ is *immediately in the scope of* a quantifier $\Diamond_x$ if it is in the scope of $\Diamond_x$ and there is no quantifier $\Diamond_y$ 'in between', that is, such that $\Diamond_y$ is in the scope of $\Diamond_x$ and φ is in the scope of $\Diamond_y$.

DEFINITION 3. A formula χ is in *normal form* if in χ every subformula immediately in the scope of a quantifier contains the quantified variable of this quantifier free.

For example, $\diamond_y \diamond_x (P(x,y) \wedge \diamond_z S(x,z))$ is in normal form, and $\diamond_x (P(x) \wedge \diamond_z S(z))$ is not (since $\diamond_z S(z)$ is immediately in the scope of $\diamond_x$ and does not have x free).

LEMMA 1. If a formula is in normal form, then each of its subformulas is.
Proof. Obvious. □

We are going to show that for a formula in normal form a tableau is always finite. This is not true for an arbitrary formula. Consider a tableau for

$$T(\diamond_x Q(x) \wedge \neg \diamond_x \neg (P(x) \wedge \diamond_z S(z)))$$

$$\begin{array}{ll}
T_{\wedge,\neg} & T\diamond_x Q(x),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
T_{\diamond} & R(d),\ TQ(d),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
F_{\diamond} & R(d),\ TQ(d),\ F\neg(P(d) \wedge \diamond_z S(z)),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
F_{\neg} & R(d),\ TQ(d),\ T(P(d) \wedge \diamond_z S(z)),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
T_{\wedge} & R(d),\ TQ(d),\ TP(d),\ T\diamond_z S(z),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
T_{\diamond} & R(d),\ TQ(d),\ TP(d),\ R(e),\ TS(e),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
F_{\diamond} & \ldots R(e),\ F\neg(P(e) \wedge \diamond_z S(z)),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
F_{\neg} & \ldots T(P(e) \wedge \diamond_z S(z)),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
T_{\wedge} & \ldots TP(e),\ T\diamond_z S(z),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
T_{\diamond} & \ldots R(e_1),\ TS(e_1),\ F\diamond_x \neg(P(x) \wedge \diamond_z S(z)) \\
 & \ldots
\end{array}$$

Here the tableau construction starts to loop. Note that $F_{\diamond}$ is the rule responsible for that: it does not decrease the complexity of the formula.

To prove that a tableau construction always stops for formulas in normal form, we need to define a notion of *dependency between variables*. This notion appears in Fine (1985). In the present context it has the following meaning:

DEFINITION 4. A variable x *depends* on a variable y in a branch of a given tableau, if in this tableau either $R(x, \ldots y \ldots)$ holds or (b) there is a variable z such that $R(x, \ldots z \ldots)$ holds and z depends on y.

For example, in

$$T\diamond_x (S(x,y) \wedge \diamond_z P(x,z))$$

$$\begin{array}{ll}
T_{\diamond} & R(x,y),\ T(S(x,y) \wedge \diamond_z P(x,z)) \\
T_{\wedge} & R(x,y),\ TS(x,y),\ T\diamond_z P(x,z) \\
T_{\diamond} & R(x,y),\ R(z,x),\ TS(x,y),\ TP(x,z)
\end{array}$$

x depends on y and z depends on x and y.

Dependency in a tableau has some obvious properties. Transitivity follows from the definition.

LEMMA 2. If x depends on y on some branch of a tableau, then y appeared free on this branch before x was introduced.

Proof. Assume that x depends on y on some branch. If $R(x, \ldots y \ldots)$ holds on this branch, then x was introduced by $T\Diamond$ rule with a premise which had y free. By definition, x is a new variable, therefore there was a stage in the tableau construction (immediately before the rule was applied) when y occurred free on the branch and x did not.

Assume that x depends on y because there are $z_1, \ldots, z_n$ such that $R(x, \ldots z_1 \ldots)$, ..., $R(z_n, \ldots y \ldots)$. As before, this means that z_n appeared after y, ..., z_1 after z_2, and x after z_1. Thus x appeared after y. □

Lemma 2 implies that dependency is asymmetric: if x depends on y, then y does not depend on x. It also implies that x cannot depend on a variable which was introduced later on the branch.

If a formula is in normal form, its nested quantifiers are 'hooked' into one another: if $\Diamond_z\varphi$ is immediately in the scope of $\Diamond_y$, then y is free in φ. This yields some important properties of tableaux with signed formulas in normal form as origins. Before formulating them, some terminology has to be defined.

We shall say that a quantifier $\Diamond_z$ in $\Diamond_z\varphi$ on some branch of a tableau is *instantiated* on a variable z_1, if Δ, $\Diamond_z\varphi$ is as a premise of the $T\Diamond$-rule and the successor of this node is Δ, $T\varphi[z/z_1]$. Observe that if $\Diamond_z$ is immediately in the scope of $\Diamond_y$, then $\Diamond_z$ will be instantiated on a variable dependent on the variable used to instantiate $\Diamond_y$.

We call a signed formula A a *result of decomposing in accordance with the tableau rules* a signed formula B (on some branch of a tableau) if

(i) either A is obtained by applying one of the tableau rules to B (for example, $A = TP(x)$ is a result of decomposing $B = F\neg P(x)$ in

$$F\neg P(x)$$

$$F_{\neg}\ TP(x)$$

(ii) or there are signed formulas $A_1, \ldots, A_n$ on this branch, such that $A = A_1$, A_i is a result of decomposing A_{i+1} ($i < n$), and $A_n = B$.

Observe that if A is a result of decomposing B, then (with T and F omitted) A is a subformula of B.

LEMMA 3. Any result of decomposing in accordance with the tableau rules of a formula $\Diamond_x \chi$ in normal form will contain a variable dependent on the variable used to instantiate $\Diamond_x$, or this variable itself.

Proof. Assume that φ is a subformula of $\Diamond_x \chi$ in normal form which is in the scope of quantifiers $\Diamond_x, \Diamond_{x_1}, \dots, \Diamond_{x_n}$ ($\Diamond_x$ being the outermost). If $n = 0$ (φ is immediately in the scope of $\Diamond_x$), then φ contains x free (since χ is in normal form) and therefore the variable used to instantiate $\Diamond_x$ will be free in φ.

Let $n \geq 1$, and $T(F)\varphi(d_1, \dots, d_m)$ be the result of decomposing χ according to the tableau rules. We show that at least one of the d_i depends on the variable used to instantiate $\Diamond_x$.

For the sake of simplicity, we assume that $\Diamond_{x_i}$ is instantiated by x_i. By ψ_i we denote the biggest subformula of χ in the scope of $\Diamond_{x_i}$. Note that φ is a subformula of ψ_n, and x_n is free in φ (since χ is in normal form). Since $\Diamond_{x_1}\psi_1$ is immediately in the scope of $\Diamond_x$, x is free in ψ_1, etc. Therefore for the variables used to instantiate the quantifiers holds

$$R(x_1, \dots x \dots), R(x_2, \dots x_1 \dots), R(x_3, \dots x_2 \dots), \dots, R(x_n, \dots x_{n-1} \dots)$$

and by transitivity x_n depends on x. But x_n is free in φ. Thus, $\varphi(d_1, \dots, d_m)$ contains at least one variable dependent on x. □

LEMMA 4. Assume that a tableau construction for a formula $T(F)\chi$ in normal form has reached a stage when the only applicable rule is $T_\Diamond$. Let $z_1, \dots, z_n$ be the list of all variables which occur free on a branch of the tableau at this stage. Any formula which appears later on this branch will contain at least one free variable which is not among $z_1, \dots, z_n$.

Proof. If the only applicable rule is $T_\Diamond$, then every formula on the branch at this stage is either atomic, or begins with $T\Diamond_x$ or $F\Diamond_x$. Since χ is in normal form, every formula in the tableau is (Lemma 1). Any result of decomposing a formula in normal form beginning with $\Diamond_x$ will contain free a variable which depends on the variable used to instantiate $\Diamond_x$, or this variable itself (Lemma 3). This variable (used to instantiate $\Diamond_x$) is not among $z_1, \dots, z_n$ since it will be introduced by $T_\Diamond$ rule at a later stage (we have assumed that the $F_\Diamond$ rule was no more applicable, therefore even applications of the $F_\Diamond$ rule to the formulas beginning with $F\Diamond_x$ will use a variable not among $z_1, \dots z_n$). Since neither of z_i can depend on a variable introduced later on the branch (Lemma 2), every result of decomposition will contain a variable which is not among $z_1, \dots, z_n$. □

LEMMA 5. Assume that a tableau construction for a formula $T(F)\chi$ in normal form has reached a stage where the only applicable rule is $T_\Diamond$. Let $F\Diamond_x\psi_1, \dots, F\Diamond_x\psi_n$ be the list of all formulas on a branch of the tableau beginning with $F\Diamond$. For every $F\Diamond_x\psi_i$, the $F_\Diamond$ rule with this formula as a premise will be applied only finitely many times.

Proof. Let $\bar{z}$ be the free variables of $\Diamond_x \psi_i$. The $F\Diamond$ rule with $F\Diamond_x \psi_i(x, \bar{z})$ as a premise can be repeated only if a new variable d with $R(d, \bar{z})$ is introduced by the $T\Diamond$ rule with a formula of the form $T\Diamond_x \alpha(x, \bar{z})$ as a premise. Assume that there are m such formulas present on the branch at this stage. No formula with $\bar{z}$ as its only free variables will appear later on the branch by Lemma 4. Therefore the $F\Diamond$ rule with $F\Diamond_x \psi_i(x, \bar{z})$ as a premise will be applied precisely m more times. □

LEMMA 6. If χ is in normal form, the tableau construction for $T(F)\chi$ always stops.

Proof. Take an arbitrary branch of a tableau. Perform all propositional rules and all applications of the $F\Diamond$ rule with respect to previously introduced variables. One can show that at any stage of constructing a tableau this process stops after a finite number of steps. At some stage we cannot proceed any further without applying the $T\Diamond$ rule. Assume that at this stage the $F\Diamond$-formulas (which give rise to repetitions of $F\Diamond$ rules) are

$$F\Diamond_{x_1}\theta_1, \ldots, F\Diamond_{x_k}\theta_k$$

Due to Lemma 5, for every $\Diamond_{x_i}\theta_i$ the rule $F\Diamond$ with this formula as a premise will be repeated only finitely many times.

In a finite number of steps the tableau construction reaches the stage when $F\Diamond$ will not be repeated with $F\Diamond_x\theta_i$ as a premise, and again only the $T\Diamond$ rule is applicable. Assume that at this point a branch of the tableau contains atomic formulas, formulas beginning with $T\Diamond_x$, new formulas beginning with $F\Diamond$, $F\Diamond_x\psi_1, \ldots, F\Diamond_x\psi_m$, and $\{F\Diamond_x\theta_i : 1 \leq i \leq k\}$. The complexity of all formulas on the branch except for $\{F\Diamond_x\theta_i : 1 \leq i \leq k\}$ is decreased. For each of the $F\Diamond_x\psi_j$ the $F\Diamond$ rule will be applied finitely many times. After that the complexity of all formulas on the branch except for the ones which will not be used any more is again decreased. An easy induction shows that the tableau construction will stop after a finite number of steps. □

COROLLARY 1. The satisfiability problem for the formulas in normal form is decidable.

LEMMA 7. Every formula has an equivalent in normal form.

Proof. Take an arbitrary formula χ. Let us call the quantifiers which have in their immediate scope subformulas not containing the quantified variable free *quasivacuous*. Our aim is to eliminate all quasivacuous quantifiers. The proof that this is always possible goes by induction on the maximal quantifier depth a quasivacuous quantifier. (The quantifier depth of a *subformula* is the number of quantifiers in whose scope this subformula is situated. By the quantifier depth of a *quantifier* I mean the quantifier depth of the subformula which begins with this quantifier.)

For example, in $\diamond_x(P(x) \wedge \diamond_z S(z))$ $\diamond_x$ is quasivacuous, because $\diamond_z S(z)$ is immediately in its scope, and it does not contain x free. The quantifier depth of $\diamond_x$ in this formula equals 0. In $\diamond_x(P(x) \wedge \diamond_y(Q(x,y) \wedge \diamond_z S(z)))$ $\diamond_x$ is not quasivacuous, since $\diamond_z S(z)$ is not immediately in the scope of $\diamond_x$: there is $\diamond_y$ in between. $\diamond_y$ is quasivacuous and its quantifier depth is 1.

Basis. If the quantifier depth of all quasivacuous quantifiers in χ equals 0, then the procedure described below (taking the subformulas, not having the quantified variable free, out of the scope of the quantifier) brings χ in normal form.

Inductive step. We show that the maximal quantifier depth of a quasivacuous quantifier in χ can be reduced by one.

Fix a quasivacuous quantifier $\diamond_x$ with maximal depth (not having in its scope other quasivacuous quantifiers). Denote the biggest subformula of χ in the scope of $\diamond_x$ as φ. We give an algorithm for finding an equivalent of $\diamond_x\varphi$, to be called φ', with the the same free variables, such that in φ' there are no more subformulas immediately in the scope of $\diamond_x$ which do not have x free, and the quantifiers in the scope of $\diamond_x$ are still not quasivacuous. Note that $\chi \equiv \chi[\diamond_x\varphi/\varphi']$.

If x is not free in φ, apply the following derivable equivalence:

$$\diamond_x\varphi(\bar{z}) \equiv \diamond_x\top(x,\bar{z}) \wedge \varphi(\bar{z})$$

(where $\bar{z}$ are all the free variables of φ, and $x \notin \bar{z}$).

Otherwise write φ as a disjunction of conjunctions, where each conjunct is either an atomic formula or its negation, or a formula beginning with a quantifier or its negation. Denote this disjunction as $\theta_1 \vee \ldots \vee \theta_n$. Obviously,

$$\diamond_x\varphi \equiv \diamond_x(\bigvee_i \theta_i),$$

$1 \leq i \leq n$. Note that the θ's may have different free variables, therefore the distributivity property

$$\diamond_x(\bigvee_i \theta_i) \equiv \bigvee_i \diamond_x\theta_i$$

does not necessarily hold. But if the free variables of φ are $x, \bar{y}$, then the following holds:

$$\diamond_x(\bigvee_i \theta_i) \equiv \diamond_x(\bigvee_i(\theta_i \wedge \top(x,\bar{y}))) \equiv \bigvee_i \diamond_x(\theta_i \wedge \top(x,\bar{y}))$$

In each disjunct θ_i, denote the conjunction of subformulas having x free (including $\top(x,\bar{y})$) as $\psi_{i1}(x)$ and the conjunction of formulas not having x free as ψ_{i2}. Our aim is to find an equivalent of

$$\bigvee_i \diamond_x(\psi_{i1}(x) \wedge \psi_{i2})$$

where no ψ_{i2} is in the scope of $\Diamond_x$. To do that, apply

$$\Diamond_x(\varphi(x,\bar{y}) \wedge \psi(\bar{z})) \equiv \Diamond_x(\varphi(x,\bar{y}) \wedge \top(x,\bar{z})) \wedge \psi(\bar{z}),$$

given that x is not free in ψ.

The result of applying this derivability is the desired formula φ'. Observe that since we have not changed the subformulas under the quantifiers in the scope of $\Diamond_x$, there are still no quasivacuous quantifiers in the scope of $\Diamond_x$. The result of substituting φ' in χ instead of $\Diamond_x\varphi$ gives an equivalent formula with maximal depth of a quasivacuous quantifier reduced by 1.

Note that this procedure can create new quasivacuous quantifiers, but with a smaller depth. Consider the example above:

$$\Diamond_x(P(x) \wedge \Diamond_y(Q(x,y) \wedge \Diamond_z S(z))).$$

In this formula $\Diamond_y$ is a quasivacuous quantifier with depth 1. After applying the algorithm to $\Diamond_y$ this formula becomes

$$\Diamond_x(P(x) \wedge \Diamond_y(Q(x,y) \wedge \top(y)) \wedge \Diamond_z S(z))$$

which is equivalent to

$$\Diamond_x(P(x) \wedge \Diamond_y Q(x,y) \wedge \Diamond_z S(z))$$

and $\Diamond_x$ is now a quasivacuous quantifier; but it has depth 0. After repeating the procedure for $\Diamond_x$, we obtain

$$\Diamond_x(P(x) \wedge \Diamond_y Q(x,y)) \wedge \Diamond_z S(z)$$

which is in normal form. □

COROLLARY 2. $L^-_{\min}$ has finite model property.

Proof. Every formula φ of $L^-_{\min}$ has an equivalent φ' in normal form, and if φ is satisfiable, then there is a *finite* open branch of a tableau for $F\varphi'$. But then a satisfying model for φ' (and therefore for φ) as constructed in the proof of the Theorem 1 is finite. □

THEOREM 2. $L^-_{\min}$ is decidable.

Proof. The theorem follows from Lemmas 6 and 7 and Theorem 1. □

COROLLARY 3. The fragment of first-order logic which is the image of $\mathcal{L}(\Diamond)$ under the standard translation, with R satisfying

$$R(a,\bar{b}) \Leftrightarrow R(a,\pi(\bar{b}))$$

$$R(a,b_1b_1\bar{b}) \Leftrightarrow R(a,b_1\bar{b})$$

is decidable.

4. Related work

Here we briefly state some more general results, obtained by different methods.

Andréka and Németi (1994) and Andréka, van Benthem and Németi (1995) study fragments of first order logic with restricted quantification. One of these fragments, called Fragment 2 in Andréka, van Benthem and Németi (1995), contains all first order formulas in which quantifiers are restricted as follows:

$$\exists y_1 \dots y_n(R(\bar{x}, \bar{y}) \wedge \varphi)$$

with $FV(\varphi) \subseteq \{\bar{x}, \bar{y}\}$; the order of the variables in R does not matter, and R may be any predicate symbol (it does not have to be the same for all formulas in the fragment, as in our case). Obviously, the image of $\mathcal{L}(\Diamond)$ under the standard translation is included in Fragment 2.

THEOREM 3. (Andréka and Németi (1994)) Fragment 2 with equality is decidable.

The proof uses the method of mosaics which was applied before in proving that cylindric relativized set algebras have decidable equation theory (cf. Németi (1992). On other connections between restricted fragments, generalized quantification and cylindric relativized algebras, see Andréka, van Benthem and Németi (1995) and Simon and van Lambalgen (1994)).

The theorem above implies decidability of $L^-_{\min}$ and various extensions of $L^-_{\min}$ which are complete with respect to the classes of models defined by first order conditions on R which are *in Fragment* 2. Given a formula φ with n (free and bound) variables, one can show that if it has a model in which the dependency relation R of all arities less or equal to n satisfies a certain condition on R, then it has a model in which this condition is satisfied for all arities. This implies that φ is satisfiable in a model with a certain condition on R if, and only if, the conjunction of the standard translation of φ and finitely many formulas defining the condition on R for arity less or equal to n, is first order satisifable. If this conjunction is in Fragment 2, then by the theorem above, the latter problem is decidable.

5. Concluding remark

The interest of restricted fragments is not only in their surprising formal properties, like decidability. The study of alternatives to standard quantification or its generalizations (taking a local point of view on models as opposed to the global one, or studying generalized dependencies between variables as opposed to ordinary Skolem functions) helps to see standard quantification in a broader context. In Alechina and van Lambalgen (1995b) it was shown that classical first order logic can be reformulated to contain a substitution rule as a structural rule, and logics for several generalized quantifiers can be obtained by weakening this rule; $L^-_{\min}$ is the weakest logic considered there. One of the directions for future work is to find a uniform framework for studying logics corresponding to cylindric relativized algebras in the same fashion.

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References

Alechina, N. and van Lambalgen, M., 1995a, "Correspondence and completeness for generalized quantifiers", *Bulletin of IGPL* **3**, 167–190.

Alechina, N. and van Lambalgen, M., 1995b, "Generalized quantification as substructural logic", ILLC Report ML-95-05, University of Amsterdam. To appear in *Journal of Symbolic Logic.*

Andréka, H. and Németi, I., 1994, "Decidability of the bounded fragment of first order logic with equality", manuscript.

Andréka, H., van Benthem, J., and Németi, I., 1995, "Back and forth between modal logic and classical logic", *Bulletin of IGPL* **3**, 685–720.

Blackburn, P. and Seligman, J., 1995, "Hybrid Languages", this issue of *Journal of Logic, Language and Information.*

Fine, K., 1985, "Natural deduction and arbitrary objects", *Journal of Philosophical Logic* **14**, 57–107.

Németi, I., 1992 "Decidability of weakened versions of first order logic and cylindric relativized set algebras", to appear in *Logic Colloquium'92*, L. Csirmaz, D.M. Gabbay, and M. de Rijke, eds., Stanford: CSLI.

Simon, A. and van Lambalgen, M., 1994, "Axiomatizing cylindric relativized set algebras with generalized quantification", manuscript.

Smullyan, R.M., 1968, *First-Order Logic*, Berlin: Springer-Verlag.

Van Benthem, J., 1994 "Modal foundations for predicate logic", *Studia Logica*, to appear.

Van Benthem, J. and Alechina, N., 1993 "Modal quantification over structured domains", ILLC Report ML-93-02, University of Amsterdam. To appear in *Advances in Intensional Logic*, M. de Rijke, ed., Dordrecht: Kluwer.

van Lambalgen, M., 1990, "The axiomatization of randomness", *Journal of Symbolic Logic* **55**, 1143–1167.

Van Lambalgen, M., 1991, "Natural deduction for generalized quantifiers", in *Generalized Quantifier Theory and Applications*, J. van der Does and J. van Eijck, eds., Amsterdam: Dutch Network for Language, Logic and Information. To appear as CSLI Lecture Notes.