

Research Article

Multiple-Attribute Decision-Making Using Fermatean Fuzzy Hamacher Interactive Geometric Operators

Gulfam Shahzadi ¹, Fariha Zafar ², and Maha Abdullah Alghamdi ³

¹Department of Mathematics, Virtual University of Pakistan, Lahore, Pakistan

²Department of Mathematics, University of Okara, Okara, Pakistan

³Department of Mathematics, Faculty of Science, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia

Correspondence should be addressed to Fariha Zafar; fariha.zafar@uo.edu.pk

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Fermatean fuzzy set (FFS) is a more efficient, flexible, and generalized model to deal with uncertainty, as compared to intuitionistic and Pythagorean fuzzy models. This research article presents a novel multiple-attribute decision-making (MADM) technique based on FFS. Aggregation operators (AOs), for example, Dombi, Einstein, and Hamacher, are frequently being used in the MADM process and are considered useful tools for evaluating the given alternatives. Among these, one of the most effective is the Hamacher operator. The salient feature of this operator is that it reduces the impact of negative information and provides more accurate results. Motivated by the primary characteristics of the Hamacher operator, we apply Hamacher interactive aggregation operators based on FFSs to determine the best alternative. Using Hamacher's norm operations, we introduce some new geometric operators, namely, Fermatean fuzzy Hamacher interactive weighted geometric (FFHIWG) operator, Fermatean fuzzy Hamacher interactive ordered weighted geometric (FFHIOWG) operator, and Fermatean fuzzy Hamacher interactive hybrid weighted geometric (FFHIHWG) operator. Some important results and properties of the proposed AOs are discussed, and to achieve the optimal alternative, the proposed MADM technique is carried out in a real-life application of the medical field. An algorithm of the proposed technique is also developed. The significance of the proposed method is that Fermatean fuzzy Hamacher interactive geometric (FFHIG) operators deal with the relationship among belongingness degree (BD) and nonbelongingness degree (NBD) of the objects, which perform a crucial role in decision-making (DM). At last, to show the exactness and validity of the proposed work, a comparative analysis of the proposed model and the existing models is presented.

1. Introduction

Ambiguous or uncertain information is one of the greatest dilemmas dealing with the MADM process. The uncertain information can be captured in different ways. In the last few years, Zadeh's fuzzy set theory (FST) [1] and its extensions, i.e., intuitionistic fuzzy set theory (IFST) [2], Pythagorean fuzzy set theory (PFST) [3], hesitant fuzzy set theory (HFST) [4, 5], and interval-valued fuzzy set theory (IVFST) [6], have been proved to be efficient tools handling uncertainty in numerous applications of MADM. However, there are some restrictions involved in all these theories, for example, FST deals with belongingness degree only, whereas IFST deals with

both BD and NBD but it restricts their sum to be less or equal to 1. To overcome this issue, PFST replaces the condition of the sum to "the sum of squares of BD and NBD to be less or equal to 1." Recently, a more generalized theory, namely, Fermatean fuzzy set theory (FFST), was introduced by Senapati and Yager [7]. The notion of FFS was initiated from IFSSs and PFSSs, where the sum of cubes of NBD and BD is less than or equal to one. Therefore, FFSs are more flexible and generalized as compared to both IFSSs and PFSSs.

Aggregation methods based on IFSSs and PFSSs were widely used in MADM. Xu [8] and Zhao et al. [9] defined aggregation operators (AOs) based on IFSSs. Wei and Lu [10] developed Pythagorean fuzzy (PF) power AOs and used

them in DM problems. Ordered weighted averaging aggregation operators (OWAAOs) for MADM were defined by Yager [11]. Many multiple-attribute decision-making models (MADMMs) use algebraic operators and these are Dombi, Einstein, and Hamacher operators. In recent years, many theories were developed based on these operators. Dombi [12] defined Dombi triangular norm and conorm operators. Many authors contributed their work to Dombi AOs. Akram et al. [13] worked on Pythagorean Dombi fuzzy AOs (PFDAOs). Wei [14] introduced interaction AOs based on PFSs and also applied these to MADMMs.

Hamacher AOs were introduced in 1978 [15]. Wei [16] defined Hamacher AOs based on PFSs and gave a comparative analysis for MADM. Garg [17, 18] presented some series of IF interactive averaging AOs by applying interactive averaging AOs on IFSs and also gave the idea of IF Hamacher AOs having entropy weight. Wu and Wei [19] presented MADMMs based on PF Hamacher AOs (PFHAOs). Wei [14] introduced the PF interaction AOs (PFIAOs) with their application to MADM. Waseem et al. [20] discussed MADM based on m -polar fuzzy Hamacher AOs (mFHAOs). Zhao and Wei [21] gave the idea of IF Einstein hybrid AOs (IFEHAOs). Wang and Liu [22] elaborated IF information AOs using Einstein operations. The idea for assessment of express service quality with entropy weight was explained by Wang et al. [23] under PF interactive Hamacher power AO.

Senapati and Yager [24] introduced Fermatean fuzzy averaging/geometric operators (FFAOS/FFFOs). They also defined operations over Fermatean fuzzy numbers (FFNs) [25]. Garg et al. [26] presented a method for the most suitable laboratory selection for COVID-19 test under Fermatean fuzzy environment (FFE). Akram et al. [27] discussed a MADMM to show the benefits of a sanitizer in COVID-19 under FFE. Shahzadi and Akram [28] proposed the idea of Fermatean fuzzy soft AOs and applied this idea in the field of group decision-making for the selection of an antivirus mask. Recently, Aydemir and Gunduz [29] explained the Fermatean fuzzy TOPSIS (FF-TOPSIS) method consisting of Dombi AOs. Shahzadi et al. [30] introduced the idea of Hamacher interactive hybrid weighted averaging operators under FFE. Feng et al. [31] investigated membership grades of q -rung orthopair fuzzy sets geometrically. For more comprehension and understanding, the readers are referred to study [24, 25, 32–35].

The motivations of this study are defined as follows:

- (i) The proposed Hamacher interactive AOs (HIAOs) deal with the relationship between the BD and NBD of an object
- (ii) The MADMM based on FFSs shows that the change in NBDs will definitely affect the BDs of the objects
- (iii) The proposed Fermatean fuzzy AOs generalize the BDs and NBDs of the objects; i.e., greater values of belongingness and nonbelongingness degrees can be taken as compared to IFST and PFST
- (iv) The HIAOs are a much convenient approach to cope with the issues in the DM process; this article

aims to define HIAOs based on FFSs to handle uncertainty associated with the choice of alternatives in MADMMs

- (v) Hamacher interactive AOs give more precise and exact choice values in decision results when applied to MADMMs

The contributions of this article are outlined as follows:

- (i) Some new HIAOs such as FFHIWG, FFHIOGW, and FFHIHWG are proposed here
- (ii) The attractive properties alongside their special cases are discussed which reduce the loopholes in the existing operators
- (iii) An algorithm for MADM using the proposed operators is described and an application is presented to show the applicability of the intended method in the real world
- (iv) A comparison is also presented which shows the innovation and importance of the contemplated model

The remaining part of the article is arranged as follows. In Section 2, some elementary notions are presented. Section 3 explains a hybrid structure of Hamacher interactive operators based on FFSs such as FFHIWG operators with a few important results and basic properties, for example, boundedness, homogeneity, idempotency, monotonicity, and shift invariance. In Section 4, the basic concept and results of the FFHIOGW operator are presented. Section 5 presents the notions related to the FFHIHWG operator. In Section 6, a MADMM under Fermatean fuzzy environment is explained through a real-life application. In Section 7, the influence of distinct values of the parameter is shown. In Section 8, a comparative analysis with existing theories is discussed which shows the efficacy and importance of the intended model. In Section 9, the presented work is summarized with concluding remarks.

2. Preliminaries

Definition 1 (see [7]). A Fermatean fuzzy set (FFS) \mathcal{L} on \mathfrak{D} (a nonempty crisp set) is defined as

$$\mathcal{L} = \{ \langle \mathbf{a}, \mu_{\mathcal{L}}(\mathbf{a}), \nu_{\mathcal{L}}(\mathbf{a}) \rangle \}, \quad (1)$$

where $\mu_{\mathcal{L}}: \mathfrak{D} \rightarrow [0, 1]$, $\nu_{\mathcal{L}}: \mathfrak{D} \rightarrow [0, 1]$ and $\omega_{\mathcal{L}}(\mathbf{a}) = \sqrt[3]{1 - (\mu_{\mathcal{L}}(\mathbf{a}))^3 - (\nu_{\mathcal{L}}(\mathbf{a}))^3}$ indicate belongingness degree, nonbelongingness degree, and indeterminacy degree, respectively.

Definition 2 (see [7]). The score function (SF) and accuracy function (AF) for a FFS $\mathcal{L} = (\mu_{\mathcal{L}}, \nu_{\mathcal{L}})$ are given by

$$\begin{aligned} S(\mathcal{L}) &= \mu_{\mathcal{L}}^3 - \nu_{\mathcal{L}}^3, & S(\mathcal{L}) &\in [-1, 1], \\ \mathcal{A}(\mathcal{L}) &= \mu_{\mathcal{L}}^3 + \nu_{\mathcal{L}}^3, & \mathcal{A}(\mathcal{L}) &\in [0, 1]. \end{aligned} \quad (2)$$

Definition 3 (see [17]). Consider two FFSs $\mathcal{L}_1 = \langle \mu_{\mathcal{L}_1}, \nu_{\mathcal{L}_1} \rangle$ and $\mathcal{L}_2 = \langle \mu_{\mathcal{L}_2}, \nu_{\mathcal{L}_2} \rangle$. Then,

- (1) If $S(\mathcal{L}_1) < S(\mathcal{L}_2)$, then $\mathcal{L}_1 < \mathcal{L}_2$.
- (2) If $S(\mathcal{L}_1) > S(\mathcal{L}_2)$, then $\mathcal{L}_1 > \mathcal{L}_2$.
- (3) If $S(\mathcal{L}_1) = S(\mathcal{L}_2)$, then
 - (a) If $\mathcal{A}(\mathcal{L}_1) < \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 < \mathcal{L}_2$.
 - (b) If $\mathcal{A}(\mathcal{L}_1) > \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 > \mathcal{L}_2$.
 - (c) If $\mathcal{A}(\mathcal{L}_1) = \mathcal{A}(\mathcal{L}_2)$, then $\mathcal{L}_1 \sim \mathcal{L}_2$.

Definition 4 (see [15]). Hamacher t-norm and t-conorm are defined by

$$T(\mathbf{a}_i, \mathbf{a}_j) = \frac{\mathbf{a}_i \mathbf{a}_j}{\zeta + (1 - \zeta)(\mathbf{a}_i + \mathbf{a}_j - \mathbf{a}_i \mathbf{a}_j)}, \tag{3}$$

$$T^*(\mathbf{a}_i, \mathbf{a}_j) = \frac{\mathbf{a}_i + \mathbf{a}_j - \mathbf{a}_i \mathbf{a}_j - (1 - \zeta)\mathbf{a}_i \mathbf{a}_j}{1 - (1 - \zeta)\mathbf{a}_i \mathbf{a}_j}.$$

- (i) For $\zeta = 1$, these operations become algebraic t-norm, $T(\mathbf{a}_i, \mathbf{a}_j) = \mathbf{a}_i \mathbf{a}_j$, and algebraic t-conorm, $T^*(\mathbf{a}_i, \mathbf{a}_j) = \mathbf{a}_i + \mathbf{a}_j - \mathbf{a}_i \mathbf{a}_j$
- (ii) For $\zeta = 2$, these operations become Einstein t-norm, $T(\mathbf{a}_i, \mathbf{a}_j) = (\mathbf{a}_i \mathbf{a}_j / 1 + (1 - \mathbf{a}_i)(1 - \mathbf{a}_j))$, and Einstein t-conorm, $T^*(\mathbf{a}_i, \mathbf{a}_j) = (\mathbf{a}_i + \mathbf{a}_j / 1 + \mathbf{a}_i \mathbf{a}_j)$

Definition 5. Let $\mathcal{L}_1 = \langle \mu_1, \nu_1 \rangle$, $\mathcal{L}_2 = \langle \mu_2, \nu_2 \rangle$, and $\mathcal{L} = \langle \mu, \nu \rangle$ be three FFSs and $\lambda > 0$; then,

$$\begin{aligned} \text{(i) } \mathcal{L}_1 \oplus \mathcal{L}_2 &= \left\langle \sqrt[\zeta]{\frac{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\mu_i^{\zeta}) - \prod_{i=1}^{\mathfrak{n}} (1 - \mu_i^{\zeta})}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\mu_i^{\zeta}) + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \mu_i^{\zeta})}}, \sqrt[\zeta]{\frac{\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta}) - \prod_{i=1}^{\mathfrak{n}} (1 - \mu_i^{\zeta} - \nu_i^{\zeta})}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\mu_i^{\zeta}) + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \mu_i^{\zeta})}} \right\rangle \\ \text{(ii) } \mathcal{L}_1 \otimes \mathcal{L}_2 &= \left\langle \sqrt[\zeta]{\frac{\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})^2 \prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta} - \mu_i^{\zeta})}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\mu_i^{\zeta}) + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})}}, \sqrt[\zeta]{\frac{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\nu_i^{\zeta}) - \prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\mu_i^{\zeta}) + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})}} \right\rangle \\ \text{(iii) } \lambda \cdot \mathcal{L} &= \left\langle \sqrt[\zeta]{\frac{((1 + (\zeta - 1)\mu^{\zeta})^{\lambda} - (1 - \mu^{\zeta})^{\lambda}) / (1 + (\zeta - 1)\mu^{\zeta}) + (\zeta - 1)(1 - \mu^{\zeta})^{\lambda}}{\zeta \{ (1 - \nu^{\zeta})^{\lambda} - (1 - \mu^{\zeta} - \nu^{\zeta})^{\lambda} \} / (1 + (\zeta - 1)\mu^{\zeta}) + (\zeta - 1)(1 - \mu^{\zeta})^{\lambda}}}, \sqrt[\zeta]{\frac{\zeta \{ (1 - \nu^{\zeta})^{\lambda} - (1 - \mu^{\zeta} - \nu^{\zeta})^{\lambda} \} / (1 + (\zeta - 1)\mu^{\zeta}) + (\zeta - 1)(1 - \mu^{\zeta})^{\lambda}}{((1 + (\zeta - 1)\nu^{\zeta})^{\lambda} - (1 - \nu^{\zeta})^{\lambda}) / (1 + (\zeta - 1)\nu^{\zeta}) + (\zeta - 1)(1 - \nu^{\zeta})^{\lambda}}} \right\rangle \\ \text{(iv) } \mathcal{L}^{\lambda} &= \left\langle \sqrt[\zeta]{\frac{\zeta \{ (1 - \nu^{\zeta})^{\lambda} - (1 - \mu^{\zeta} - \nu^{\zeta})^{\lambda} \} / (1 + (\zeta - 1)\nu^{\zeta}) + (\zeta - 1)(1 - \nu^{\zeta})^{\lambda}}{((1 + (\zeta - 1)\nu^{\zeta})^{\lambda} - (1 - \nu^{\zeta})^{\lambda}) / (1 + (\zeta - 1)\nu^{\zeta}) + (\zeta - 1)(1 - \nu^{\zeta})^{\lambda}}}, \sqrt[\zeta]{\frac{((1 + (\zeta - 1)\nu^{\zeta})^{\lambda} - (1 - \nu^{\zeta})^{\lambda}) / (1 + (\zeta - 1)\nu^{\zeta}) + (\zeta - 1)(1 - \nu^{\zeta})^{\lambda}}{\zeta \{ (1 - \nu^{\zeta})^{\lambda} - (1 - \mu^{\zeta} - \nu^{\zeta})^{\lambda} \} / (1 + (\zeta - 1)\mu^{\zeta}) + (\zeta - 1)(1 - \mu^{\zeta})^{\lambda}}} \right\rangle \end{aligned}$$

3. Fermatean Fuzzy Hamacher Interactive Weighted Geometric Operators

In this section, we introduce the Fermatean fuzzy Hamacher interactive weighted geometric operator (FFHIWGO) and describe its some important characteristics.

Definition 6. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ ($i = 1, 2, \dots, \mathfrak{n}$) be a family of FFSs and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{\mathfrak{n}})^T$ be its weight vector (WV) such that $\kappa_i > 0$ and $\sum_{i=1}^{\mathfrak{n}} \kappa_i = 1$, then FFHIWG: $\Omega^{\mathfrak{n}} \rightarrow \Omega$ is defined as

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{n}}) = \kappa_1 \mathcal{L}_1 \otimes \kappa_2 \mathcal{L}_2 \otimes \dots \otimes \kappa_{\mathfrak{n}} \mathcal{L}_{\mathfrak{n}}. \tag{4}$$

Theorem 1. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs; then,

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{n}}) = \left\langle \sqrt[\zeta]{\frac{\zeta \{ \prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})^{\kappa_i} - \prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta} - \mu_i^{\zeta})^{\kappa_i} \}}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\nu_i^{\zeta})^{\kappa_i} + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})^{\kappa_i}}}, \sqrt[\zeta]{\frac{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\nu_i^{\zeta})^{\kappa_i} - \prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})^{\kappa_i}}{\prod_{i=1}^{\mathfrak{n}} (1 + (\zeta - 1)\nu_i^{\zeta})^{\kappa_i} + (\zeta - 1)\prod_{i=1}^{\mathfrak{n}} (1 - \nu_i^{\zeta})^{\kappa_i}}} \right\rangle. \tag{5}$$

Proof. For $\mathfrak{n} = 1, \kappa = \kappa_1 = 1$,

$$\begin{aligned} \text{FFHIWG}(\mathcal{L}_1) &= \kappa_1 \mathcal{L}_1 \\ &= \mathcal{L}_1 \\ &= (\mu_1, \nu_1) \\ &= \left\langle \sqrt[\zeta]{\frac{\zeta \{ (1 - \nu_1^{\zeta}) - (1 - \nu_1^{\zeta} - \mu_1^{\zeta}) \}}{(1 + (\zeta - 1)\nu_1^{\zeta}) + (\zeta - 1)(1 - \nu_1^{\zeta})}}, \sqrt[\zeta]{\frac{(1 + (\zeta - 1)\nu_1^{\zeta}) - (1 - \nu_1^{\zeta})}{(1 + (\zeta - 1)\nu_1^{\zeta}) + (\zeta - 1)(1 - \nu_1^{\zeta})}} \right\rangle. \end{aligned} \tag{6}$$

Thus, the result is true for $\eta = 1$. Suppose that result holds for $\eta = p$, i.e.,

$$FFHIWG(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_p) = \left\langle \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^p (1 - \nu_i^3)^{K_i} - \prod_{i=1}^p (1 - \nu_i^3 - \mu_i^3)^{K_i}\}}{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^p (1 - \nu_i^3)^{K_i}}}, \sqrt[3]{\frac{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^p (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^p (1 - \nu_i^3)^{K_i}}} \right\rangle. \tag{7}$$

Now, for $\eta = p + 1$,

$$\begin{aligned} FFHIWG(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{p+1}) &= \oplus_{i=1}^{p+1} \kappa_i \mathcal{L}_i \\ &= \left\langle \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^p (1 - \nu_i^3)^{K_i} - \prod_{i=1}^p (1 - \nu_i^3 - \mu_i^3)^{K_i}\}}{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^p (1 - \nu_i^3)^{K_i}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^p (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^p (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^p (1 - \nu_i^3)^{K_i}}} \right\rangle \\ &\quad \otimes \left\langle \sqrt[3]{\frac{\varsigma\{(1 - \nu_{p+1}^3)^{K_{p+1}} - (1 - \nu_{p+1}^3 - \mu_{p+1}^3)^{K_{p+1}}\}}{(1 + (\varsigma - 1)\nu_{p+1}^3)^{K_{p+1}} + (\varsigma - 1)(1 - \nu_{p+1}^3)^{K_{p+1}}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{(1 + (\varsigma - 1)\nu_{p+1}^3)^{K_{p+1}} - (1 - \nu_{p+1}^3)^{K_{p+1}}}{(1 + (\varsigma - 1)\nu_{p+1}^3)^{K_{p+1}} + (\varsigma - 1)(1 - \nu_{p+1}^3)^{K_{p+1}}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^{p+1} (1 - \nu_i^3)^{K_i} - \prod_{i=1}^{p+1} (1 - \nu_i^3 - \mu_i^3)^{K_i}\}}{\prod_{i=1}^{p+1} (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^{p+1} (1 - \nu_i^3)^{K_i}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{\prod_{i=1}^{p+1} (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^{p+1} (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^{p+1} (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^{p+1} (1 - \nu_i^3)^{K_i}}} \right\rangle. \end{aligned} \tag{8}$$

\Rightarrow The result holds, $\forall \eta$.

□ (i) For $\varsigma = 1$, FFHIWGO becomes Fermatean fuzzy interactive weighted geometric operator (FFIWGO):

Remark 1. Here are cases of the FFHIWGO.

$$FFIWG(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \left\langle \sqrt[3]{\prod_{i=1}^\eta (1 - \nu_i^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_i^3 - \mu_i^3)^{K_i}}, \sqrt[3]{1 - \prod_{i=1}^\eta (1 - \nu_i^3)^{K_i}} \right\rangle. \tag{9}$$

(ii) For $\varsigma = 2$, FFHIWG operator becomes Fermatean fuzzy Einstein interactive weighted geometric operator (FFEIWGO):

$$\text{FFEIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_b) = \left\langle \sqrt[3]{\frac{2\{\prod_{i=1}^b (1 - \nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3 - \mu_i^3)^{K_i}\}}{\prod_{i=1}^b (1 + \nu_i^3)^{K_i} + \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}}, \sqrt[3]{\frac{\prod_{i=1}^b (1 + \nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^b (1 + \nu_i^3)^{K_i} + \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}} \right\rangle. \quad (10)$$

Theorem 2. The clumped value of FFSs $\mathcal{L}_i = (\mu_i, \nu_i)$, by using FFHIWGO, is a FFS, i.e.,

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_b) \in \text{FFS}. \quad (11)$$

Proof. As \mathcal{L}_i 's are FFSs, $0 \leq \mu_i, \nu_i \leq 1$, and $0 \leq \mu_i^3 + \nu_i^3 \leq 1$. Therefore,

$$\begin{aligned} & \frac{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}} \\ &= 1 - \frac{\varsigma \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}} \\ &\leq 1 - \prod_{i=1}^b (1 - \nu_i^3)^{K_i} \leq 1. \end{aligned} \quad (12)$$

Also, $(1 + (\varsigma - 1)\nu_i^3) \geq (1 - \nu_i^3) \implies \prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3) - \prod_{i=1}^b (1 - \nu_i^3) \geq 0$. Therefore,

$$\begin{aligned} & \frac{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}} \geq 0 \\ &\implies \sqrt[3]{\frac{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3)^{K_i}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}}} \geq 0. \end{aligned} \quad (13)$$

Thus, $0 \leq \nu_{\text{FFHIWG}} \leq 1$.

Moreover,

$$\begin{aligned} & \frac{\varsigma\{\prod_{i=1}^b (1 - \nu_i^3)^{K_i} - \prod_{i=1}^b (1 - \nu_i^3 - \mu_i^3)^{K_i}\}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}} \\ &\leq \frac{\varsigma\{\prod_{i=1}^b (1 - \nu_i^3)^{K_i}\}}{\prod_{i=1}^b (1 + (\varsigma - 1)\nu_i^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^b (1 - \nu_i^3)^{K_i}} \\ &\leq \prod_{i=1}^b (1 - \nu_i^3)^{K_i} \\ &\leq 1. \end{aligned} \quad (14)$$

Also,

$$\prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3 - \mu_i^3)^{\kappa_i} \geq 0,$$

$$\frac{\varsigma \left\{ \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3 - \mu_i^3)^{\kappa_i} \right\}}{\prod_{i=1}^{\mathfrak{h}} (1 + (\varsigma - 1)\nu_i^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3)^{\kappa_i}} \geq 0, \quad (15)$$

$$\sqrt[3]{\frac{\varsigma \left\{ \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3 - \mu_i^3)^{\kappa_i} \right\}}{\prod_{i=1}^{\mathfrak{h}} (1 + (\varsigma - 1)\nu_i^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{h}} (1 - \nu_i^3)^{\kappa_i}}} \geq 0.$$

Thus, $0 \leq \mu_{\text{FFHIWG}} \leq 1$. \square

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{h}}) = \mathcal{L}_o. \quad (16)$$

Property 1 (idempotency). If $\mathcal{L}_i = \mathcal{L}_o = (\mu_o, \nu_o), \forall i$, then

Proof. Since $\mathcal{L}_i = \mathcal{L}_o = (\mu_o, \nu_o) (\forall i = 1, 2, \dots, \mathfrak{h})$ and $\sum_{i=1}^{\mathfrak{h}} \kappa_i = 1$, by Theorem 1,

$$\begin{aligned} \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{h}}) &= \left\langle \sqrt[3]{\frac{\varsigma \left\{ \prod_{i=1}^{\mathfrak{h}} (1 - \nu_o^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{h}} (1 - \nu_o^3 - \mu_o^3)^{\kappa_i} \right\}}{\prod_{i=1}^{\mathfrak{h}} (1 + (\varsigma - 1)\nu_o^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{h}} (1 - \nu_o^3)^{\kappa_i}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{h}} (1 + (\varsigma - 1)\nu_o^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{h}} (1 - \nu_o^3)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{h}} (1 + (\varsigma - 1)\nu_o^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{h}} (1 - \nu_o^3)^{\kappa_i}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\varsigma \left\{ (1 - \nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i} - (1 - \nu_o^3 - \mu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i} \right\}}{(1 + (\varsigma - 1)\nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i} + (\varsigma - 1)(1 - \nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{(1 + (\varsigma - 1)\nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i} - (1 - \nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i}}{(1 + (\varsigma - 1)\nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i} + (\varsigma - 1)(1 - \nu_o^3)^{\sum_{i=1}^{\mathfrak{h}} \kappa_i}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\varsigma \left\{ (1 - \nu_o^3) - (1 - \nu_o^3 - \mu_o^3) \right\}}{(1 + (\varsigma - 1)\nu_o^3) + (\varsigma - 1)(1 - \nu_o^3)}}, \sqrt[3]{\frac{(1 + (\varsigma - 1)\nu_o^3) - (1 - \nu_o^3)}{(1 + (\varsigma - 1)\nu_o^3) + (\varsigma - 1)(1 - \nu_o^3)}} \right\rangle \\ &= (\mu_o, \nu_o). \end{aligned} \quad (17)$$

Property 2 (boundedness). Let $\mathcal{L}^- = (\min_i(\mu_i), \max_i(\nu_i))$ and $\mathcal{L}^+ = (\max_i(\mu_i), \min_i(\nu_i))$; then,

$$\mathcal{L}^- \leq \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{h}}) \leq \mathcal{L}^+. \quad (18)$$

Proof. Let $f(\mathbf{a}) = (1 - \mathbf{a}/1 + (\varsigma - 1)\mathbf{a})$, $\mathbf{a} \in [0, 1]$; then $f'(\mathbf{a}) = -(\varsigma/(1 + (\varsigma - 1)\mathbf{a})^2) < 0$, so $f(\mathbf{a})$ is a decreasing function (DF). As $\nu_{i,\min}^3 \leq \nu_i^3 \leq \nu_{i,\max}^3, \forall i = 1, 2, \dots, \mathfrak{h}$, then $f(\nu_{i,\max}^3) \leq f(\nu_i^3) \leq f(\nu_{i,\min}^3), \forall i$; that is, $(1 - \nu_{i,\max}^3)/$

$1 + (\zeta - 1)\nu_{i,\max}^3 \leq (1 - \nu_i^3 / 1 + (\zeta - 1)\nu_i^3) \leq (1 - \nu_{i,\min}^3 / 1 + (\zeta - 1)\nu_{i,\min}^3), \forall i$. Let $\kappa_i \in [0, 1]$ and $\sum_{i=1}^n \kappa_i = 1$; we have

$$\begin{aligned} & \left(\frac{1 - \nu_{i,\max}^3}{1 + (\zeta - 1)\nu_{i,\max}^3} \right)^{\kappa_i} \leq \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq \left(\frac{1 - \nu_{i,\min}^3}{1 + (\zeta - 1)\nu_{i,\min}^3} \right)^{\kappa_i} \\ \prod_{i=1}^n & \left(\frac{1 - \nu_{i,\max}^3}{1 + (\zeta - 1)\nu_{i,\max}^3} \right)^{\kappa_i} \leq \prod_{i=1}^n \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq \prod_{i=1}^n \left(\frac{1 - \nu_{i,\min}^3}{1 + (\zeta - 1)\nu_{i,\min}^3} \right)^{\kappa_i} \\ & \Leftrightarrow \left(\frac{1 - \nu_{i,\max}^3}{1 + (\zeta - 1)\nu_{i,\max}^3} \right)^{\sum_{i=1}^n \kappa_i} \leq \prod_{i=1}^n \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq \left(\frac{1 - \nu_{i,\min}^3}{1 + (\zeta - 1)\nu_{i,\min}^3} \right)^{\sum_{i=1}^n \kappa_i} \\ & \Leftrightarrow \left(\frac{1 - \nu_{i,\max}^3}{1 + (\zeta - 1)\nu_{i,\max}^3} \right) \leq \prod_{i=1}^n \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq \left(\frac{1 - \nu_{i,\min}^3}{1 + (\zeta - 1)\nu_{i,\min}^3} \right) \\ & \Leftrightarrow (\zeta - 1) \left(\frac{1 - \nu_{i,\max}^3}{1 + (\zeta - 1)\nu_{i,\max}^3} \right) \leq (\zeta - 1) \prod_{i=1}^n \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq (\zeta - 1) \left(\frac{1 - \nu_{i,\min}^3}{1 + (\zeta - 1)\nu_{i,\min}^3} \right) \quad (19) \\ & \Leftrightarrow \left(\frac{\zeta}{1 + (\zeta - 1)\nu_{i,\max}^3} \right) \leq 1 + (\zeta - 1) \prod_{i=1}^n \left(\frac{1 - \nu_i^3}{1 + (\zeta - 1)\nu_i^3} \right)^{\kappa_i} \leq \left(\frac{\zeta}{1 + (\zeta - 1)\nu_{i,\min}^3} \right) \\ & \Leftrightarrow \left(\frac{1 + (\zeta - 1)\nu_{i,\min}^3}{\zeta} \right) \leq \frac{1}{1 + (\zeta - 1) \prod_{i=1}^n (1 - \nu_i^3 / 1 + (\zeta - 1)\nu_i^3)^{\kappa_i}} \leq \left(\frac{1 + (\zeta - 1)\nu_{i,\max}^3}{\zeta} \right) \\ & \Leftrightarrow (1 + (\zeta - 1)\nu_{i,\min}^3) \leq \frac{\zeta}{1 + (\zeta - 1) \prod_{i=1}^n (1 - \nu_i^3 / 1 + (\zeta - 1)\nu_i^3)^{\kappa_i}} \leq (1 + (\zeta - 1)\nu_{i,\max}^3) \\ & \Leftrightarrow (\zeta - 1)\nu_{i,\min}^3 \leq \frac{\zeta}{1 + (\zeta - 1) \prod_{i=1}^n (1 - \nu_i^3 / 1 + (\zeta - 1)\nu_i^3)^{\kappa_i}} - 1 \leq (\zeta - 1)\nu_{i,\max}^3 \\ & \Leftrightarrow \nu_{i,\min}^3 \leq \frac{\prod_{i=1}^n (1 + (\zeta - 1)\nu_i^3)^{\kappa_i} - \prod_{i=1}^n (1 - \nu_i^3)^{\kappa_i}}{\prod_{i=1}^n (1 + (\zeta - 1)\nu_i^3)^{\kappa_i} + (\zeta - 1) \prod_{i=1}^n (1 - \nu_i^3)^{\kappa_i}} \leq \nu_{i,\max}^3 \end{aligned}$$

Thus,

$$\nu_{i,\min} \leq \sqrt[3]{\frac{\prod_{i=1}^n (1 + (\zeta - 1)\nu_i^3)^{\kappa_i} - \prod_{i=1}^n (1 - \nu_i^3)^{\kappa_i}}{\prod_{i=1}^n (1 + (\zeta - 1)\nu_i^3)^{\kappa_i} + (\zeta - 1) \prod_{i=1}^n (1 - \nu_i^3)^{\kappa_i}}} \leq \nu_{i,\max} \quad (20)$$

Consider $g(\mathbf{b}) = (\zeta - (\zeta - 1)\mathbf{b}) / (\zeta - 1)\mathbf{b}$, $\mathbf{b} \in (0, 1]$, then $g'(\mathbf{b}) = -(\zeta / (\zeta - 1)\mathbf{b}^2)$; i.e., $g(\mathbf{b})$ is a DF on $(0, 1]$. Since $1 - \nu_{i,\max}^3 \leq 1 - \nu_i^3 \leq 1 - \nu_{i,\min}^3, \forall i$, then $g(1 - \nu_{i,\max}^3) \leq$

$g(1 - \nu_i^3) \leq g(1 - \nu_{i,\min}^3), \forall i$; that is, $(\zeta - (\zeta - 1)(1 - \nu_{i,\min}^3)) / (\zeta - 1)(1 - \nu_{i,\min}^3) \leq (\zeta - (\zeta - 1)(1 - \nu_i^3)) / (\zeta - 1)(1 - \nu_i^3) \leq (\zeta - (\zeta - 1)(1 - \nu_{i,\max}^3)) / (\zeta - 1)(1 - \nu_{i,\max}^3)$. Then,

$$\begin{aligned}
& \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\min}^3)}{(\varsigma - 1)(1 - \nu_{i,\min}^3)} \right)^{\kappa_i} \leq \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_i^3)}{(\varsigma - 1)(1 - \nu_i^3)} \right)^{\kappa_i} \leq \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\max}^3)}{(\varsigma - 1)(1 - \nu_{i,\max}^3)} \right)^{\kappa_i} \\
& \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\min}^3)}{(\varsigma - 1)(1 - \nu_{i,\min}^3)} \right)^{\kappa_i} \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_i^3)}{(\varsigma - 1)(1 - \nu_i^3)} \right)^{\kappa_i} \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\max}^3)}{(\varsigma - 1)(1 - \nu_{i,\max}^3)} \right)^{\kappa_i} \\
& \Rightarrow \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\min}^3)}{(\varsigma - 1)(1 - \nu_{i,\min}^3)} \right)^{\sum_{i=1}^{\mathfrak{b}} \kappa_i} \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_i^3)}{(\varsigma - 1)(1 - \nu_i^3)} \right)^{\kappa_i} \leq \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\max}^3)}{(\varsigma - 1)(1 - \nu_{i,\max}^3)} \right)^{\sum_{i=1}^{\mathfrak{b}} \kappa_i} \\
& \Rightarrow \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\min}^3)}{(\varsigma - 1)(1 - \nu_{i,\min}^3)} \right) \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_i^3)}{(\varsigma - 1)(1 - \nu_i^3)} \right)^{\kappa_i} \leq \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_{i,\max}^3)}{(\varsigma - 1)(1 - \nu_{i,\max}^3)} \right) \\
& \Rightarrow \left(\frac{\varsigma}{(\varsigma - 1)(1 - \nu_{i,\min}^3)} \right) \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{\varsigma - (\varsigma - 1)(1 - \nu_i^3)}{(\varsigma - 1)(1 - \nu_i^3)} \right)^{\kappa_i} + 1 \leq \left(\frac{\varsigma}{(\varsigma - 1)(1 - \nu_{i,\max}^3)} \right) \\
& \Rightarrow \left(\frac{(\varsigma - 1)(1 - \nu_{i,\max}^3)}{\varsigma} \right) \leq \frac{1}{\prod_{i=1}^{\mathfrak{b}} (\varsigma - (\varsigma - 1)(1 - \nu_i^3)/(\varsigma - 1)(1 - \nu_i^3))^{\kappa_i} + 1} \leq \left(\frac{(\varsigma - 1)(1 - \nu_{i,\min}^3)}{\varsigma} \right) \\
& \Rightarrow (1 - \nu_{i,\max}^3) \leq \frac{\varsigma}{(\varsigma - 1) \prod_{i=1}^{\mathfrak{b}} (\varsigma - (\varsigma - 1)(1 - \nu_i^3)/(\varsigma - 1)(1 - \nu_i^3))^{\kappa_i} + (\varsigma - 1)} \leq (1 - \nu_{i,\min}^3) \\
& \Rightarrow (1 - \nu_{i,\max}^3) \leq \frac{\varsigma}{(\varsigma - 1) \prod_{i=1}^{\mathfrak{b}} (\varsigma - (\varsigma - 1)\mu_i^3/(\varsigma - 1)\mu_i^3)^{\kappa_i} + (\varsigma - 1)} \leq (1 - \nu_{i,\min}^3).
\end{aligned} \tag{21}$$

Also,

$$\begin{aligned}
& 1 - \nu_{i,\max}^3 - \mu_{i,\min}^3 \leq 1 - \nu_i^3 - \mu_i^3 \leq 1 - \nu_{i,\min}^3 - \mu_{i,\max}^3 \\
& \Leftrightarrow \frac{1 - \nu_{i,\max}^3 - \mu_{i,\min}^3}{1 - \nu_{i,\min}^3} \leq \frac{1 - \nu_i^3 - \mu_i^3}{1 - \nu_i^3} \leq \frac{1 - \nu_{i,\min}^3 - \mu_{i,\max}^3}{1 - \nu_{i,\max}^3} \\
& \Leftrightarrow \frac{1 - \nu_{i,\max}^3 - \mu_{i,\min}^3}{1 - \nu_{i,\min}^3} \leq \prod_{i=1}^{\mathfrak{b}} \left(\frac{1 - \nu_i^3 - \mu_i^3}{1 - \nu_i^3} \right)^{\kappa_i} \leq \frac{1 - \nu_{i,\min}^3 - \mu_{i,\max}^3}{1 - \nu_{i,\max}^3} \\
& \Leftrightarrow \frac{-\nu_{i,\max}^3 + \nu_{i,\min}^3 + \mu_{i,\max}^3}{1 - \nu_{i,\max}^3} \leq 1 - \prod_{i=1}^{\mathfrak{b}} \left(\frac{1 - \nu_i^3 - \mu_i^3}{1 - \nu_i^3} \right)^{\kappa_i} \leq \frac{-\nu_{i,\min}^3 + \nu_{i,\max}^3 + \mu_{i,\min}^3}{1 - \nu_{i,\min}^3} \\
& \Leftrightarrow -\nu_{i,\max}^3 + \nu_{i,\min}^3 + \mu_{i,\max}^3 \leq \frac{\varsigma \{1 - \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3 - \mu_i^3/1 - \nu_i^3)^{\kappa_i}\}}{(\varsigma - 1) \prod_{i=1}^{\mathfrak{b}} (1 + (\varsigma - 1)\nu_i^3/(\varsigma - 1)(1 - \nu_i^3))^{\kappa_i} + (\varsigma - 1)} \leq -\nu_{i,\min}^3 + \nu_{i,\max}^3 + \mu_{i,\min}^3 \\
& \Leftrightarrow \mu_{i,\max}^3 \leq \frac{\varsigma \{ \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3 - \mu_i^3)^{\kappa_i} \}}{\prod_{i=1}^{\mathfrak{b}} (1 + (\varsigma - 1)\nu_i^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3)^{\kappa_i}} \leq \mu_{i,\min}^3 \\
& \Leftrightarrow \mu_{i,\max} \leq \sqrt[3]{\frac{\varsigma \{ \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3)^{\kappa_i} - \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3 - \mu_i^3)^{\kappa_i} \}}{\prod_{i=1}^{\mathfrak{b}} (1 + (\varsigma - 1)\nu_i^3)^{\kappa_i} + (\varsigma - 1) \prod_{i=1}^{\mathfrak{b}} (1 - \nu_i^3)^{\kappa_i}}} \leq \mu_{i,\min}.
\end{aligned} \tag{22}$$

Let $\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n) = \mathcal{L} = \langle \mu_{\mathcal{L}}, \nu_{\mathcal{L}} \rangle$; then, from inequalities (20) and (22), $\mu_{\min} \leq \mu_{\mathcal{L}} \leq \mu_{\max}$, $\nu_{\max} \leq \nu_{\mathcal{L}} \leq \nu_{\min}$, where $\mu_{\min} = \min_i \{\mu_i\}$, $\mu_{\max} = \max_i \{\mu_i\}$, $\nu_{\min} = \min_i \{\nu_i\}$, and $\nu_{\max} = \max_i \{\nu_i\}$. So, $S(\mathcal{L}) = \mu_{\mathcal{L}}^3 - \nu_{\mathcal{L}}^3 \leq \mu_{\max}^3 - \nu_{\min}^3 = S(\mathcal{L}^+)$ and $S(\mathcal{L}) = \mu_{\mathcal{L}}^3 - \nu_{\mathcal{L}}^3 \geq \mu_{\min}^3 - \nu_{\max}^3 = S(\mathcal{L}^-)$. As $S(\mathcal{L}) < S(\mathcal{L}^+)$ and $S(\mathcal{L}) > S(\mathcal{L}^-)$,

$$\mathcal{L}^- \leq \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n) \leq \mathcal{L}^+. \quad (23)$$

Property 3 (monotonicity). If $\mathcal{L}_i \leq \mathcal{T}_i, \forall i$, then

$$\text{FFHIWG}(\mathcal{L}_1 \otimes \mathcal{T}, \mathcal{L}_2 \otimes \mathcal{T}, \dots, \mathcal{L}_n \otimes \mathcal{T}) = \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n) \otimes \mathcal{T}. \quad (25)$$

Proof. As $\mathcal{L}_i, \mathcal{T} \in \text{FFSs}$, so

$$\mathcal{L}_i \otimes \mathcal{T} = \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma \{ (1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3) - (1 - \nu_i^3 - \mu_i^3)(1 - \nu_{\mathcal{T}}^3 - \mu_{\mathcal{T}}^3) \}}{(1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3) + (\varsigma - 1)(1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3)}}, \\ \sqrt[3]{\frac{(1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3) - (1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3)}{(1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3) + (\varsigma - 1)(1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3)}}} \end{array} \right\rangle. \quad (26)$$

Therefore,

$$\begin{aligned} & \text{FFHIWG}(\mathcal{L}_1 \otimes \mathcal{T}, \mathcal{L}_2 \otimes \mathcal{T}, \dots, \mathcal{L}_n \otimes \mathcal{T}) \\ &= \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma \{ \prod_{i=1}^n ((1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3))^{k_i} - \prod_{i=1}^n ((1 - \nu_i^3 - \mu_i^3)(1 - \nu_{\mathcal{T}}^3 - \mu_{\mathcal{T}}^3))^{k_i} \}}{\prod_{i=1}^n ((1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3))^{k_i} + (\varsigma - 1) \prod_{i=1}^n ((1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3))^{k_i}}}, \\ \sqrt[3]{\frac{\prod_{i=1}^n ((1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3))^{k_i} - \prod_{i=1}^n ((1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3))^{k_i}}{\prod_{i=1}^n ((1 + (\varsigma - 1)\nu_i^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}}^3))^{k_i} + (\varsigma - 1) \prod_{i=1}^n ((1 - \nu_i^3)(1 - \nu_{\mathcal{T}}^3))^{k_i}}} \end{array} \right\rangle \\ &= \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma \{ \prod_{i=1}^n (1 - \nu_i^3)^{k_i} (1 - \nu_{\mathcal{T}}^3)^{k_i} - \prod_{i=1}^n (1 - \nu_i^3 - \mu_i^3)^{k_i} (1 - \nu_{\mathcal{T}}^3 - \mu_{\mathcal{T}}^3)^{k_i} \}}{\prod_{i=1}^n (1 + (\varsigma - 1)\nu_i^3)^{k_i} (1 + (\varsigma - 1)\nu_{\mathcal{T}}^3)^{k_i} + (\varsigma - 1) \prod_{i=1}^n (1 - \nu_i^3)^{k_i} (1 - \nu_{\mathcal{T}}^3)^{k_i}}}, \\ \sqrt[3]{\frac{\prod_{i=1}^n (1 + (\varsigma - 1)\nu_i^3)^{k_i} (1 + (\varsigma - 1)\nu_{\mathcal{T}}^3)^{k_i} - \prod_{i=1}^n (1 - \nu_i^3)^{k_i} (1 - \nu_{\mathcal{T}}^3)^{k_i}}{\prod_{i=1}^n (1 + (\varsigma - 1)\nu_i^3)^{k_i} (1 + (\varsigma - 1)\nu_{\mathcal{T}}^3)^{k_i} + (\varsigma - 1) \prod_{i=1}^n (1 - \nu_i^3)^{k_i} (1 - \nu_{\mathcal{T}}^3)^{k_i}}} \end{array} \right\rangle \end{aligned}$$

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n) \leq \text{FFHIWG}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n). \quad (24)$$

Proof. It can be proved on similar lines to the above. \square

Property 4 (shift invariance). If $\mathcal{T} = \langle \mu_{\mathcal{T}}, \nu_{\mathcal{T}} \rangle$ is another FFS, then

$$\begin{aligned}
 &= \left\langle \sqrt[3]{\frac{\varsigma\left(\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}\right\}\left(1-\nu_{\mathcal{F}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3-\mu_i^3\right)^{k_i}\right\}\left(1-\nu_{\mathcal{F}}^3-\mu_{\mathcal{F}}^3\right)\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}\right\}\left(1+\left(\varsigma-1\right)\nu_{\mathcal{F}}^3\right)+\left(\varsigma-1\right)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}\right\}\left(1-\nu_{\mathcal{F}}^3\right)}}, \right. \\
 &\left. \sqrt[3]{\frac{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}\right\}\left(1+\left(\varsigma-1\right)\nu_{\mathcal{F}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}\right\}\left(1-\nu_{\mathcal{F}}^3\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}\right\}\left(1+\left(\varsigma-1\right)\nu_{\mathcal{F}}^3\right)+\left(\varsigma-1\right)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}\right\}\left(1-\nu_{\mathcal{F}}^3\right)}}}\right\rangle \tag{27} \\
 &= \text{FFHIWG}\left(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{y}}\right) \otimes \mathcal{F}.
 \end{aligned}$$

Property 5 (homogeneity). Let $\lambda > 0$, then

$$\text{FFHIWG}\left(\lambda \mathcal{L}_1, \lambda \mathcal{L}_2, \dots, \lambda \mathcal{L}_{\mathfrak{y}}\right) = \lambda \text{FFHIWG}\left(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{y}}\right). \tag{28}$$

Proof. Since $\mathcal{L}_i = (\mu_i, \nu_i)$ are FFSs and $\lambda > 0$, therefore

$$\lambda \mathcal{L}_i = \left\langle \sqrt[3]{\frac{\varsigma\left\{\left(1-\nu^3\right)^{\lambda}-\left(1-\nu^3-\mu^3\right)^{\lambda}\right\}}{\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}+\left(\varsigma-1\right)\left(1-\nu^3\right)^{\lambda}}}, \sqrt[3]{\frac{\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}-\left(1-\nu^3\right)^{\lambda}}{\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}+\left(\varsigma-1\right)\left(1-\nu^3\right)^{\lambda}}}\right\rangle. \tag{29}$$

Therefore,

$$\begin{aligned}
 \text{FFHIWG}\left(\lambda \mathcal{L}_1, \lambda \mathcal{L}_2, \dots, \lambda \mathcal{L}_{\mathfrak{y}}\right) &= \left\langle \sqrt[3]{\frac{\varsigma\left\{\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\nu^3\right)^{\lambda}\right)^{k_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\nu^3-\mu^3\right)^{\lambda}\right)^{k_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}\right)^{k_i}+\left(\varsigma-1\right)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\nu^3\right)^{\lambda}\right)^{k_i}}}, \right. \\
 &\left. \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}\right)^{k_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\nu^3\right)^{\lambda}\right)^{k_i}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+\left(\varsigma-1\right)\nu^3\right)^{\lambda}\right)^{k_i}+\left(\varsigma-1\right)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\nu^3\right)^{\lambda}\right)^{k_i}}}\right\rangle \\
 &= \left\langle \sqrt[3]{\frac{\varsigma\left\{\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\nu^3\right)^{k_i}\right)^{\lambda}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\nu^3-\mu^3\right)^{k_i}\right)^{\lambda}\right\}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu^3\right)^{k_i}\right)^{\lambda}+\left(\varsigma-1\right)\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\nu^3\right)^{k_i}\right)^{\lambda}}}, \right. \\
 &\left. \sqrt[3]{\frac{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu^3\right)^{k_i}\right)^{\lambda}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\nu^3\right)^{k_i}\right)^{\lambda}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu^3\right)^{k_i}\right)^{\lambda}+\left(\varsigma-1\right)\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\nu^3\right)^{k_i}\right)^{\lambda}}}\right\rangle \tag{30} \\
 &= \lambda \left\langle \sqrt[3]{\frac{\varsigma\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3-\mu_i^3\right)^{k_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}+\left(\varsigma-1\right)\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}}}, \right. \\
 &\left. \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\left(\varsigma-1\right)\nu_i^3\right)^{k_i}+\left(\varsigma-1\right)\prod_{i=1}^{\mathfrak{y}}\left(1-\nu_i^3\right)^{k_i}}}\right\rangle \\
 &= \lambda \text{FFHIWG}\left(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{\mathfrak{y}}\right).
 \end{aligned}$$

Property 6. Let $\mathcal{L}_i = (\mu_{\mathcal{L}_i}, \nu_{\mathcal{L}_i})$ and $\mathcal{T}_i = (\mu_{\mathcal{T}_i}, \nu_{\mathcal{T}_i})$ be two collections of FFSSs; then

$$\text{FFHIWG}(\mathcal{L}_1 \otimes \mathcal{T}_1, \mathcal{L}_2 \otimes \mathcal{T}_2, \dots, \mathcal{L}_\eta \otimes \mathcal{T}_\eta) = \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \otimes \text{FFHIWG}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_\eta). \quad (31)$$

Proof. As $\mathcal{L}_i = (\mu_{\mathcal{L}_i}, \nu_{\mathcal{L}_i})$ and $\mathcal{T}_i = (\mu_{\mathcal{T}_i}, \nu_{\mathcal{T}_i})$ are two collections of FFSSs, then

$$\mathcal{L}_i \otimes \mathcal{T}_i = \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma\{(1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3) - (1 - \nu_{\mathcal{L}_i}^3 - \mu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3 - \mu_{\mathcal{T}_i}^3)\}}{(1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3) + (\varsigma - 1)(1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3)}} \\ \sqrt[3]{\frac{(1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3) - (1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3)}{(1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3) + (\varsigma - 1)(1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3)}} \end{array} \right\rangle. \quad (32)$$

Therefore,

$$\begin{aligned} & \text{FFHIWG}(\mathcal{L}_1 \otimes \mathcal{T}_1, \mathcal{L}_2 \otimes \mathcal{T}_2, \dots, \mathcal{L}_\eta \otimes \mathcal{T}_\eta) \\ &= \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^\eta ((1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3))^{K_i} - \prod_{i=1}^\eta ((1 - \nu_{\mathcal{L}_i}^3 - \mu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3 - \mu_{\mathcal{T}_i}^3))^{K_i}\}}{\prod_{i=1}^\eta ((1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3))^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta ((1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3))^{K_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^\eta ((1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3))^{K_i} - \prod_{i=1}^\eta ((1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3))^{K_i}}{\prod_{i=1}^\eta ((1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)(1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3))^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta ((1 - \nu_{\mathcal{L}_i}^3)(1 - \nu_{\mathcal{T}_i}^3))^{K_i}}} \end{array} \right\rangle \\ &= \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3 - \mu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3 - \mu_{\mathcal{T}_i}^3)^{K_i}\}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}} \\ \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i} \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}} \end{array} \right\rangle \quad (33) \\ &= \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3 - \mu_{\mathcal{L}_i}^3)^{K_i}\}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i}}} \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{L}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{L}_i}^3)^{K_i}}} \\ \otimes \left\langle \begin{array}{l} \sqrt[3]{\frac{\varsigma\{\prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3 - \mu_{\mathcal{T}_i}^3)^{K_i}\}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}} \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} - \prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}{\prod_{i=1}^\eta (1 + (\varsigma - 1)\nu_{\mathcal{T}_i}^3)^{K_i} + (\varsigma - 1)\prod_{i=1}^\eta (1 - \nu_{\mathcal{T}_i}^3)^{K_i}}} \end{array} \right\rangle \end{array} \right\rangle \\ &= \text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \otimes \text{FFHIWG}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_\eta). \quad \square \end{aligned}$$

Property 7. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ and $\mathcal{T} = (\mu, \nu)$ be FFSs and $\eta > 0$; then

$$\text{FFHIWG}(\eta\mathcal{L}_1 \otimes \mathcal{T}, \eta\mathcal{L}_2 \otimes \mathcal{T}, \dots, \eta\mathcal{L}_\eta \otimes \mathcal{T}) = \eta\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \otimes \mathcal{T}. \tag{34}$$

Proof. By applying Properties 1, 5, and 6, we can prove it. \square

Definition 7. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\eta)^T$ be its WV, such that $\kappa_i > 0$ and $\sum_{i=1}^\eta \kappa_i = 1$; then FFHIOWG: $\Omega^\eta \rightarrow \Omega$ is defined as

4. Fermatean Fuzzy HIOWG Operators

This section elaborates the notion of Hamacher interactive ordered weighted geometric operators (HIOWGO) under FFSs.

$$\text{FFHIWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \kappa_1 \mathcal{L}_{\varrho(1)} \otimes \kappa_2 \mathcal{L}_{\varrho(2)} \otimes \dots \otimes \kappa_\eta \mathcal{L}_{\varrho(\eta)}, \tag{35}$$

where $(\varrho(1), \varrho(2), \dots, \varrho(\eta))$ is a permutation of $(1, 2, \dots, \eta)$, such that $\varrho(i-1) \geq \varrho(i)$ for any i .

Theorem 3. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs; then

$$\text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \left\langle \sqrt[3]{\frac{\zeta \left\{ \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i} - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3 - \mu_{\varrho(i)}^3)^{\kappa_i} \right\}}{\prod_{i=1}^\eta (1 + (\zeta - 1)\nu_{\varrho(i)}^3)^{\kappa_i} + (\zeta - 1) \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}}, \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + (\zeta - 1)\nu_{\varrho(i)}^3)^{\kappa_i} - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}{\prod_{i=1}^\eta (1 + (\zeta - 1)\nu_{\varrho(i)}^3)^{\kappa_i} + (\zeta - 1) \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}} \right\rangle. \tag{36}$$

Proof. The proof is similar to that of Theorem 1. \square

(i) For $\zeta = 1$, FFHIOWGO becomes FF interactive ordered weighted geometric operator (FFIOWGO):

Remark 2. The two cases of the FFHIOWG operator are as follows:

$$\text{FFIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \left\langle \sqrt[3]{\prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i} - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3 - \mu_{\varrho(i)}^3)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}} \right\rangle. \tag{37}$$

(ii) For $\zeta = 2$, FFHIOWGO becomes FF Einstein interactive ordered weighted geometric operator (FFEIOWGO) operator:

$$\text{FFEIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \left\langle \sqrt[3]{\frac{2\{\prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i} - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3 - \mu_{\varrho(i)}^3)^{\kappa_i}\}}{\prod_{i=1}^\eta (1 + \nu_{\varrho(i)}^3)^{\kappa_i} + \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}}, \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + \nu_{\varrho(i)}^3)^{\kappa_i} - \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}{\prod_{i=1}^\eta (1 + \nu_{\varrho(i)}^3)^{\kappa_i} + \prod_{i=1}^\eta (1 - \nu_{\varrho(i)}^3)^{\kappa_i}}}} \right\rangle. \quad (38)$$

Property 8. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\eta)^T$ its WV such that $\kappa_i > 0$ and $\sum_{i=1}^\eta \kappa_i = 1$.

(i) Idempotency: if $\mathcal{L}_i = \mathcal{L}_o = (\mu_o, \nu_o), \forall i$, then

$$\text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \mathcal{L}_o. \quad (39)$$

(ii) Boundedness: let $\mathcal{L}^- = (\min_i(\mu_i), \max_i(\nu_i))$ and $\mathcal{L}^+ = (\max_i(\mu_i), \min_i(\nu_i))$; then

$$\mathcal{L}^- \leq \text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \leq \mathcal{L}^+. \quad (40)$$

(iii) Monotonicity: when $\mathcal{L}_i \leq \mathcal{T}_i, \forall i$, then

$$\text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \leq \text{FFHIOWG}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_\eta). \quad (41)$$

(iv) Shift invariance: if $\mathcal{T} = (\mu_{\mathcal{T}}, \nu_{\mathcal{T}})$ is another FFS, then

$$\text{FFHIOWG}(\mathcal{L}_1 \otimes \mathcal{T}, \mathcal{L}_2 \otimes \mathcal{T}, \dots, \mathcal{L}_\eta \otimes \mathcal{T}) = \text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) \otimes \mathcal{T}. \quad (42)$$

(v) Homogeneity: let $\lambda > 0$; then,

$$\text{FFHIOWG}(\lambda \mathcal{L}_1, \lambda \mathcal{L}_2, \dots, \lambda \mathcal{L}_\eta) = \lambda \text{FFHIOWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta). \quad (43)$$

Proof. The proof is similar to that for FFHIWG operator. \square

5. Fermatean Fuzzy Hamacher Interactive Hybrid Weighted Geometric Operators

Definition 8. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs; then, FFHIHWG: $\Omega^\eta \rightarrow \Omega$ is defined as

$$\text{FFHIHWG}(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \kappa_1 \dot{\mathcal{L}}_1 \otimes \kappa_2 \dot{\mathcal{L}}_2 \otimes \dots \otimes \kappa_\eta \dot{\mathcal{L}}_\eta, \quad (44)$$

where 'WV' associated with FFHIHWG is $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\eta)^T$ and $\phi = (\phi_1, \phi_2, \dots, \phi_\eta)^T$ is the WV of \mathcal{L}_i , such that $\phi_i \in [0, 1]$ and $\sum_{i=1}^\eta \phi_i = 1$. Consider \mathcal{L} is the i th largest among all the weighted FFSs ($\mathcal{L} = \eta \phi_i \mathcal{L}_i$) and $(\varrho(1), \varrho(2), \dots, \varrho(\eta))$ is a permutation of $(1, 2, \dots, \eta)$, such that $\varrho(i-1) \geq \varrho(i)$ for any i .

Theorem 4. Let $\mathcal{L}_i = (\mu_i, \nu_i)$ be a collection of FFSs; then,

$$FFHIHWG(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_\eta) = \left\langle \sqrt[3]{\frac{\zeta \left\{ \prod_{i=1}^\eta (1 - \dot{\nu}_{\varrho(i)}^3)^{K_i} - \prod_{i=1}^\eta (1 - \dot{\nu}_{\varrho(i)}^3 - \dot{\mu}_{\varrho(i)}^3)^{K_i} \right\}}{\prod_{i=1}^\eta (1 + (\zeta - 1) \dot{\nu}_{\varrho(i)}^3)^{K_i} + (\zeta - 1) \prod_{i=1}^\eta (1 - \dot{\nu}_{\varrho(i)}^3)^{K_i}}}, \sqrt[3]{\frac{\prod_{i=1}^\eta (1 + (\zeta - 1) \dot{\nu}_{\varrho(i)}^3)^{K_i} - \prod_{i=1}^\eta (1 - \dot{\nu}_{\varrho(i)}^3)^{K_i}}{\prod_{i=1}^\eta (1 + (\zeta - 1) \dot{\nu}_{\varrho(i)}^3)^{K_i} + (\zeta - 1) \prod_{i=1}^\eta (1 - \dot{\nu}_{\varrho(i)}^3)^{K_i}}} \right\rangle. \quad (45)$$

Proof. It is the same as that of Theorem 1. □

Remark 3. The FFHIHWGO also satisfies the same results as given in Property 8.

6. Decision-Making Analysis under FFE

In a MADMM, it is a difficult task for decision makers to select the most feasible alternative among the given choices. Let $\{\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_\eta\}$ be η distinct alternatives that can be characterized under a set of m different criteria $\{c_1, c_2, \dots, c_m\}$ taken by decision makers. Suppose that decision makers express their preference values in terms of FFSs $\alpha_{ij} = (\mu_{ij}, \nu_{ij})$ ($i = 1, 2, \dots, \eta; j = 1, 2, \dots, m$), where μ_{ij} and ν_{ij} are the BD and NBD of each alternative, respectively, corresponding to the given attribute such that $0 \leq \mu_{ij}^3 + \nu_{ij}^3 \leq 1$. There are different steps of MADM problem, given as follows:

Step 1. Decision criteria can be grouped into two opposite categories, usually called the “benefit” and the “cost” criteria. Benefit criteria may be called “reward” criteria and cost criteria “regret” or “loss” criteria. A benefit criterion means that the higher an alternative scores in terms of it, the better the alternative is. The opposite is considered true for the cost criteria. The normalized Fermatean fuzzy decision matrix (FFDMx) is obtained by interchanging the assessment values of cost attributes (CAs) with benefit attributes (BAs) [36]; that is,

$$\mathcal{P}_{ij} = \begin{cases} \alpha_{ij}^c, & \text{for CAs,} \\ \alpha_{ij}, & \text{for BAs.} \end{cases} \quad (46)$$

Step 2. By using the DMx given in Step 1, the overall aggregated value of alternatives \mathfrak{S}_i under different

choices of attributes c_j is obtained by using any of FFHIWG, FFHIOWG, or FFHIHWG operators.

Step 3. The score function is used to determine the score values of the alternatives.

Step 4. The alternatives $\{\mathfrak{S}_1, \mathfrak{S}_2, \dots, \mathfrak{S}_\eta\}$ are firstly arranged in descending order according to the score values and after it, the most appropriate alternative is selected.

A flowchart for the proposed MADM is shown in Figure 1.

6.1. Numerical Example. This section presents an application of the proposed MADM model in the field of medical science. Suppose that a doctor wants to select a patient who is more affected by cancer. Let $\{A_1, A_2, A_3, A_4, A_5\}$ be a set of patients and $\{C_1, C_2, C_3, C_4\}$ be a set of attributes with WV $\{0.3, 0.3, 0.2, 0.2\}$. The following are the most important factors which increase the chances of cancer in a patient:

- C_1 stands for tobacco,
 - C_2 stands for alcohol,
 - C_3 stands for unhealthy diet,
 - C_4 stands for physical inactivity.
- (47)

By FFHIWG operator, see the following

Step 1: since the criteria set is of the same type, DMx cannot be normalized.

Step 2: to determine the combined assessment of each alternative, we apply the FFHIWGO for $\zeta = 2$ as follows.

For \mathcal{P}_1 ,

$$\begin{aligned} &= FFHIWG(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}, \mathfrak{S}_{14}) \\ &= \left\langle \sqrt[3]{\frac{2 \left((1 - 0.6^3)^{0.3} (1 - 0.1^3)^{0.3} (1 - 0.4^3)^{0.2} (1 - 0.3^3)^{0.2} - (1 - 0.6^3 - 0.9^3)^{0.3} (1 - 0.1^3 - 0.6^3)^{0.3} (1 - 0.4^3 - 0.9^3)^{0.2} (1 - 0.3^3 - 0.5^3)^{0.2} \right)}{(1 + 0.6^3)^{0.3} (1 + 0.1^3)^{0.3} (1 + 0.4^3)^{0.2} (1 + 0.3^3)^{0.2} + (1 - 0.6^3)^{0.3} (1 - 0.1^3)^{0.3} (1 - 0.4^3)^{0.2} (1 - 0.3^3)^{0.2}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{(1 + 0.6^3)^{0.3} (1 + 0.1^3)^{0.3} (1 + 0.4^3)^{0.2} (1 + 0.3^3)^{0.2} - (1 - 0.6^3)^{0.3} (1 - 0.1^3)^{0.3} (1 - 0.4^3)^{0.2} (1 - 0.3^3)^{0.2}}{(1 + 0.6^3)^{0.3} (1 + 0.1^3)^{0.3} (1 + 0.4^3)^{0.2} (1 + 0.3^3)^{0.2} + (1 - 0.6^3)^{0.3} (1 - 0.1^3)^{0.3} (1 - 0.4^3)^{0.2} (1 - 0.3^3)^{0.2}}} \right\rangle \\ &= (0.87, 0.56). \end{aligned} \quad (48)$$

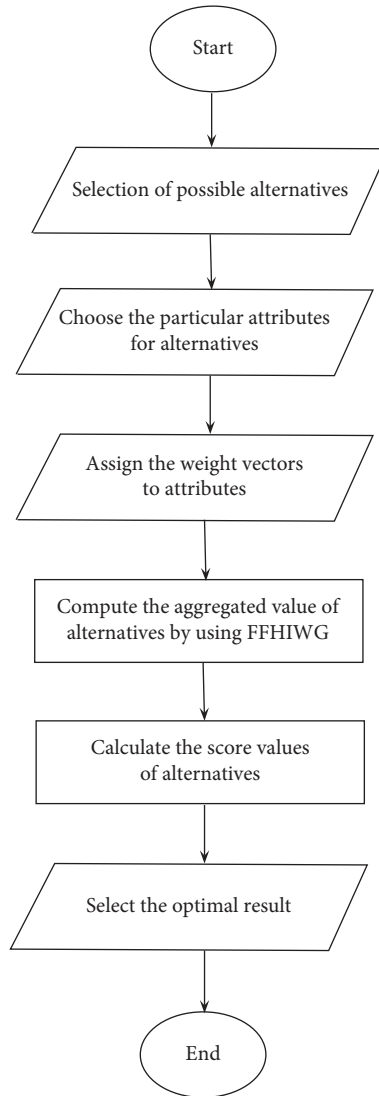


FIGURE 1: Flowchart for the selection of the best alternative.

For \mathcal{P}_2 ,

$$= \left\langle \sqrt[3]{\frac{2\left(\left(1-0.4^3\right)^{0.3}\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.2}\left(1-0.2^3\right)^{0.2}-\left(1-0.4^3-0.7^3\right)^{0.3}\left(1-0.4^3-0.8^3\right)^{0.3}\left(1-0.5^3-0.8^3\right)^{0.2}\left(1-0.2^3-0.7^3\right)^{0.2}\right)}{\left(1+0.4^3\right)^{0.3}\left(1+0.4^3\right)^{0.3}\left(1+0.5^3\right)^{0.2}\left(1+0.2^3\right)^{0.2}+\left(1-0.4^3\right)^{0.3}\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.2}\left(1-0.2^3\right)^{0.2}}}, \sqrt[3]{\frac{\left(1+0.4^3\right)^{0.3}\left(1+0.4^3\right)^{0.3}\left(1+0.5^3\right)^{0.2}\left(1+0.2^3\right)^{0.2}-\left(1-0.4^3\right)^{0.3}\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.2}\left(1-0.2^3\right)^{0.2}}{\left(1+0.4^3\right)^{0.3}\left(1+0.4^3\right)^{0.3}\left(1+0.5^3\right)^{0.2}\left(1+0.2^3\right)^{0.2}+\left(1-0.4^3\right)^{0.3}\left(1-0.4^3\right)^{0.3}\left(1-0.5^3\right)^{0.2}\left(1-0.2^3\right)^{0.2}}}\right\rangle$$

$= (0.76, 0.37).$

(49)

For \mathcal{P}_3 ,

$$= \left\langle \sqrt[3]{\frac{2\left((1-0.6^3)^{0.3}(1-0.6^3)^{0.3}(1-0.6^3)^{0.2}(1-0.4^3)^{0.2} - (1-0.6^3-0.7^3)^{0.3}(1-0.6^3-0.4^3)^{0.3}(1-0.6^3-0.9^3)^{0.2}(1-0.4^3-0.8^3)^{0.2}\right)}{(1+0.6^3)^{0.3}(1+0.6^3)^{0.3}(1+0.6^3)^{0.2}(1+0.4^3)^{0.2} + (1-0.6^3)^{0.3}(1-0.6^3)^{0.3}(1-0.6^3)^{0.2}(1-0.4^3)^{0.2}}}, \right. \\ \left. \sqrt[3]{\frac{(1+0.6^3)^{0.3}(1+0.6^3)^{0.3}(1+0.6^3)^{0.2}(1+0.4^3)^{0.2} - (1-0.6^3)^{0.3}(1-0.6^3)^{0.3}(1-0.6^3)^{0.2}(1-0.4^3)^{0.2}}{(1+0.6^3)^{0.3}(1+0.6^3)^{0.3}(1+0.6^3)^{0.2}(1+0.4^3)^{0.2} + (1-0.6^3)^{0.3}(1-0.6^3)^{0.3}(1-0.6^3)^{0.2}(1-0.4^3)^{0.2}}} \right\rangle \\ = (0.78, 0.57). \quad (50)$$

For \mathcal{P}_4 ,

$$= \text{FFHIWG}(\mathfrak{E}_{11}, \mathfrak{E}_{12}, \mathfrak{E}_{13}, \mathfrak{E}_{14}) \\ = \left\langle \sqrt[3]{\frac{2\left((1-0.2^3)^{0.3}(1-0.1^3)^{0.3}(1-0.4^3)^{0.2}(1-0.2^3)^{0.2} - (1-0.2^3-0.9^3)^{0.3}(1-0.1^3-0.9^3)^{0.3}(1-0.4^3-0.7^3)^{0.2}(1-0.2^3-0.4^3)^{0.2}\right)}{(1+0.2^3)^{0.3}(1+0.1^3)^{0.3}(1+0.4^3)^{0.2}(1+0.2^3)^{0.2} + (1-0.2^3)^{0.3}(1-0.1^3)^{0.3}(1-0.4^3)^{0.2}(1-0.2^3)^{0.2}}}, \right. \\ \left. \sqrt[3]{\frac{(1+0.2^3)^{0.3}(1+0.1^3)^{0.3}(1+0.4^3)^{0.2}(1+0.2^3)^{0.2} - (1-0.2^3)^{0.3}(1-0.1^3)^{0.3}(1-0.4^3)^{0.2}(1-0.2^3)^{0.2}}{(1+0.2^3)^{0.3}(1+0.1^3)^{0.3}(1+0.4^3)^{0.2}(1+0.2^3)^{0.2} + (1-0.2^3)^{0.3}(1-0.1^3)^{0.3}(1-0.4^3)^{0.2}(1-0.2^3)^{0.2}}} \right\rangle \\ = (0.92, 0.27). \quad (51)$$

For \mathcal{P}_5 ,

$$= \text{FFHIWG}(\mathfrak{E}_{11}, \mathfrak{E}_{12}, \mathfrak{E}_{13}, \mathfrak{E}_{14}) \\ = \left\langle \sqrt[3]{\frac{2\left((1-0.1^3)^{0.3}(1-0.4^3)^{0.3}(1-0.3^3)^{0.2}(1-0.4^3)^{0.2} - (1-0.1^3-0.8^3)^{0.3}(1-0.4^3-0.7^3)^{0.3}(1-0.3^3-0.6^3)^{0.2}(1-0.4^3-0.7^3)^{0.2}\right)}{(1+0.1^3)^{0.3}(1+0.4^3)^{0.3}(1+0.3^3)^{0.2}(1+0.4^3)^{0.2} + (1-0.1^3)^{0.3}(1-0.4^3)^{0.3}(1-0.3^3)^{0.2}(1-0.4^3)^{0.2}}}, \right. \\ \left. \sqrt[3]{\frac{(1+0.1^3)^{0.3}(1+0.4^3)^{0.3}(1+0.3^3)^{0.2}(1+0.4^3)^{0.2} - (1-0.1^3)^{0.3}(1-0.4^3)^{0.3}(1-0.3^3)^{0.2}(1-0.4^3)^{0.2}}{(1+0.1^3)^{0.3}(1+0.4^3)^{0.3}(1+0.3^3)^{0.2}(1+0.4^3)^{0.2} + (1-0.1^3)^{0.3}(1-0.4^3)^{0.3}(1-0.3^3)^{0.2}(1-0.4^3)^{0.2}}} \right\rangle \\ = (0.72, 0.34). \quad (52)$$

Step 3: the score values for each alternative are given as

$$\begin{aligned} S(\mathfrak{E}_1) &= 0.56, \\ S(\mathfrak{E}_2) &= 0.37, \\ S(\mathfrak{E}_3) &= 0.29, \\ S(\mathfrak{E}_4) &= 0.76, \\ S(\mathfrak{E}_5) &= 0.33. \end{aligned} \quad (53)$$

Step 4: as $\mathfrak{E}_4 > \mathfrak{E}_1 > \mathfrak{E}_2 > \mathfrak{E}_5 > \mathfrak{E}_3$, hence, C_4 is the person, who is most affected by cancer.

7. Influence of Distinct Parameter's Values

An attribute ζ performs a crucial role in ranking results. In this section, we observe the score functions and ranking results based on FFHIWGO, under different values of ζ . Here, we observed some of the following influences:

- (1) The influence of attributes ζ on alternatives' ranking by applying the FFHIWGO
- (2) The effect of attributes ζ on alternatives' ranking by using the FFHIOWGO

TABLE 1: Ranking order using the FFHIWG operator.

ς	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Ranking order
1	0.54	0.37	0.32	0.76	0.33	$A_4 > A_1 > A_2 > A_5 > A_3$
2	0.56	0.37	0.29	0.76	0.33	$A_4 > A_1 > A_2 > A_5 > A_3$
3	0.56	0.37	0.29	0.76	0.32	$A_4 > A_1 > A_2 > A_5 > A_3$
4	0.58	0.37	0.29	0.76	0.34	$A_4 > A_1 > A_2 > A_5 > A_3$
5	0.58	0.37	0.31	0.76	0.33	$A_4 > A_1 > A_2 > A_5 > A_3$

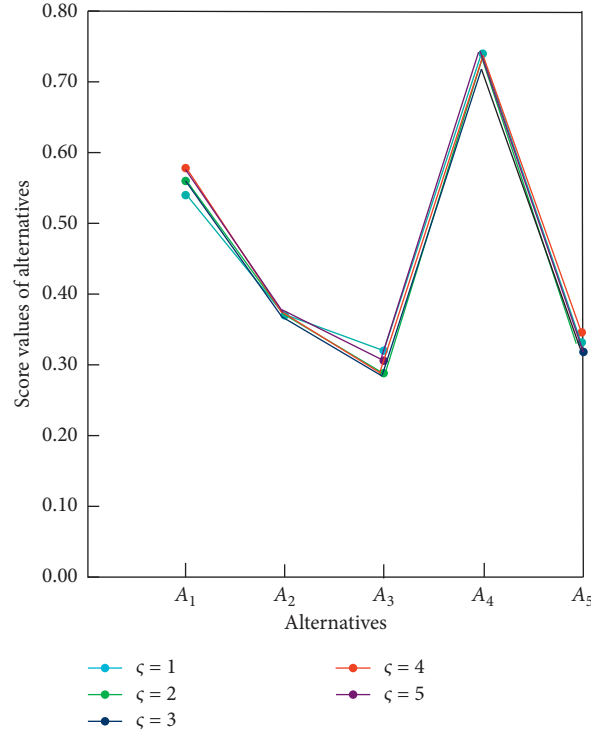


FIGURE 2: Ranking order for different values of ς .

(3) The influence of attributes ς on alternatives' ranking by applying FFHIHGO

From Table 1 and Figure 2, we can see that alternatives may have the same or different score values but the ranking order remains the same for all the attributes.

8. A Comparative Analysis and Discussion

To view the importance and validity of proposed AOs, we give a comparative analysis of the proposed model with the different existing models. From Table 2, it is clear that there are some values for which $\mu + \nu > 1$ and $\mu^2 + \nu^2 > 1$. Therefore, the methods presented in [18, 23] failed to handle such problems.

8.1. Comparison with Fermatean Fuzzy TOPSIS Method. This section presents the comparison of the proposed work with the FF-TOPSIS method [7]. The steps are given as follows:

(1) Table 2 provides the FFDM in which each entry is a FFS.

(2) The Fermatean fuzzy positive ideal solution (FFPIS) \mathcal{H}^+ and Fermatean fuzzy negative ideal solution (FFNIS) \mathcal{H}^- are

$$\begin{aligned} \mathcal{H}^+ &= \{(0.9, 0.2), (0.9, 0.1), (0.9, 0.4), (0.8, 0.4)\}, \\ \mathcal{H}^- &= \{(0.7, 0.6), (0.4, 0.6), (0.6, 0.3), (0.4, 0.2)\}. \end{aligned} \tag{54}$$

(3) The distance of alternatives A_i from FFPIS \mathcal{H}^+ and FFNIS \mathcal{H}^- is given in Table 3.

(4) The revised closeness degree of each alternative is given:

$$\begin{aligned} \xi(A_1) &= -0.92, \\ \xi(A_2) &= -0.45, \\ \xi(A_3) &= -1.038, \\ \xi(A_4) &= 0, \\ \xi(A_5) &= -1.16. \end{aligned} \tag{55}$$

(5) Arrange the alternatives in descending order with respect to $\xi(A_i)$:

TABLE 2: FF decision matrix.

α_{ij}	C_1	C_2	C_3	C_4
A_1	(0.9, 0.6)	(0.6, 0.1)	(0.9, 0.4)	(0.5, 0.3)
A_2	(0.7, 0.4)	(0.8, 0.4)	(0.8, 0.5)	(0.7, 0.2)
A_3	(0.7, 0.6)	(0.4, 0.6)	(0.9, 0.6)	(0.8, 0.4)
A_4	(0.9, 0.2)	(0.9, 0.1)	(0.7, 0.4)	(0.4, 0.2)
A_5	(0.8, 0.1)	(0.7, 0.4)	(0.6, 0.3)	(0.7, 0.4)

TABLE 3: Distance of alternatives from FFPIS and FFNIS.

$D(A_i, \mathcal{H}^+)$	$D(A_i, \mathcal{H}^-)$
0.298	0.242
0.245	0.293
0.307	0.222
0.174	0.306
0.314	0.196

$$A_4 > A_2 > A_1 > A_3 > A_5. \quad (56)$$

(6) Thus, A_4 is the best alternative.

It is clear that ranking lists is slightly different from both methods but the most suitable alternative from both methods is A_4 . Though, FF-TOPSIS method is a useful technique to solve MADMMs, but there are also some issues which cannot be resolved by adopting this method. Therefore, such issues can be tackled through the proposed model and the results obtained from it are closer to the original results.

Advantages and limitations of intended AOs: the benefit and motivation behind this intended approach are as follows:

- (i) The other membership grades in the aggregated value affect the other grades even if BD of any alternative is zero.
- (ii) There is a relationship between the BD and NBD of an alternative.
- (iii) However, there are some limitations of this model. It cannot be applied in situations where we take the parameters for the evaluation of anything. It means this theory has a lack of parameterization property.

9. Conclusions

A FFS having emerging applications in MADM, is a more efficient technique to cope with ambiguities involved in the given data, as compared to IFSs and PFSs. AOs are very useful for evaluating the given alternatives in DM process because they integrate the evaluation values of all the given individuals into a unified form. By comparing other AOs, the structure of Hamacher's norms is a more general framework, which effectively integrates complex information. In this article, we have intended some FF-Hamacher interactive geometric AOs such as FFHIWGO, FFHIOWGO, and FFHIHWGO. These intended operators have some useful

characteristics, for example, boundedness, homogeneity, idempotency, monotonicity, and shift invariance. Some special cases of the proposed AOs have been also explained. The interaction between belongingness and non-belongingness degrees has been discussed in the proposed HIAOs. We have explained a MADM algorithm to cope with uncertainties present in decision problems. To show the efficacy and applicability of the intended MADM, we have applied this method in a real-life system. We have presented a comparative analysis of our model with some of the existing models and concluded that our model is more flexible and it depicts ambiguous and inexact information in complex structures. Thus, the AOs due to their highly adaptable nature are very crucial. Therefore, taking into account this direction further we will explore some more properties and types of these operators in Fermatean fuzzy soft sets and q -rung orthopair fuzzy soft sets. Moreover, we will extend our work on the following decision-making problems:

- (1) An in-depth study of the Hamacher AOs for Fermatean fuzzy information such as induced Fermatean fuzzy Hamacher interactive AOs and q -rung picture fuzzy Hamacher interactive AOs will be a hot topic in the future
- (2) A MADM problem in medical diagnosis under Fermatean fuzzy soft data using Hamacher interactive AOs will be discussed
- (3) A MADM problem for the selection of a smartphone under Fermatean fuzzy soft data using Hamacher interactive AOs will be discussed

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare no conflicts of interest.

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