

# Multidimensional Integral Transformations

Yu.A. Brychkov

*Computing Center, Russian Academy of Sciences, Moscow*

H.-J. Glaeske

*Friedrich Schiller University, Jena, Germany*

A.P. Prudnikov

*Computing Center, Russian Academy of Sciences, Moscow*

Vu Kim Tuan

*Hanoi Institute of Mathematics, Vietnam*

GORDON AND BREACH SCIENCE PUBLISHERS  
Philadelphia • Reading • Paris • Montreux • Tokyo • Melbourne

# Contents

Preface .....	xi
<b>0. Notations and Recapitulations .....</b>	<b>1</b>
0.1. Numbers .....	1
0.2. Euclidean and Unitary Spaces .....	1
0.3. Multiindices .....	3
0.4. Functions .....	4
0.5. Spaces of Functions .....	5
0.6. Some Theorems for $L_1$ -Functions .....	7
<b>1. The Fourier Transformation .....</b>	<b>9</b>
1.1. The Fourier Transform of $L_1$ -Functions .....	10
1.1.1. Definition and Basic Properties .....	10
1.1.2. Operational Properties .....	13
1.1.3. Fourier Transforms of Radially Symmetric Functions .....	18
1.1.4. The Inversion of the Fourier Transform .....	20
1.1.4.1. Preliminaries .....	20
1.1.4.2. Inversion Formulas .....	24
1.2. The Fourier Transform of $L_2$ -Functions .....	30
1.3. The Fourier Transform of Distributions .....	36
1.3.1. Spaces of Test Functions and Distributions .....	36
1.3.2. The Fourier Transform of Rapidly Decreasing Functions .....	43
1.3.3. The Fourier Transform of Tempered Distributions .....	45
1.3.3.1. Definition .....	45

1.3.3.2.	Rules of Operational Calculus .....	47
1.3.4.	Fourier Transforms and Entire Functions.....	50
<b>1.4.</b>	<b>Applications to Partial Differential Equations .....</b>	<b>53</b>
1.4.1.	General Remarks .....	53
1.4.2.	The Heat Equation .....	57
1.4.3.	The Fundamental Solution of the Schrödinger Operator .....	61
1.4.4.	The Wave Equation .....	63
1.4.5.	The Laplace and the Poisson Equations .....	70
1.4.6.	The Biharmonic Equation .....	73
1.4.7.	The Helmholtz Equation .....	75
<b>2.</b>	<b>The Laplace Transformation.....</b>	<b>81</b>
<b>2.1.</b>	<b>Foundations .....</b>	<b>82</b>
2.1.1.	Definition and Basic Properties .....	82
2.1.2.	The Domain of Convergence .....	87
2.1.3.	Uniqueness Theorems .....	91
2.1.4.	Laplace Transform and Development in Series.....	92
<b>2.2.</b>	<b>Operational Properties .....</b>	<b>93</b>
2.2.1.	Elementary Operations .....	93
2.2.2.	Derivatives and Integrals in the Original and in the Image Domains .....	95
2.2.3.	The Formula of Parseval .....	100
<b>2.3.</b>	<b>The Inversion of the Laplace Transform .....</b>	<b>100</b>
2.3.1.	The Complex Inversion Formula .....	100
2.3.2.	Inversion Formulas of the Limit Type .....	106
2.3.3.	Practical Determination of the Original .....	110
2.3.4.	Inversion by Development into Series .....	111
<b>2.4.</b>	<b>Convolution Theorems .....</b>	<b>114</b>
2.4.1.	Convolution in the Original and in the Image Domains .....	114
2.4.2.	Generalized Convolutions .....	117
2.4.3.	Transforms of Fractional Integrals .....	120
<b>2.5.</b>	<b>Asymptotic Behaviour .....</b>	<b>121</b>
2.5.1.	Abelian Theorems .....	121
2.5.2.	Tauberian Theorems.....	129

<b>2.6.</b>	<b>General Relations of Operational Calculus</b> .....	136
2.6.1.	From the One-Dimensional to the Two-Dimensional Case .....	136
2.6.2.	From the One-Dimensional to the $n$ -Dimensional Case .....	144
2.6.3.	Two-Dimensional Examples .....	146
2.6.4.	From the $n$ -Dimensional to the One-Dimensional Case .....	149
<b>2.7.</b>	<b>Some Remarks on the Bilateral Laplace Transformation</b> .....	155
<b>2.8.</b>	<b>The Laplace Transform of Distributions</b> .....	157
2.8.1.	Definition and Basic Properties .....	157
2.8.2.	Operational Rules .....	160
2.8.3.	The Bilateral Laplace Transform .....	162
2.8.4.	Laplace Transformation in $\mathcal{D}'_+$ .....	164
<b>2.9.</b>	<b>Applications</b> .....	166
2.9.1.	Calculation of Integrals .....	166
2.9.2.	Special Functions .....	169
2.9.2.1.	Series Expansions .....	169
2.9.2.2.	Laguerre and Hermite Polynomials of $n$ Variables .....	174
2.9.2.3.	Hyper Bessel Functions .....	175
2.9.3.	Partial Differential Equations .....	176
2.9.3.1.	General Remarks .....	176
2.9.3.2.	Partial Differential Equations of the First Order .....	178
2.9.3.3.	The Wave Equation .....	180
2.9.3.4.	The Heat Equation .....	182
2.9.3.5.	The Poisson Equation .....	185
2.9.4.	Nonlinear Systems and Differential Equations .....	187
<b>3.</b>	<b>The Mellin Transformation</b> .....	<b>193</b>
<b>3.1.</b>	<b>The Mellin Transformation in Weighted <math>L_1</math>-Spaces</b> .....	<b>194</b>
3.1.1.	Definition and Basic Properties .....	194
3.1.2.	The Inversion of the Mellin Transform .....	197
3.1.3.	Operational Rules .....	200
<b>3.2.</b>	<b>The Theorem of Plancherel</b> .....	<b>207</b>
<b>3.3.</b>	<b>The Mellin Transform of Distributions</b> .....	<b>210</b>
3.3.1.	Definition and Basic Properties .....	210
3.3.2.	Operational Properties .....	212

3.3.3.	The Mellin Transform in $\mathcal{D}'(\mathbb{R}_+^n)$ .....	216
<b>4.</b>	<b>Other Integral Transformations</b> .....	<b>219</b>
4.1.	<b>The Mellin Convolution Type Transformation</b> .....	219
4.1.1.	Definition and Main Properties.....	219
4.1.2.	The Spaces $\mathcal{M}_2(\mathbb{K})$ and $\mathcal{M}'_2(\mathbb{K})$ .....	220
4.1.3.	The Spaces $\mathcal{M}(\mathbb{D})$ and $\mathcal{M}'(\mathbb{D})$ .....	223
4.1.4.	Mellin Convolution Type Transformations in $\mathcal{M}_2(\mathbb{K})$ , $\mathcal{M}(\mathbb{D})$ , $\mathcal{M}'_2(\mathbb{K})$ , $\mathcal{M}'(\mathbb{D})$ .....	224
4.2.	<b>Modified Laplace Transformations</b> .....	225
4.2.1.	Definition and Basic Properties.....	225
4.2.2.	Complex Inversion Formula.....	228
4.2.3.	Real Inversion Formula.....	230
4.2.4.	Modified Laplace Transformation in $\mathcal{M}'_2(\mathbb{K})$ and in $\mathcal{M}'(\mathbb{D})$ ...	232
4.2.5.	Second Modified Laplace Transformation.....	232
4.3.	<b>The H-Transformation</b> .....	233
4.3.1.	Definition and Basic Properties.....	233
4.3.2.	Factorization of H-Transformations.....	234
4.3.3.	The Stieltjes Transformations.....	235
4.3.4.	The Hankel Transformations.....	239
4.3.5.	Operators of Integration and Differentiation.....	243
4.3.6.	Other Special Cases of the H-Transformation.....	248
4.4.	<b>Watson Transformations</b> .....	250
4.4.1.	Definition.....	250
4.4.2.	Characterization of Watson Kernels.....	252
4.4.3.	The Hankel Transformation.....	255
4.4.4.	Nonsymmetrical Formulas.....	257
4.4.5.	Unitary Transformations.....	260
4.5.	<b>Bessel and Riesz Potentials</b> .....	264
4.5.1.	Bessel Potentials.....	264
4.5.2.	The Space $\mathcal{L}_p^\alpha$ of Bessel Potentials.....	267
4.5.3.	The Spaces $\Phi(\mathbb{R}^n)$ and $\Phi'(\mathbb{R}^n)$ .....	267
4.5.4.	Riesz Potentials.....	268
4.5.5.	Hypersingular Integrals.....	270
4.5.6.	Riesz-Bessel Potentials.....	272

4.6.	<b>Other Transformations</b> .....	273
4.6.1.	<b>Pyramidal Fractional Integrals</b> .....	273
4.6.2.	<b>A Non-Convolutional Transformation</b> .....	275
4.6.3.	<b>The Radon Transformation</b> .....	278
<b>Appendix I. Tables of Two-Dimensional Laplace Transforms</b> .....		<b>281</b>
1.	<b>Algebraic Functions</b> .....	282
2.	<b>The Exponential Function</b> .....	307
3.	<b>Hyperbolic Functions</b> .....	311
4.	<b>Trigonometric Functions</b> .....	314
5.	<b>The Logarithmic Function</b> .....	318
6.	<b>Inverse Trigonometric Functions</b> .....	322
7.	<b>The Error Function</b> .....	327
8.	<b>The Euler Dilogarithm</b> .....	333
9.	<b>The Bessel Function</b> .....	334
10.	<b>The Modified Bessel Function</b> .....	336
11.	<b>Orthogonal Polynomials</b> .....	340
12.	<b>The Elliptic Integrals</b> .....	347
13.	<b>Generalized Hypergeometric Functions</b> .....	355
<b>Appendix II. Tables of Associated Transforms</b> .....		<b>357</b>
<b>References</b> .....		<b>369</b>
<b>List of Notation</b> .....		<b>377</b>
<b>Index of Symbols</b> .....		<b>381</b>
<b>Index</b> .....		<b>383</b>