

# Multi-Sensor Generalized Sequential Probability Ratio Test Using Level-Triggered Sampling

**Shang Li**

Joint work with: Xiaou Li, Xiaodong Wang, Jingchen Liu

Electrical Engineering Department  
Columbia University

December 14, 2015

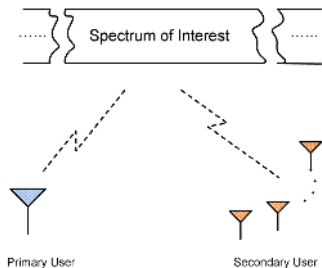
# Motivations

- Hypothesis testing
  - Multiple sensors: Diversity
  - Sequential test: Quick decision
- Applications:
  - Medical sensors: Remote diagnosis and health monitoring



# Motivations

- Hypothesis testing:
  - Multiple sensors: Diversity
  - Sequential test: Quick decision
- Applications:
  - Medical sensors: Remote diagnosis and health monitoring
  - Cognitive radio: Idle spectrum sensing



# Outline

- 1 Problem Statement
- 2 Proposed Multi-Sensor Sequential Test
- 3 Numerical Results

# Outline

- 1 Problem Statement
- 2 Proposed Multi-Sensor Sequential Test
- 3 Numerical Results

# Problem Statement

- Sequential hypothesis testing: Composite v.s. composite

$$\mathcal{H}_0 : Y_t \sim h_\gamma(y), \quad \gamma \in \Gamma, \quad t = 1, 2, \dots$$

$$\mathcal{H}_1 : Y_t \sim f_\theta(y), \quad \theta \in \Theta, \quad t = 1, 2, \dots$$

- Stopping time  $T : \{Y_1, Y_2, \dots\} \rightarrow \mathbb{N}$
- Decision function  $\delta : \{Y_1, \dots, Y_T\} \rightarrow \{0, 1\}$

- Formulation:

$$\inf_{\{T, \delta\}} \mathbb{E}_x T, \quad x \in \Gamma \cup \Theta$$

$$\text{subject to } \sup_{\gamma} \mathbb{P}_\gamma(\delta = 1) \leq \alpha, \quad \sup_{\theta} \mathbb{P}_\theta(\delta = 0) \leq \beta$$

⇒ Asymptotic optimality (as  $\alpha, \beta \rightarrow 0$ )

# Problem Statement

- Define the feasible class of sequential tests:

$$\mathcal{T}(\alpha, \beta) \triangleq \left\{ \{T, \delta\} : \sup_{\gamma} \mathbb{P}_{\gamma}(\delta = 1) \leq \alpha, \sup_{\theta} \mathbb{P}_{\theta}(\delta = 0) \leq \beta \right\}$$

## Theorem (Li&Liu'14)

For any sequential test  $\{T, \delta\} \in \mathcal{T}(\alpha, \beta)$ , its expected sample size is lower bounded by

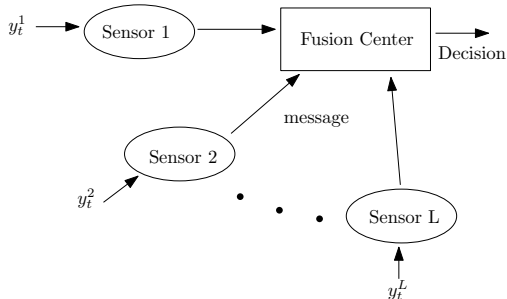
$$\mathbb{E}_{\gamma} (T) \geq \frac{-\log \beta}{\inf_{\theta \in \Theta} D(h_{\gamma} || f_{\theta})} + o(-\log \beta)$$

$$\mathbb{E}_{\theta} (T) \geq \frac{-\log \alpha}{\inf_{\gamma \in \Gamma} D(f_{\theta} || h_{\gamma})} + o(-\log \alpha)$$

Kullback-Leibler divergence:  $D(f || h) = \mathbb{E}_f \left( \log \frac{f(X)}{h(X)} \right)$

# Problem Statement

- Challenges:
  - Low battery at sensors
  - Limited wireless bandwidth
- Remedies:
  - Lower precision (**one-bit message**)
  - Lower frequency (**every  $T_0$  steps, fixed or on average**)



⇒ **Can we devise a good test under this setup?**

# Outline

- 1 Problem Statement
- 2 Proposed Multi-Sensor Sequential Test**
- 3 Numerical Results

# Generalized Sequential Probability Ratio Test

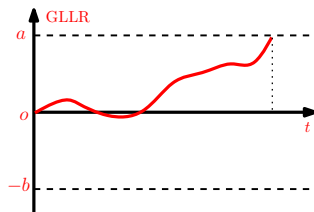
- Statistic: Generalized log-likelihood ratio (GLLR)

$$\text{GLLR}(1, t) \triangleq \max_{\theta \in \Theta} \sum_{j=1}^t \log f_{\theta}(y_j^{\ell}) - \max_{\gamma \in \Gamma} \sum_{j=1}^t \log h_{\gamma}(y_j^{\ell})$$

- Stopping rule:  $T \triangleq \inf \{t : \text{GLLR}(1, t) \notin (-b, a)\}$

- Decision function:  $\delta_T \triangleq \mathbb{1}_{\{\text{GLLR}(1, T) \geq a\}}$

⇒ Asymptotically optimal



# Proposed: Level-Triggered Sampling

- **At sensors:**

- When to transmit:

$$t_n \triangleq \inf \{t : \text{GLLR}(t_{n-1}, t) \notin (-b, a)\}, \quad n = 1, 2, \dots, t_0 = 0$$

- What to transmit:

$$u_n \triangleq \begin{cases} +1, & \text{if } \text{GLLR}(t_{n-1}, t_n) \geq a \\ -1, & \text{if } \text{GLLR}(t_{n-1}, t_n) \leq -b \end{cases}$$

- In brief: Repeat GSPRT  $\rightarrow$  local decisions  $\rightarrow$  binary messages

# Proposed: Level-Triggered Sampling

- **At fusion center:**

- Global statistic:

$$V_t \triangleq \sum_{\ell=1}^L \sum_{n=1}^{N_t^\ell} \left( a \mathbb{1}_{\{u_n^\ell=1\}} - b \mathbb{1}_{\{u_n^\ell=-1\}} \right)$$

- Stopping rule & decision rule:

$$T_p \triangleq \inf \{t : V_t \notin (-B, A)\}, \quad \delta_p \triangleq \mathbb{1}_{\{V_{T_p} \geq A\}}$$

- In brief:

Sum up the local boundary values, and make decision as the sum hits either global boundary.

# Performance

## Lemma (Expected Stopping Time)

*In the asymptotic regime where  $b, a \rightarrow \infty$  and  $A/a, B/b \rightarrow \infty$ , LTS-GSPRT yields the following expected sample size:*

$$\mathbb{E}_\gamma (\mathsf{T}_p) \sim \frac{B}{\inf_\theta D(h_\gamma \| f_\theta) L}, \quad \mathbb{E}_\theta (\mathsf{T}_p) \sim \frac{A}{\inf_\gamma D(f_\theta \| h_\gamma) L}.$$

## Lemma (Error Probabilities)

*In the asymptotic regime where  $b, a \rightarrow \infty$  and  $A/a, B/b \rightarrow \infty$ , the LTS-GSPRT yields the following type-I and type-II error probabilities:*

$$\sup_{\gamma \in \Gamma} \log \mathbb{P}_\gamma (\delta_p = 1) \sim -A, \quad \sup_{\theta \in \Theta} \log \mathbb{P}_\theta (\delta_p = 0) \sim -B.$$

# Performance

- Asymptotic performance:

$$\mathbb{E}_\gamma(T_p) \sim \frac{-\log \beta}{\inf_{\theta \in \Theta} D(h_\gamma \| f_\theta) L}, \quad \mathbb{E}_\theta(T_p) \sim \frac{-\log \alpha}{\inf_{\gamma \in \Gamma} D(f_\theta \| h_\gamma) L}$$

## Theorem

The decentralized LTS-GSPRT  $\{T_p, \delta_p\}$  is asymptotically optimal, in the asymptotic regime where  $b, a \rightarrow \infty$  and  $A/a, B/b \rightarrow \infty$ :

$$\mathbb{E}_x(T_p) \sim \inf_{\{T, \delta\} \in \mathcal{T}(\alpha, \beta)} \mathbb{E}_x(T), \quad x \in \Gamma \cup \Theta.$$

# Outline

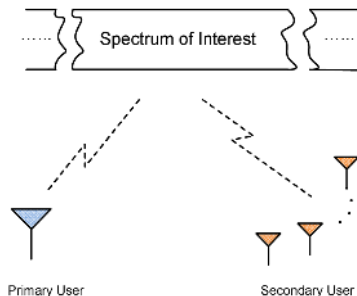
- 1 Problem Statement
- 2 Proposed Multi-Sensor Sequential Test
- 3 Numerical Results**

# Application: Cooperative Spectrum Sensing

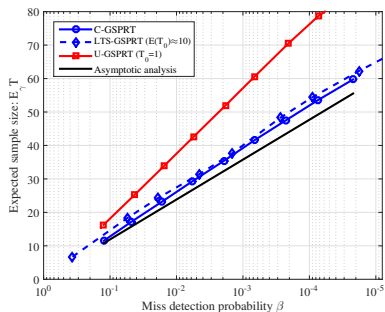
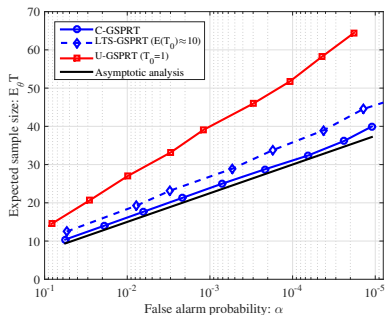
- Spectrum Sensing Model [Sergura&Wang'10]:

$$\mathcal{H}_0 : Y_t^\ell \sim \mathcal{N}(0, \gamma), \quad 0 < \gamma_0 \leq \gamma \leq \gamma_1, \quad \ell \in \mathcal{L}, t = 1, 2, \dots,$$

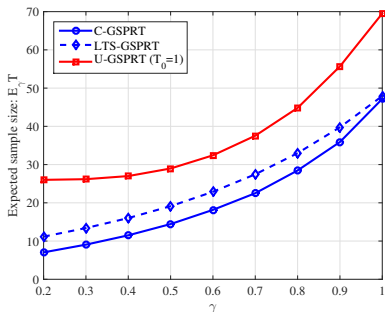
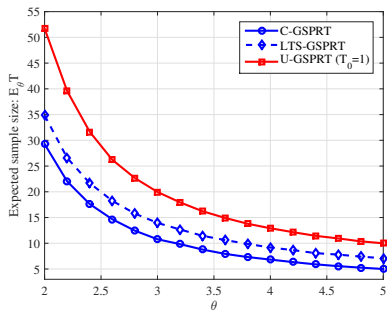
$$\mathcal{H}_1 : Y_t^\ell \sim \mathcal{N}(0, \theta), \quad \gamma_1 < \theta_0 \leq \theta \leq \theta_1 \quad \ell \in \mathcal{L}, t = 1, 2, \dots,$$



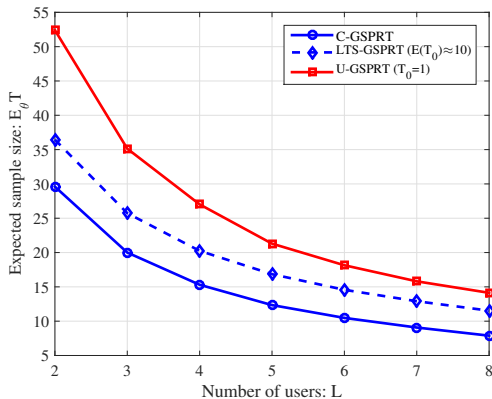
## Results



## Results



# Results



# Q&A

