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Minimum Aberration Two-Level Split-Plot Designs
by
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A Thesis Submitted
in
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ABSTRACT

In this thesis a certain aspect of fractionated two-level split-plot designs is associated with a subset of the 2^{n-k} fractional factorial designs. The concept of aberration is then extended to these split-plot designs in order to compare designs. Finally, two methods are presented for constructing two-level minimum aberration split-plot designs.

1. INTRODUCTION

In multifactor experiments in which it is not practical to run all the factors in a completely random order, a split-plot design can be used as an efficient tool. Usually a split-plot design is obtained by combining two separate designs, one for the whole plot and one for the subplot. In this thesis the concept of aberration is used as a way of selecting "good" fractional factorial split-plot designs with two levels. It will be seen in this paper that a good split-plot design is usually the one where the subplot design is constructed by using all the factors involved in the subplot and some factors from the whole plot. Section 2 of this paper will establish the relation between split-plot designs and fractional factorial designs. This relation is best seen through the generating matrices by which split-plot designs are well represented. Through this relation, the concept of minimum aberration for fractional factorial designs can be used to study split-plot designs. Section 3 will present two methods for constructing two-level minimum aberration

split-plot designs. Using the first method, one can use minimum aberration fractional factorial designs to construct minimum aberration split-plot designs. In order to compare two split-plot designs, the aberration function is defined so that comparing two split-plot designs is equivalent to comparing the two values of the aberration function. This aberration function, linear integer programming, and some known properties on the wordlength pattern are used to present the second method. The second method will essentially reduce the problem of finding a minimum aberration split-plot design to a linear integer programming problem where the aberration function is to be minimized.

2. RELATION BETWEEN SPLIT-PLOT AND FRACTIONAL FACTORIAL DESIGNS

The defining relation of a 2^{n-k} fractional factorial design D can be represented by the generating matrix (Franklin, 1984), and without loss of generality the generating matrix G can be written in the form $G=(I C)$, where I is the $k \times k$ identity matrix, and C is a $k \times (n-k)$ matrix with its elements equal to 0 or 1.

Now we apply this notation to two-level fractional factorial split-plot designs. Suppose that we have n_1 factors in the whole plot with fractionation element k_1 and n_2 factors in the subplot with fractionation element k_2 . Then there are $2^{n_1-k_1}$ treatment combinations in the whole plot and $2^{(n_1+n_2)-(k_1+k_2)}$ treatment combinations for the total split-plot design. Let G_1 be the generating matrix of a $2^{n_1-k_1}$ design D_1 for the whole plot

and G_2 be the generating matrix of a $2^{n_2-k_2}$ design D_2 for the subplot. Then $G_1 = (I_1 \ C_1)$ and $G_2 = (I_2 \ C_2)$, where I_1 is the $k_1 \times k_1$ identity matrix, C_1 is a $k_1 \times (n_1 - k_1)$ matrix, I_2 is the $k_2 \times k_2$ identity matrix, and C_2 is a $k_2 \times (n_2 - k_2)$ matrix. The design matrix for the total split-plot design can be generated through the *generating matrix*

$$G = \begin{pmatrix} G_1 & \mathbf{O} \\ \mathbf{O} & G_2 \end{pmatrix} = \begin{pmatrix} I_1 & C_1 & \mathbf{O}_1 & \mathbf{O}_2 \\ \mathbf{O}_3 & \mathbf{O}_4 & I_2 & C_2 \end{pmatrix} \begin{matrix} k_1 & n_1 - k_1 & k_2 & n_2 - k_2 \\ k_1 \\ k_2 \end{matrix}, \quad (1)$$

where all the elements in $\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4$ are zero.

To illustrate the above idea, let us consider an example. (Our example is too small a design to normally be used, but it succinctly illustrates some key ideas.) Suppose there are 3 factors, numbered 1, 2, and 3, in the whole plot with fractionation element 1, and 4 factors, numbered 4, 5, 6, and 7, in the subplot with fractionation element 2. Then we have a 2^{3-1} design for the whole plot and a 2^{4-2} design for the subplot. Let the 2^{3-1} design in the whole plot be D_1 with generating relation $I=123$ and the 2^{4-2} design for the subplot be D_2 with generating relation $I=46=567$. Then the corresponding generating matrices for D_1 and D_2 are respectively

$$G_1 = (1 \quad 1 \quad 1)$$

and

$$G_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

The design matrix for the split-plot design is then

$$\mathbf{M} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
-1 & -1 & +1 & -1 & -1 & -1 & +1 \\
-1 & -1 & +1 & +1 & -1 & +1 & -1 \\
-1 & -1 & +1 & -1 & +1 & -1 & -1 \\
-1 & -1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & -1 & -1 & -1 & -1 & +1 \\
+1 & -1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & -1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & +1 & +1 \\
-1 & +1 & -1 & -1 & -1 & -1 & +1 \\
-1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -1 & -1 & -1 & +1 \\
+1 & +1 & +1 & +1 & -1 & +1 & -1 \\
+1 & +1 & +1 & -1 & +1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1
\end{pmatrix} .$$

Note that although \mathbf{M} is the design matrix of the design $2^{7-3} = 2^{(3+4)-(1+2)}$ with the generating relation $I=123=46=567$, this fractional factorial design is not equivalent to the above split-plot design. The reason is that the split-plot design consists of both a *treatment design* (here a 2^{7-3}) and an *error-control* design, which defines how the treatment combinations are randomized and blocked. (The italicized phrases are those used by Hinkelmann and Kempthorne (1994).). We only consider the case where the error-control design for the whole plot units are completely randomized, and the error-control design for the subplot units are completely randomized for each treatment combination in the whole plot units. Since the sequel only discusses the treatment-design aspects of the split-plot design, we will be able to regard these split-plot restrictions of the treatment design as creating an important subset of fractional factorial designs.

Since $I=123=46=567$ is the generating relation of the 2^{7-3} design, we can easily obtain its generating matrix as:

$$\mathbf{G} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \end{matrix}$$

which follows immediately from (1) and the expressions for \mathbf{G}_1 and \mathbf{G}_2 .

The above example has shown us the motivation for using a generating matrix for a split-plot design. This example also tells us how the fractional factorial 2^{7-3} design and the split-plot design are related. This relation can be readily extended to the following general case.

Equivalence 1. Given a split-plot design with n_1 factors and k_1 fractionation in the whole plot and n_2 factors and k_2 fractionation in the subplot, suppose the generating matrix \mathbf{G} of this split-plot design is given in (1). Let $\mathbf{D} = 2^{(n_1+n_2)-(k_1+k_2)}$ be the fractional factorial design with the generating matrix \mathbf{G} . Then the split-plot design is equivalent to this fractional factorial $2^{(n_1+n_2)-(k_1+k_2)}$ design, in the sense that their treatment designs are identical.

This equivalence will be extended in a useful way. Let us reconsider our above example. The design \mathbf{D}_1 for the whole plot and the design \mathbf{D}_2 for the subplot were constructed separately. That is to say, the generators in \mathbf{D}_1 were the words formed only from the letters (factors) 1, 2, and 3 in the whole plot, and the generators in \mathbf{D}_2 were the words formed only from the letters (factors) 4, 5, 6, and 7 in the subplot. If we instead fix the whole plot design \mathbf{D}_1 as before, but now modify the subplot design \mathbf{D}_2 as \mathbf{D}_2^* with

generating relation $I=146=567$, then, as in the above, this modified split-plot design has the generating matrix

$$\mathbf{G}^* = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

and this split-plot design can now be identified with the design 2^{7-3} having the generating relation $I=123=146=567$, in the sense that their treatment designs are identical. Comparing the original split-plot design with the modified one, we find that the second design is better than the first one. In the second design no single-factor effect is confounded with any other single-factor effect — the defining relations is $I=123=146=567=2346=123567=1457=23457$ —but in the original design the single effect 4 is confounded with the single effect 6.

The idea of using \mathbf{D}_2^* , whose generators include the letter 1 from the whole plot, is mentioned in Cochran and Cox (1957). \mathbf{D}_2^* is still referred to as a subplot design, and throughout the remainder of the thesis, “subplot design” is used in the extended sense and, for simplicity’s sake, is denoted by \mathbf{D}_2 . The generating matrix \mathbf{G} , then, of a split-plot design with n_1 factors and k_1 fractionation in the whole plot and n_2 factors and k_2 fractionation in the subplot will have the form

$$\mathbf{G} = \begin{pmatrix} & k_1 & n_1 - k_1 & k_2 & n_2 - k_2 \\ \mathbf{I}_1 & \mathbf{C}_1 & \mathbf{O}_1 & \mathbf{O}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{I}_2 & \mathbf{C}_2 \end{pmatrix} \begin{matrix} k_1 \\ k_2 \end{matrix}. \quad (2)$$

Here, $\mathbf{I}_1, \mathbf{I}_2, \mathbf{C}_1, \mathbf{C}_2, \mathbf{O}_1, \mathbf{O}_2$ are the matrices with the same structures as in (1), \mathbf{B}_1 and \mathbf{B}_2 are matrices with elements 0 or 1, $(\mathbf{I}_1 \mathbf{C}_1)$, denoted by \mathbf{G}_1 , represents the generating matrix of the whole plot design, and $(\mathbf{B}_1, \mathbf{B}_2, \mathbf{I}_2, \mathbf{C}_2)$, represents the generating matrix of the subplot design. The \mathbf{G} in (2) is a natural generalization of the \mathbf{G} in (1). We write a corresponding extension of Equivalence 1 as follows.

Equivalence 2. Given a split-plot design with n_1 factors and k_1 fractionation in the whole plot, n_2 factors in the subplot, and k_2 fractionation in a subplot design, suppose that the generating matrix \mathbf{G} of this split-plot design is given in (2). If $\mathbf{D} = 2^{(n_1+n_2)-(k_1+k_2)}$ is the fractional factorial design with the generating matrix \mathbf{G} , then the split-plot design is equivalent to this fractional factorial design, in the sense that their treatment designs are identical.

3. MINIMUM ABERRATION SPLIT-PLOT DESIGNS

3.1. Definition

To construct a split-plot design of form (2), it is natural to first select a whole plot design \mathbf{D}_1 with good properties. For example, the designs displayed in Table 12.15 of Box, Hunter and Hunter (1978) can be used since they are of minimum aberration. Now, then, we need a subplot design \mathbf{D}_2 that will result in good properties for the entire split-plot

design. Hence a natural question is how the subplot design is obtained, given a specific whole plot design. In fact, this question is equivalent, using Equivalence 2, to asking how to construct $\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}_2$ in (2), given \mathbf{C}_1 , so that the corresponding fractional factorial $2^{(n_1+n_2)-(k_1+k_2)}$ design is good under some criterion. We solve this problem by using the concept of minimum aberration about fractional factorial designs introduced by Fries and Hunter (1980), extending this idea to minimum aberration split-plot designs. For convenience, we first introduce the following notation.

Notation: Given a $k_1 \times n_1$ matrix $\mathbf{G}_1 = (\mathbf{I}_1 \ \mathbf{C}_1)$, the symbol $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ will be used to denote a fractional factorial $2^{(n_1+n_2)-(k_1+k_2)}$ design whose generating matrix is in the form of (2) with the prescribed matrix \mathbf{C}_1 , and thus by Equivalence 2, represents a generalized split-plot design.

Now consider a design $\mathbf{D} = 2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$. The number of letters in a word appearing in the defining relation of the design \mathbf{D} is called the *wordlength* of the word. Let $A_i(\mathbf{D})$ be the number of words of length i in the design \mathbf{D} . The *wordlength pattern* of the design \mathbf{D} is defined as $W(\mathbf{D}) = (A_1(\mathbf{D}), A_2(\mathbf{D}), \dots, A_{n_1+n_2}(\mathbf{D}))$. For example, the split-plot design \mathbf{D} corresponding to the generating matrix \mathbf{G}^* in Section 2 has defining relation $\mathbf{I} = 123 = 146 = 567 = 2346 = 123567 = 1457 = 23457$, and so $W(\mathbf{D}) = (0, 0, 3, 2, 1, 1, 0)$.

Definition. Suppose that \mathbf{D}' and \mathbf{D}'' are two $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ designs. Let r be the smallest i such that $A_i(\mathbf{D}') \neq A_i(\mathbf{D}'')$. Then \mathbf{D}' is said to have less aberration than \mathbf{D}'' if $A_r(\mathbf{D}') < A_r(\mathbf{D}'')$. If no such i exists, then \mathbf{D}' and \mathbf{D}'' are said to have equal aberration. A

design $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ is said to have minimum aberration if no other design $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ has less aberration. A split-plot design with generating matrix \mathbf{G} in (2) is said to have minimum aberration if its corresponding $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ design has minimum aberration.

Remark 1. Our definition of minimum aberration $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ designs is similar to the definition of minimum aberration 2^{n-k} designs (see Chen and Wu, 1991). However, it should be noted that all the designs $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ already have k_1 generators described by $\mathbf{G}_1=(\mathbf{I}_1 \mathbf{C}_1)$. Thus, finding a minimum aberration $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ design requires a choice of only k_2 generators.

Remark 2. Designs with more shorter wordlength words are inferior to those with more longer wordlength words. For example, consider two Resolution IV designs, one of which is a minimum aberration design. When it is safe to assume that the effects of three-factor interactions and higher order interactions will be zero, then the minimum aberration design will estimate more unconfounded two-factor interactions than the other Resolution IV design.

Remark 3. The class of $2^{(n_1+n_2)-(k_1+k_2)}(\mathbf{G}_1)$ designs for a given \mathbf{G}_1 are a subset of the class of 2^{n-k} designs for $n=n_1+n_2$ and $k=k_1+k_2$.

3.2 Finding minimum aberration split-plot designs

We examine two methods to obtain a minimum aberration split-plot design. Method I, based on remark 3, finds such designs from available minimum aberration fractional factorial designs. Method II employs linear integer programming.

Method I

For any n the minimum aberration 2^{n-k} designs for $k \leq 4$ and $k=5$ are given in Chen and Wu (1991), and Chen (1992), respectively. Also for some n and some $k \geq 6$ the minimum aberration 2^{n-k} designs are given in Franklin (1984). All of the minimum aberration fractional factorial designs with resolution III or higher and most of the minimum aberration fractional factorial designs with resolution II can be used to construct minimum aberration split-plot designs.

For this purpose, we first want to get a simplified version of form (2). Let g_{ij} be a typical element of G of (2), which lies in the i th row and j th column. If one nonzero element g_{ij} is in B_1 , then adding the j th row of G to the i th row with reduction modulo 2, to replace the i th row, will reduce the element g_{ij} to 0. Therefore, performing similar row operations for all the non-zero elements of B_1 will reduce B_1 to O_3 , a matrix with all elements equal to 0, and thus the following new matrix is obtained:

$$\begin{pmatrix} \mathbf{I}_1 & \mathbf{C}_1 & \mathbf{O}_1 & \mathbf{O}_2 \\ \mathbf{O}_3 & \mathbf{B}_3 & \mathbf{I}_2 & \mathbf{C}_2 \end{pmatrix} \begin{matrix} k_1 \\ k_2 \end{matrix}.$$

Since the sum of two rows with reduction modulo 2 represents the product of the two generators corresponding to these two rows, it is seen that the above matrix is also a generating matrix of the split-plot design corresponding to the G in (2). Switching some columns of the above matrix will give us the following matrix

$$\begin{pmatrix} & k_1 & k_2 & n_1 - k_1 & n_2 - k_2 \\ \mathbf{I}_1 & \mathbf{O}_1 & \mathbf{C}_1 & \mathbf{O}_2 \\ \mathbf{O}_3 & \mathbf{I}_2 & \mathbf{B}_3 & \mathbf{C}_2 \end{pmatrix} \begin{matrix} k_1 \\ k_2 \end{matrix}, \quad (3)$$

which, of course, is a generating matrix of the same split-plot design except for relabeling of the factors.

Now let us use an example to show how to obtain minimum aberration split-plot designs from minimum aberration fractional factorial designs. The strategy here is to rearrange the matrix to be of the form (3), whose two key features are that the left portion is the identity matrix of dimension $k_1 + k_2$ and the upper right portion is a $k_1 \times (n_2 - k_2)$ matrix of 0's. For convenience, we will use the symbol $[i] \leftrightarrow [j]$ to denote the elementary row operation that the positions of the i th row and j th row of a matrix G have been switched.

The following fractional factorial 2^{9-4} design with generating relation $I=12346=12357=2458=3459$ is a minimum aberration design (see Chen and Wu, 1991). The generating matrix of this design is

$$\mathbf{G} = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us perform the following elementary operations:

$$\mathbf{G} \text{ (switch columns)} \rightarrow \begin{pmatrix} 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} = (\mathbf{I C}).$$

Thus comparing $(\mathbf{I C})$ with form (3), one gets $k_1=1$, $k_2=3$, $n_1-k_1=4$ (so $n_1=4+k_1=5$), and $n_2-k_2=1$ (so $n_2=1+k_2=4$). Therefore, the minimum aberration split-plot design represented by $(\mathbf{I C})$ is easily stated as follows: factors 1, 2, ..., 9; whole plot design $2^{n_1-k_1}=2^{5-1}$ with factors 1, 2, 3, 4, and 6, and generating relation $I=12346$; subplot design $2^{n_2-k_2}=2^{4-3}$ with factors 5, 7, 8, and 9, and generating relation $I=12357=2458=3459$. Of course, one can make a relabeling of the factors (for example, just switching the positions of factors 5 and 6) to get the following minimum aberration split-plot design stated in a somewhat standard form: factors 1, 2, ..., 9; whole plot design $2^{n_1-k_1}=2^{5-1}$ with factors 1, 2, 3, 4, and 5, and generating relation $I=12345$; subplot design $2^{n_2-k_2}=2^{4-3}$ with factors 6, 7, 8, and 9, and generating relation $I=12367=2468=3469$.

Note that another two minimum aberration split-plot designs can be derived from the above 2^{9-4} design as well. Observe that the column under factor 1 in $(\mathbf{I C})$ has two 0 elements. We want to “move” them to the upper right hand corner and so perform the following elementary operations:

$$(\mathbf{I C}) \text{ (switch columns)} \rightarrow \begin{pmatrix} 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}
& \left([3] \leftrightarrow [1], [4] \leftrightarrow [2] \right) \rightarrow \begin{pmatrix} 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \\
& \text{(switch columns)} \rightarrow \begin{pmatrix} 8 & 9 & 6 & 7 & 2 & 3 & 4 & 5 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} = \mathbf{G}^*.
\end{aligned}$$

Comparing \mathbf{G}^* with form (3), one has $k_1=2$, $k_2=2$, $n_1=4+k_1=6$, $n_2=1+k_2=3$, and a minimum aberration split-plot design is obtained as follows: factors 1, 2, ..., 9; whole plot design $2^{n_1-k_1}=2^{6-2}$ with factors 2, 3, 4, 5, 8, and 9, and generating relation $I=2458=3459$; subplot design $2^{n_2-k_2}=2^{3-2}$ with factors 1, 6, and 7, and generating relation $I=12346=12357$. Now, the third row of \mathbf{C} has two 0 elements, and we want to "move" them to the upper right hand corner. To do so, we perform the following operations:

$$\begin{aligned}
& \text{(IC) (switch columns)} \rightarrow \begin{pmatrix} 6 & 7 & 8 & 9 & 1 & 3 & 2 & 4 & 5 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \\
& \left([3] \leftrightarrow [1] \right) \rightarrow \begin{pmatrix} 6 & 7 & 8 & 9 & 1 & 3 & 2 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}
\end{aligned}$$

$$(\text{switch columns}) \rightarrow \begin{pmatrix} & 8 & 7 & 6 & 9 & 2 & 4 & 5 & 1 & 3 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Therefore, as before, one obtains a minimum aberration split-plot design as follows:

factors 1, 2, ..., 9; whole plot design $2^{n_1-k_1}=2^{4-1}$ with factors 2, 4, 5 and 8, and generating relation $I=2458$; subplot design $2^{n_2-k_2}=2^{5-3}$ with factors 1, 3, 6, 7, and 9, and generating relation $I=12357=12346=3459$.

Generally speaking, a minimum aberration fractional factorial 2^{n-k} design with resolution III or higher often gives us at least two different minimum aberration split-plot designs. In fact, let G be a generating matrix of such a 2^{n-k} design. Switch the columns of G to get a generating matrix in the form $(I C)$. If one column (row) of C has k_1 (r) zeros, then the elementary operations of switching rows and columns of $(I C)$ as in the above example will lead to a generating matrix G^* of a minimum aberration split-plot design, where there are n factors in total, $n-k+k_1-1$ ($n-k-r+1$) factors and k_1 (1) fractionation in the whole plot, and $k-k_1+1$ ($k+r-1$) factors and $k-k_1$ ($k-1$) fractionation in the subplot design. The generating relations of the whole plot and subplot designs of the minimum aberration split-plot design can be written down directly from G^* . Of course, the collection of the words appearing in both generating relations, along with I , constitute the generating relation of the original fractional factorial 2^{n-k} design. It should be noted that quite often one can obtain a minimum aberration split-plot design from a minimum

aberration fractional factorial 2^{n-k} design with resolution II, and only in the rare case where all the elements of C are equal to 1 can one *not* do so. (Of course, such designs by themselves have little use in practice.)

Method II (An algorithm)

To get an algorithm for constructing the minimum aberration split-plot designs, we first look at some known properties of the wordlength patterns. As in the case of unrestricted fractional factorial designs, we can safely assume in the following that each of the $n_1 + n_2$ letters in a $2^{(n_1+n_2)-(k_1+k_2)}$ (G_1) design must appear in the defining relation. From Chen (1992), the wordlength pattern $(A_1, A_2, \dots, A_{n_1+n_2})$ of a $2^{(n_1+n_2)-(k_1+k_2)}$ (G_1) design must satisfy

- (i) $\sum A_i = 2^k - 1$, where $k = k_1 + k_2$.
- (ii) $\sum A_{2i-1} = 2^{k-1}$ or 0.
- (iii) $\sum i A_i = n 2^{k-1}$, where $n = n_1 + n_2$.
- (iv) $\sum i^2 A_i \geq 2^{k-2} [n^2 + q^2(2^k - 1) + 2qr + r]$, where $n = q(2^k - 1) + r$ with $n = n_1 + n_2$.
- (v) $\sum i^2 A_i$ is divisible by 2^{k-1} .
- (vi) $\sum i A_{2i}$ is divisible by 2^{k-3} .

Of course, if we let $(a_1, a_2, \dots, a_{n_1})$ be the wordlength pattern of the whole plot design D_1 , then we must have

- (vii) $A_j \geq a_j$, $j = 1, 2, \dots, n_1$; $A_j \geq 0$, $j = n_1 + 1, n_1 + 2, \dots, n_1 + n_2$.

Given the wordlength pattern $W=(A_1, A_2, \dots, A_{n_1+n_2})$ of a design $D=2^{(n_1+n_2)-(k_1+k_2)}(G_1)$,

define the *aberration function*

$$f(D) = f(W) = A_n + A_{n-1}2^k + A_{n-2}2^{2k} + \dots + A_12^{(n-1)k},$$

where $n=n_1+n_2$ and $k=k_1+k_2$. Let $W'=(A_1', A_2', \dots, A'_{n_1+n_2})$ be the wordlength pattern of

another design D' . Then the judgment about which of the two designs is better can be

made through the aberration function. Precisely, we have

Theorem. (a) $W=W'$ if and only if $f(D) = f(D')$.

(b) D has less aberration than D' if and only if $f(D) < f(D')$.

A proof of the theorem is given in the Appendix.

The basic algorithm is now set as follows.

Step 1. Find a subplot design D_2 so that the corresponding split-plot design

$D_0=2^{(n_1+n_2)-(k_1+k_2)}(G_1)$ is likely to have minimum aberration. Set $B=0$.

Step 2. Let $W=(A_1, A_2, \dots, A_n)$ with $n=n_1+n_2$. Find the solution to the following linear integer programming problem

$$\begin{aligned} & \text{minimize } f(W) \\ & \text{subject to (i), (ii), (iii), (iv), (vii),} \\ & B < f(W) < f(D_0). \end{aligned} \tag{4}$$

If there is no solution, then D_0 has minimum aberration, so stop. If there is a solution, proceed to the next step.

Step 3. Check conditions (v) and (vi) for the solution W to (4). If W does not satisfy (v) or (vi), then set $B=f(W)$ and go back to step 2. If the solution W satisfies (v) and (vi) and corresponds to a design D , then this design D has minimum aberration, so stop. If the solution W satisfies (v) and (vi) but does not correspond to a design, simply set $B=f(W)$ and go back to step 2.

In the above algorithm, the linear integer programming method is used to find the possible wordlength pattern, which enables the aberration function to achieve its minimum value. Thus, minimum aberration is obtained. (For more details on linear integer programming, see Walukiewicz (1991).) Understanding of the algorithm may be aided by the following example.

For the example in Section 2, we have seven factors 1, 2, 3, 4, 5, 6, 7. The whole plot design is $D_1=2^{3-1}$ with three factors 1, 2, and 3, and defining relation $I=123$. The subplot contains four factors 4, 5, 6, and 7. We want to construct a minimum aberration split-plot design with 16 runs; that is, we need to construct a minimum aberration $2^{(3+4)-(1+2)}$ (G_1) design, where G_1 is the generating matrix of D_1 . In Section 2 we used, as a subplot design, $D_2^*=2^{4-2}$ with factors 4, 5, 6, and 7 and generating relation $I=146=567$. Now we combine D_1 and D_2^* to form an initial split-plot design $D_0=2^{(3+4)-(1+2)}$ (G_1) with generating relation $I=123=146=567$. The solution to the linear integer programming problem (4) with $B=0$ is $W=(0, 0, 2, 3, 2, 0, 0)$. This W satisfies (v) and (vi) and corresponds to a design $D = 2^{(3+4)-(1+2)}$ (G_1) with generating relation $I=123=146=2567$. Therefore, this split-plot design D has minimum aberration. The subplot design is 2^{4-2} with four factors 4, 5, 6, and 7, and

generating relation $I=146=2567$. Note that the above minimum aberration split-plot design **D** can not be obtained by using Method I since the minimum aberration fractional factorial 2^{7-3} design has resolution IV (see Chen and Wu, 1991).

4. SUMMARY

The collection of split-plot designs generated by the matrices of form (2) or (3) is an extension of the collection of commonly used split-plot designs represented by the matrices of form (1). The representation of split-plot designs in the matrices of (2) clearly indicates that the treatment-design aspect of split-plot designs can be treated in the same way as in the case of fractional factorial designs. In particular, the concept of aberration can be applied, which allows us to compare designs in a natural way. We have presented two methods for constructing minimum aberration split-plot designs. Method I can be done by hand quite easily, while Method II requires integer programming.

Most of the results presented in this paper can be readily extended to the case of s -level split plot designs, where s is a prime. The definition of minimum aberration s^{n-k} fractional factorial designs can be found in Franklin (1984) or Chen and Wu (1991). The obvious modification of the definition given in this paper will lead to the definition of minimum aberration s -level split plot designs. Note that for any s^{n-k} fractional factorial design, its generating matrix can be written in the form $G=(I C)$, where I is the $k \times k$

identity matrix and C is a $k \times (n-k)$ matrix with its elements from the Galois field $G(s) = \{0, 1, 2, 3, \dots, s-1\}$ (See Chen and Wu (1991).), and that for s -level split plot designs, the elements of C_1, C_2, B_1, B_2 and B_3 in the generating matrices (1), (2), and (3) belong to $G(s)$. Therefore it is straight forward to use Method I. Also note that the aberration function now is $f(D) = f(W) = A_n + A_{n-1} s^k + A_{n-2} s^{2k} + \dots + A_1 s^{(n-1)k}$, and the theorem stated in Section 3 of this paper is true in the general case. However, it seems to us that enough conditions about the wordlength patterns of s -level fractional factorial designs have not been found. Hence it is hard to use method II.

APPENDIX

Proof of Theorem 2: For (a) the necessity is obvious. So let us consider the sufficiency.

Suppose $f(D) = f(D')$; that is, $(A_1 - A_1')2^{-k} + (A_2 - A_2')2^{-2k} + \dots + (A_n - A_n')2^{-nk} = 0$. We want to show $W = W'$. It suffices to show $A_1 = A_1'$. In fact, if $A_1 \neq A_1'$, we have $|A_1 - A_1'| \geq 1$. Therefore

$$\begin{aligned}
 2^{-k} &\leq |(A_1 - A_1')2^{-k}| \\
 &= |(A_2 - A_2')2^{-2k} + \dots + (A_n - A_n')2^{-nk}| \\
 &\leq (2^k - 1)2^{-2k} + \dots + (2^k - 1)2^{-nk} \\
 &< (2^k - 1)2^{-2k} (1 + 2^{-k} + 2^{-2k} + \dots) \\
 &= 2^{-k}.
 \end{aligned}$$

This contradiction shows that $A_1 = A_1'$.

Now consider the necessity part of (b). If \mathbf{D} has less aberration than \mathbf{D}' , then there exists some m such that $A_1=A_1', A_2=A_2', \dots, A_{m-1}=A_{m-1}', A_m < A_m'$. Therefore

$$\begin{aligned}
 (f(\mathbf{D}') - f(\mathbf{D}))2^{-nk} &= (A_m' - A_m)2^{-mk} + \sum_{i=m+1}^n (A_i' - A_i)2^{-ik} \\
 &\geq 2^{-mk} - \sum_{i=m+1}^n (2^k - 1)2^{-ik} \\
 &> 2^{-mk} - (2^k - 1)(2^{-(m+1)k} + 2^{-(m+2)k} + \dots) \\
 &= 0.
 \end{aligned}$$

For the sufficiency part of (b), suppose $f(\mathbf{D}) < f(\mathbf{D}')$. We want to show that \mathbf{D} has less aberration than \mathbf{D}' . If \mathbf{D} and \mathbf{D}' have the same wordlength pattern, then by (a), $f(\mathbf{D}) = f(\mathbf{D}')$, which is impossible. If \mathbf{D}' has less aberration, then, similar to the necessity part in (b), one has $f(\mathbf{D}') < f(\mathbf{D})$, which is also impossible. Therefore \mathbf{D} must have less aberration than \mathbf{D}' .

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