

Kernel and eigenfunction estimates for some second order elliptic operators

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Outline

Introduction

Heat kernel
estimates

1 Introduction

2 Heat kernel estimates

Schrödinger Operators

Consider

$$\begin{aligned} Au &= - \sum_{k,j=1}^n \partial_k (a_{kj} \partial_j u) + Vu, \\ Bu &= -\Delta u + Vu. \end{aligned}$$

The associated quadratic forms

$$\begin{aligned} \mathfrak{a}(u, u) &:= \sum_{k,j=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j u + \int_{\mathbb{R}^n} V |u|^2 \\ \mathfrak{b}(u, u) &:= \int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} V |u|^2, \end{aligned}$$

$$u \in D(\mathfrak{a}) = D(\mathfrak{b}) = \{u \in W^{1,2}(\mathbb{R}^n); \int_{\mathbb{R}^n} V |u|^2 < \infty\}.$$

Assumptions

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eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

$$(H) \left\{ \begin{array}{l} a_{kj} = a_{jk} \in W_{loc}^{1,\infty}(\mathbb{R}^n, \mathbb{R}), \partial_j a_{kj} = o(|x|^{\frac{\alpha}{2}}) \text{ as } |x| \rightarrow \infty, \\ \eta|\xi|^2 \leq \sum_{j,k=1}^n a_{kj}(x) \xi_k \xi_j \leq \Lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \\ V \in L_{loc}^1(\mathbb{R}^n) \text{ such that } V(x) \geq |x|^\alpha, \quad \alpha > 2. \end{array} \right.$$

Spectrum

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

$\lim_{|x| \rightarrow +\infty} V(x) = +\infty$ implies that A and B have compact resolvents. Thus,

$$\begin{aligned}\sigma(A) &= \{\mu_i; i = 0, 1, \dots\} \\ \sigma(B) &= \{\lambda_i; i = 0, 1, \dots\}.\end{aligned}$$

Let $(\psi_i)_{i \geq 0}$ and $(\varphi_i)_{i \geq 0}$ the corresponding normalized eigenfunctions of A and B , respectively.

Gaussian estimates 1

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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It is known that A and B have heat kernels $k_t(x, y)$ and $p_t(x, y)$ satisfying

$$p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}, \quad k_t(x, y) \leq C t^{-n/2} e^{-c \frac{|x-y|^2}{t}}$$

for $t > 0$ and constants $c, C > 0$

Outline

Introduction

Heat kernel
estimates

Gaussian estimates 2

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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See [E.M. Ouhabaz: Proc. Amer. Math. Soc. **134**, 2006].

$$p_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\lambda_0 t} e^{-\frac{|x-y|^2}{4t}} \left[1 + \lambda_0 t + \frac{|x-y|^2}{t} \right]^{\frac{n}{2}}$$

and

$$k_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\mu_0 t} e^{-\frac{\rho^2(x,y)}{4t}} \left[1 + \mu_0 t + \frac{\rho^2(x, y)}{t} \right]^{\frac{n}{2}}, \quad t > 0,$$

where

$$\begin{aligned} \rho(x, y) &:= \sup \{ \phi(x) - \phi(y) : \phi \in C_c^\infty(\mathbb{R}^n), \\ &\quad \sum_{k,j=1}^n a_{kj} \partial_k \phi \partial_j \phi \leq 1 \text{ a.e. on } \mathbb{R}^n \}. \end{aligned}$$

Outline

Introduction

Heat kernel
estimates

Intrinsic ultracontractivity for B

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

E.B. Davies in 1984 showed

$$p_t(x, y) \leq C e^{ct-b} \varphi_0(x) \varphi_0(y), \quad (1)$$

$x, y \in \mathbb{R}^n$, $0 < t \leq 1$, where C, c are constants and $b > \frac{\alpha+2}{\alpha-2}$.
Using Lyapunov functions techniques Metafune and Spina [JEE
7, 2007] obtained (1) with $b = \frac{\alpha+2}{\alpha-2}$.

Intrinsic ultracontractivity for B

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

Davies showed also

$$c_1|x|^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}} \leq \varphi_0(x) \leq c_2|x|^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}} \quad (2)$$

for large $|x|$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$. By (1),

$$p_t(x, y) \leq Ce^{ct-b}(|x||y|)^{-\beta}e^{-\frac{|x|^\gamma}{\gamma}}e^{-\frac{|y|^\gamma}{\gamma}} \quad (3)$$

for large $|x|$, $|y|$, and $0 < t \leq 1$.

The main theorem

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eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

Theorem 1. Assume (H) with $\Lambda < 1$. Then,

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct-b} (|x||y|)^{-\beta} e^{-\frac{|x|^\gamma}{\gamma} - \frac{|y|^\gamma}{\gamma}}$$

for large $|x|$, $|y|$ and all $t > 0$. Here

$C, c > 0$, $b > \frac{\alpha+2}{\alpha-2}$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.

Sketch of the proof

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

On $L^2_\varphi := L^2(\mathbb{R}^n, \varphi^2 dx)$ define

$$\begin{aligned}\tilde{\mathbf{a}}(u, v) &:= \mathbf{a}(\varphi u, \varphi v), \quad \tilde{\mathbf{b}}(u, v) := \mathbf{b}(\varphi u, \varphi v) \\ D(\tilde{\mathbf{a}}) &= D(\tilde{\mathbf{b}}) = \{u \in L^2_\varphi; \varphi u \in D(\mathbf{a}) = D(\mathbf{b})\}.\end{aligned}$$

their associated kernels are

$$\tilde{k}_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)}, \quad \tilde{p}_t(x, y) = \frac{p_t(x, y)}{\varphi(x)\varphi(y)}.$$

Using $\varphi \approx \varphi_0$, $|\nabla \varphi| \approx |\nabla \varphi_0|$ and the Beurling-Deny criterion for $(\varphi_0^{-1} e^{-tB} \varphi_0)$ on $L^2_{\varphi_0}$ we deduce

$$1 \wedge u \in D(\tilde{\mathbf{a}}), \quad \forall 0 \leq u \in D(\tilde{\mathbf{a}}).$$

Sketch of the proof

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

We have

$$\tilde{a}(u, v) = \sum_{j,k=1}^n \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j v \varphi^2 dx + \int_{\mathbb{R}^n} W_a u v \varphi^2 dx,$$

where $W_a = V - \sum_{j,k=1}^n \partial_j a_{kj} \frac{\partial_k \varphi}{\varphi} - \sum_{j,k=1}^n a_{kj} \frac{\partial_k \partial_j \varphi}{\varphi}$. Using
(H) and $\Lambda < 1$ we deduce

$$W_a(x) \geq -\lambda_a.$$

Sketch of the proof

Kernel and
eigenfunction
estimates for
some second
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Thus,

$$\begin{aligned}\tilde{\mathfrak{a}}(1 \wedge u, (u-1)^+) &= \int_{\mathbb{R}^n} W_{\mathfrak{a}}(1 \wedge u)(u-1)^+ \varphi^2 dx \\ &\geq -\lambda_{\mathfrak{a}} \int_{\mathbb{R}^n} (1 \wedge u)(u-1)^+ \varphi^2 dx.\end{aligned}$$

Applying Beurling-Deny

$$\|e^{-t\tilde{A}}\|_{\mathcal{L}(L^\infty)} \leq e^{\lambda_{\mathfrak{a}}t}, \quad t \geq 0.$$

Outline

Introduction

Heat kernel
estimates

Sketch of the proof

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

Since $\tilde{b}$ satisfies a log-Sobolev inequality (see E.B. Davies) and by ellipticity, $\tilde{b}(u, u) \leq \max\{\frac{1}{\eta}, 1\} \tilde{a}(u, u)$, it follows that $\tilde{a}$ satisfies the same log-Sobolev inequality. Hence, with the L^∞ -contractivity, we deduce that $e^{-t\tilde{A}}$ is ultracontractive and

$$\tilde{k}_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)} \leq Ce^{ct^{-b}}, \quad 0 < t \leq 1.$$

Sketch of the proof

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

For $t \geq 1$, by

$$k_{s+r}(x, y) = \int_{\mathbb{R}^n} k_s(x, z) k_r(y, z) dz, \quad s, r > 0,$$

$$\begin{aligned} k_t(x, x) &= \left\| e^{-(\frac{t}{2} - \frac{1}{2})A} k_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \\ &\leq e^{-2\mu_0(\frac{t}{2} - \frac{1}{2})} \left\| k_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \\ &\leq M e^{-\mu_0 t} \varphi^2(x). \end{aligned}$$

The result follows from

$$k_t(x, y) \leq \sqrt{k_t(x, x)} \sqrt{k_t(y, y)}, \quad x, y \in \mathbb{R}^n.$$

Estimates for the eigenfunctions

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estimates for
some second
order elliptic
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Corollary 2.

$$|\psi_j(x)| \leq C|x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

for large $|x|$ a $C > 0$ with $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$.

Outline

Introduction

Heat kernel
estimates

Proof

Kernel and
eigenfunction
estimates for
some second
order elliptic
operators

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Outline

Introduction

Heat kernel
estimates

$$\begin{aligned} |\psi_j(x)| e^{-\mu_j t} &= |e^{-tA} \psi_j(x)| \\ &= \left| \int_{\mathbb{R}^n} k_t(x, y) \psi_j(y) dy \right| \\ &\leq \left(\int_{\mathbb{R}^n} k_t(x, y)^2 dy \right)^{1/2} \|\psi_j\|_2 \\ &= (k_{2t}(x, x))^{1/2}. \end{aligned}$$

The general case

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eigenfunction
estimates for
some second
order elliptic
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Outline

Introduction

Heat kernel
estimates

If $\Lambda \geq 1$, we study first

$$H := -\Delta + \theta|x|^\alpha$$

with $0 < \theta < 1$. One proves that its ground state ϕ_0 satisfies

$$\phi_0(x) \approx \phi(x), \quad |\nabla \phi_0(x)| \approx |\nabla \phi(x)|,$$

where $\phi(x) = |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma}$ for large $|x|$.

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estimates for
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operators

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Outline

Introduction

Heat kernel
estimates

Theorem 2. Assume (H) and $\theta > 0$ s.t. $\theta\Lambda < 1$. Then,

$$k_t(x, y) \leq C e^{-\mu_0 t} e^{ct-b} (|x||y|)^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma}|x|^\gamma} e^{-\frac{\sqrt{\theta}}{\gamma}|y|^\gamma}, \quad t > 0$$

for large $|x|, |y|$. Here $C, c > 0, b > \frac{\alpha+2}{\alpha-2}, \beta = \frac{\alpha}{4} + \frac{n-1}{2}$ and $\gamma = 1 + \frac{\alpha}{2}$.