

KÄHLER QUANTIZATION AND REDUCTION

JOHANNES HUEBSCHMANN

Université des Sciences et Technologies de Lille
U. F. R. de Mathématiques
CNRS-UMR 8524
F-59 655 VILLENEUVE D'ASCQ, France
Johannes.Huebschmann@math.univ-lille1.fr

April 5, 2004

ABSTRACT. Exploiting a notion of Kähler structure on a stratified space introduced elsewhere we show that, in the Kähler case, reduction after quantization coincides with quantization after reduction: Key tools developed for that purpose are stratified polarizations and stratified prequantum modules, the latter generalizing prequantum bundles. These notions encapsulate, in particular, the behaviour of a polarization and that of a prequantum bundle across the strata. Our main result says that, for a positive Kähler manifold with a hamiltonian action of a compact Lie group, when suitable additional conditions are imposed, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the (invariant) unreduced and reduced quantum observables as well. Over a stratified space, the appropriate quantum phase space is a costratified Hilbert space in such a way that the costratified structure reflects the stratification. Examples of stratified Kähler spaces arise from the closures of holomorphic nilpotent orbits including angular momentum zero reduced spaces and from representations of compact Lie groups. For illustration, we carry out Kähler quantization on various spaces of that kind including singular Fock spaces.

Introduction

The relationship between unitary representations of a compact Lie group G and Kähler quantization on smooth compact hamiltonian G -spaces has received much attention. In this paper, we will develop a similar theory for hamiltonian G -spaces which are not necessarily smooth manifolds. Our motivation comes from physics: Given a quantizable system with constraints, the question arises whether reduction after quantization coincides with quantization after reduction, so that it would then make no difference whether reduction is imposed before or after quantization. This question goes back to the early days of quantum mechanics and appears already in

2000 *Mathematics Subject Classification.* 14L24 14L30 17B63 17B65 17B66 17B81 32C20 32Q15 32S05 32S60 53D17 53D20 53D50 81S10 .

Key words and phrases. Poisson manifold, Poisson algebra, Poisson cohomology, holomorphic quantization, reduction and quantization, stratified Kähler space, geometric quantization, quantization on a space with singularities, normal complex analytic space, constrained system, invariant theory, real Lie algebra of hermitian type.

DIRAC’S work on the electron and positron. In the present paper we will show that, indeed, in a suitable sense, in the framework of Kähler quantization, reduction after quantization is equivalent to quantization after reduction.

When the unreduced phase space is a quantizable smooth symplectic manifold and when the symmetries can be quantized as well reduction after quantization is well defined within the usual framework of geometric quantization. Up to now, the available methods have been insufficient to attack the problem of quantization of reduced observables, though, once the reduced phase space is no longer a smooth manifold; we will refer to this situation as the *singular case*. The singular case is the rule rather than the exception. For example, simple classical mechanical systems and the solution spaces of classical field theories involve singularities; see e. g. [1] and the references there. In the presence of singularities, restricting quantization to a smooth open dense part, the “top stratum”, can result in a loss of information and may in fact lead to inconsistent results, cf. (4.12) below. This kind of phenomenon has been long known, perhaps as a folk-lore observation, but it does not seem easy to trace it down explicitly in the literature; cf. e. g. the discussion in [4] and [5]. To overcome these difficulties on the classical level, in a predecessor to this paper [18], we isolated a certain class of “Kähler spaces with singularities”, which we call stratified Kähler spaces. On such a space, the complex analytic structure alone is unsatisfactory for issues related with quantization because it overlooks the requisite Poisson structures. In this paper, we generalize ordinary Kähler quantization to a quantization scheme over (complex analytic) stratified Kähler spaces.

We now explain briefly and informally our approach: Consider a stratified symplectic space $(N, C^\infty(N), \{\cdot, \cdot\})$. The Poisson structure encapsulates the mutual positions of the symplectic structures on the strata of N . Likewise, a *stratified Kähler polarization* (cf. [18]) induces ordinary Kähler polarizations on the strata and *encapsulates the mutual positions of these polarizations on the strata*. A complex polarization can no longer be thought of as being given by the $(0,1)$ -vectors of a complex structure, though; see Remark 3.7.3 in [18]. A suitable notion of *prequantization*, phrased in terms *prequantum modules* introduced in [14], yields the requisite representation of the Poisson algebra; in particular, this representation satisfies the Dirac condition. The concept of stratified Kähler polarization then takes care of the irreducibility problem, as does an ordinary polarization in the smooth case. Over a stratified space, the appropriate quantum phase space is what we call a *costratified Hilbert space*; this is a system of Hilbert spaces, one for each stratum, which arises from quantization on the closure of that stratum, the stratification provides linear maps between these Hilbert spaces reversing the partial ordering among the strata, and these linear maps are compatible with the quantizations. After these preparations we show in Section 3 below that, for a positive Kähler manifold with a hamiltonian action of a compact Lie group, when suitable additional conditions are imposed, reduction after quantization coincides with quantization after reduction in the sense that not only the invariant unreduced and reduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well. The correspondence between the invariant unreduced and reduced quantum phase spaces is given by a suitable map which we will refer to as a *state space comparison map*; in the literature, the phrase ‘quantization commutes with reduction’ is usually interpreted as the requirement that a state space comparison map be an isomorphism (of complex vector spaces). The

comparison of invariant unreduced and reduced quantizable classical observables we will give below (Theorem 3.6) does not rely on the state space comparison map (written as ρ in Section 3 below) being an isomorphism, though, that is, the phrase ‘quantization commutes with reduction’ may be given a consistent meaning whether or not the state space comparison map is an isomorphism, and the question arises whether there are reduced states which do *not* arise from (invariant) unreduced states. See Remark 3.9 below.

We illustrate our approach with a number of examples involving what we call *singular Fock spaces*. See Section 4 below for details. In particular, exploiting a suitable notion of momentum mapping which is Poisson even when defined on a space with singularities, we show how the relationship between unitary representations of a compact Lie group and Kähler quantization extends to certain singular cases. Momentum mappings defined on not necessarily smooth spaces occur already in the literature, for example in [2] and [26] (at least implicitly), but these momentum mappings do not involve global Poisson structures encapsulating the singular behaviour. See Remark 4.14 below. Issues related with the metaplectic correction in singular situations will be addressed elsewhere.

In recent years considerable work has been devoted to the comparison of the reduced and unreduced quantum phase spaces and to variants thereof, see [31] for an overview and the literature there, as well as e. g. [25] and [27], [36] for generalizations to higher dimensional sheaf cohomology. The actual quantization of corresponding classical observables has received much less attention, though; cf. e. g. [6] and [34] where the problem has been studied for hamiltonian G -spaces endowed with a G -invariant real polarization.

This paper is part of a program aimed at developing a satisfactory quantization procedure on certain moduli spaces including spaces of possibly twisted representations of the fundamental group of a surface in a compact Lie group. In [18] we have shown that these spaces are indeed normal Kähler spaces; see our expository papers [15–17] and the literature there for the stratified symplectic structure. In physics language, the relevant quantum phase spaces are spaces of conformal blocks.

I am indebted to A. Weinstein for discussions, and for his encouragement to carry out the research program the present paper is part of. I am grateful to T. R. Ramadas for having pointed out to me some relevant literature and to M. Schmidt and J. Śniatycki for a number of questions which helped improve the exposition.

1. Prequantization on spaces with singularities

To develop prequantization over stratified symplectic spaces and to describe the behaviour of prequantization under reduction, we will introduce *stratified prequantum modules* over stratified symplectic spaces. A stratified prequantum module determines what we call a *costratified prequantum space* but the two notions, though closely related, should not be confused. For intelligibility, we reproduce first the concept of prequantum bundle in a language tailored to our purposes.

(1.1) *Prequantum bundles.* Let (N, σ) be a quantizable symplectic manifold, let $(C^\infty(N), \{\cdot, \cdot\})$ be its symplectic Poisson algebra, and let $\zeta: \Lambda \rightarrow N$ be a prequantum bundle for (N, σ) . Thus, when the operator of covariant derivative for the requisite connection is written as $\nabla: \Omega^0(N, \zeta) \rightarrow \Omega^1(N, \zeta)$, the curvature K_∇ coincides with $-i\sigma$. Here the convention is that the curvature K_∇ of the connection ∇ is characterized

by the formula $[\nabla_X, \nabla_Y] = \nabla_{[X, Y]} + K_{\nabla}(X, Y)$ where X and Y are arbitrary smooth vector fields on N and, for a smooth vector field X on N , ∇_X is the operator which assigns the smooth complex valued section $\nabla_X s = (\nabla(s))(X)$ to a smooth complex valued section s of ζ . Henceforth we will often write the space $\Omega^0(N, \zeta)$ of *smooth* complex sections of ζ as $\Gamma^\infty(\zeta)$. Ordinary prequantization proceeds by means of Kostant's formula

$$(1.1.1) \quad \widehat{f}(s) = -i\nabla_{\{f, \cdot\}}(s) + fs, \quad f \in C^\infty(N), \quad s \in \Gamma^\infty(\zeta).$$

(We write $\nabla_{\{f, \cdot\}}$ rather than a corresponding expression involving the hamiltonian vector field X_f of the function f since, in accordance with Hamilton's equations, the Hamiltonian vector field of f is given by the operator $\{\cdot, f\}$.) Associating $\widehat{f}$ to f yields a representation on $\Gamma^\infty(\zeta)$ of the real Lie algebra which underlies $C^\infty(N)$; here $\Gamma^\infty(\zeta)$ is viewed as a complex vector space, and the elements of $C^\infty(N)$ are represented by $\mathbb{C}$ -linear transformations so that the constants in $C^\infty(N)$ act by *multiplication* and the *Dirac condition holds* (see (1.2.6) and (1.2.7) below). The physical constant $\hbar$ is absorbed in the symplectic or, what amounts to the same, Poisson structure; see Remark 1.2.10 below.

(1.2) *Prequantum modules.* Let $(A, \{\cdot, \cdot\})$ be an arbitrary real Poisson algebra. Recall from [13] and [14] that the Poisson structure $\{\cdot, \cdot\}$ determines an $(\mathbb{R}, A)$ -Lie algebra structure $([\cdot, \cdot], \pi^\sharp)$ on the A -module D_A of formal differentials for A . Here $\pi = \pi_{\{\cdot, \cdot\}}: D_A \otimes D_A \rightarrow A$ is the 2-form given by $\pi(da, db) = \{a, b\}$ ($a, b \in A$), the morphism $\pi^\sharp$ from D_A to $\text{Der}(A) = \text{Hom}_A(D_A, A)$ is the adjoint of π , and the bracket $[\cdot, \cdot]$ on D_A is given by the formula

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\}, \quad a, b, u, v \in A.$$

See [13] and [14] for details. We write the resulting $(\mathbb{R}, A)$ -Lie algebra as $D_{\{\cdot, \cdot\}}$; thus the pair $(A, D_{\{\cdot, \cdot\}})$ is a Lie-Rinehart algebra. For intelligibility we recall that, given a Lie-Rinehart algebra (A, L) , the Lie algebra L together with the additional structure is referred to as an (R, A) -Lie algebra.

The Poisson 2-form $\pi_{\{\cdot, \cdot\}}$ determines an *extension*

$$(1.2.1) \quad 0 \rightarrow A \rightarrow \overline{L}_{\{\cdot, \cdot\}}^a \rightarrow D_{\{\cdot, \cdot\}} \rightarrow 0$$

of $(\mathbb{R}, A)$ -Lie algebras which is central as an extension of ordinary Lie algebras; in particular, on the kernel A , the Lie bracket is trivial. At the risk of making a mountain out of a molehill we note that here and below, according to standard conventions, “ $0 \rightarrow U \rightarrow V$ ” and “ $V \rightarrow W \rightarrow 0$ ” signify that $U \rightarrow V$ and $V \rightarrow W$ are injective and surjective morphisms of A -modules, respectively; in particular, 0 is *not* considered as an $(\mathbb{R}, A)$ -Lie algebra. Moreover, as A -modules,

$$(1.2.2) \quad \overline{L}_{\{\cdot, \cdot\}}^a = A \oplus D_{\{\cdot, \cdot\}},$$

and the Lie bracket on $\overline{L}_{\{\cdot, \cdot\}}^a$ is given by

$$(1.2.3) \quad [(a, du), (b, dv)] = (\{u, b\} + \{a, v\} - \{u, v\}, d\{u, v\}), \quad a, b, u, v \in A.$$

The superscript “...^a” is intended to refer to “algebraic” (this superscript does not occur in [13] and [14]), and we have written “ $\overline{L}^a$ ” rather than simply L^a to indicate that the extension (1.2.1) represents the *negative* of the class of $\pi_{\{\cdot, \cdot\}}$ in Poisson cohomology $H^2_{\text{Poisson}}(A, A)$, cf. [13]. When $(A, \{\cdot, \cdot\})$ is the smooth symplectic Poisson algebra of an ordinary smooth symplectic manifold, cf. (1.1), (perhaps) up to sign, the class of $\pi_{\{\cdot, \cdot\}}$ comes down to the cohomology class represented by the symplectic structure. Plainly the extension (1.2.1) determines an extension of Lie-Rinehart algebras as well, the algebra variable being fixed.

Extending terminology introduced in [14], given an $(A \otimes \mathbb{C})$ -module M , we refer to an $(A, \overline{L}^a_{\{\cdot, \cdot\}})$ -module structure

$$(1.2.4) \quad \chi: \overline{L}^a_{\{\cdot, \cdot\}} \longrightarrow \text{End}_{\mathbb{R}}(M)$$

on M as an *algebraic prequantum module structure* for $(A, \{\cdot, \cdot\})$ provided (i) the values of χ lie in $\text{End}_{\mathbb{C}}(M)$, that is to say, for $a \in A$ and $\alpha \in D_{\{\cdot, \cdot\}}$, the operators $\chi(a, \alpha)$ are complex linear transformations, and (ii) for every $a \in A$, with reference to the decomposition (1.2.2), we have

$$(1.2.5) \quad \chi(a, 0) = i a \text{Id}_M.$$

In [13], the terminology ‘prequantum module structure’ is used for what we here call algebraic prequantum module structure. A pair (M, χ) consisting of an $(A \otimes \mathbb{C})$ -module M and an algebraic prequantum module structure will henceforth be referred to as an *algebraic prequantum module* (for $(A, \{\cdot, \cdot\})$).

Prequantization now proceeds in the following fashion, cf. [13]: The assignment to $a \in A$ of $(a, da) \in \overline{L}^a_{\{\cdot, \cdot\}}$ yields a morphism ι of real Lie algebras from A to $\overline{L}^a_{\{\cdot, \cdot\}}$; thus, for any algebraic prequantum module (M, χ) , the composite of ι with $-i\chi$ is a representation $a \mapsto \widehat{a}$ of the A underlying real Lie algebra having M , viewed as a complex vector space, as its representation space; this is a representation by $\mathbb{C}$ -linear operators so that any constant acts by multiplication, that is, for any real number r , viewed as a member of A ,

$$(1.2.6) \quad \widehat{r} = r \text{Id}$$

and so that, for $a, b \in A$,

$$(1.2.7) \quad \widehat{\{a, b\}} = i [\widehat{a}, \widehat{b}] \quad (\text{the Dirac condition}).$$

More explicitly, these operators are given by the formula

$$(1.2.8) \quad \widehat{a}(x) = \frac{1}{i} \chi(0, da)(x) + ax, \quad a \in A, \quad x \in M,$$

which we shall henceforth refer to as the *prequantization formula*.

The interpretation of quantum mechanics requires observables to be represented by symmetric operators (after introduction of a suitable Hilbert space structure), and this forces the factor i in the Dirac condition (1.2.7) (since the ordinary commutator

of two symmetric operators is skew); this factor i , in turn, forces multiplication of the structure map χ of a prequantum module by $-i$.

Symmetries are to be quantized by skew symmetric operators, though, that is, when a classical observable $a \in A$ is viewed as an *infinitesimal symmetry*, the corresponding infinitesimal quantum symmetry is given by the operator $\tilde{a} = i\hat{a}$, that is,

$$(1.2.9) \quad \tilde{a}(x) = (\chi(a, da))(x) = (\chi(0, da))(x) + iax, \quad a \in A, \quad x \in M.$$

Thus, when $\hat{a}$ is self-adjoint (with reference to an appropriate Hilbert space structure, perhaps on a suitable subspace of M), it generates a 1-parameter group of unitary transformations.

REMARK 1.2.10. In the situation of (1.1), let $\omega = \hbar\sigma$, so that $\sigma = \frac{\omega}{\hbar}$ ($= \frac{2\pi\omega}{\hbar}$). Then, using superscripts to indicate with reference to which symplectic structure hamiltonian vector fields and Poisson brackets are to be taken, given functions f and g , we have $\{f, g\}^\sigma = \hbar\{f, g\}^\omega$, the prequantization formula (1.1.1) may be written as

$$(1.2.11) \quad \hat{f}(s) = -i\hbar\nabla_{\{f, \cdot\}^\omega}(s) + fs,$$

and the formula (1.1.5) is equivalent to

$$(1.2.12) \quad i[\hat{a}, \hat{b}] = \hbar\widehat{\{a, b\}^\omega}$$

which is more common in the physics literature. Now the quantizability of σ , i. e. the existence of a prequantum bundle ζ , is equivalent to the requirement that, under the integration map $H_{\text{deRham}}^2(N, \mathbb{R}) \rightarrow H_{\text{singular}}^2(N, \mathbb{R})$, the class of σ go to 2π times the image of an integral class, and we may then think of the class of $\frac{\omega}{\hbar}$ as this integral class. However, the original phase space Poisson bracket is that coming from σ and *not* from ω nor $\frac{\omega}{\hbar}$. We therefore believe that our formula (1.2.9) is more appropriate than corresponding ones in the literature involving a factor 2π (and hence the Poisson bracket coming from $\frac{\omega}{\hbar}$).

(1.3) *Stratified symplectic spaces.* Let N be a stratified symplectic space, and let $(A, \{\cdot, \cdot\})$ be its stratified symplectic Poisson algebra; a special case would be the ordinary symplectic Poisson algebra of a smooth symplectic manifold. Dividing out the formal differentials in D_A that *vanish at every point of N* yields the A -module $\Omega^1(N)$ which serves as a module of differentials for A as well [23]. Here a formal differential α in D_A *vanishes at the point w of N* provided α goes to zero under the epimorphism $D_A \rightarrow \mathbb{R} \otimes_A D_A$ induced by the point w . Equivalently, $\Omega^1(N)$ is the quotient of D_A by the formal differentials α having the property that (X, α) is zero for *every* derivation X of A . The $(\mathbb{R}, A)$ -Lie algebra structure $([\cdot, \cdot], \pi^\#)$ on D_A descends to an $(\mathbb{R}, A)$ -Lie algebra structure on $\Omega^1(N)$; we write the resulting $(\mathbb{R}, A)$ -Lie algebra as $\Omega^1(N)_{\{\cdot, \cdot\}}$. Thus the $(\mathbb{R}, A)$ -Lie algebra $\Omega^1(N)_{\{\cdot, \cdot\}}$ consists of the A -module $\Omega^1(N)$ endowed with the induced bracket $[\cdot, \cdot]$ and structure map $\pi^\#$ from $\Omega^1(N)$ to $\text{Der}(A)$ where the notation $[\cdot, \cdot]$ and $\pi^\#$ is slightly abused. For example, when N is an ordinary smooth manifold and f a smooth function on N , in local coordinates $(x_1, \dots, x_n)$, we have the formal differential $\alpha = df - \sum \partial_j f dx_j$, and the Taylor theorem entails that this formal differential vanishes at every point of N . In [13], we have pointed out that, in this particular case, the fact that the $(\mathbb{R}, A)$ -Lie

algebra structure on D_A descends to $\Omega^1(N)$ amounts to the nowadays familiar Lie algebroid structure on the cotangent bundle of a smooth Poisson manifold N . When N is a space with singularities, the above description in terms of the quotient $\Omega^1(N)$ of D_A by the differentials which vanish at every point of N is more general, though, and cannot be given in terms of Lie algebroids.

(1.4) *The stratified symplectic structure on the closure of any stratum.* Let $(N, C^\infty(N), \{\cdot, \cdot\})$ be a stratified symplectic space. The closure $\overline{Y}$ of any stratum Y of N inherits a stratified symplectic structure $(C^\infty(\overline{Y}), \{\cdot, \cdot\}^{\overline{Y}})$ in the following fashion: Let $C^\infty(\overline{Y})$ be the algebra of continuous functions on $\overline{Y}$ which arise from restriction to $\overline{Y}$ of functions in $C^\infty(N)$. Since the inclusion $Y \subseteq N$ is a Poisson map the ideal of functions in $C^\infty(N)$ which vanish on Y and hence on $\overline{Y}$ is a Poisson ideal. Consequently the Poisson structure on $C^\infty(N)$ descends to a Poisson structure on $C^\infty(\overline{Y})$ which we denote by $\{\cdot, \cdot\}^{\overline{Y}}$; thus $(Y, C^\infty(\overline{Y}), \{\cdot, \cdot\}^{\overline{Y}})$ is a stratified symplectic space, and we refer to its structure as being *induced from the stratified symplectic structure of N* . The projection mapping from $C^\infty(N)$ to $C^\infty(\overline{Y})$ is plainly a morphism $\phi: C^\infty(N) \rightarrow C^\infty(\overline{Y})$ of Poisson algebras which, in view of the Addendum 3.8.4 in [13], induces a morphism

$$(1.4.1) \quad (C^\infty(N), D_{\{\cdot, \cdot\}}) \rightarrow (C^\infty(\overline{Y}), D_{\{\cdot, \cdot\}^{\overline{Y}}})$$

of Lie-Rinehart algebras. For the record, we spell out the following, the proof of which is straightforward and left to the reader.

Proposition 1.4.2. *Given a stratified symplectic space N , for any stratum Y , the closure $\overline{Y}$ being endowed with the induced stratified symplectic structure explained above, the morphism (1.4.1) passes to a morphism*

$$(1.4.3) \quad (\phi, \phi_*): (C^\infty(N), \Omega^1(N)_{\{\cdot, \cdot\}}) \rightarrow (C^\infty(\overline{Y}), \Omega^1(\overline{Y})_{\{\cdot, \cdot\}^{\overline{Y}}})$$

of Lie-Rinehart algebras.

(1.5) *Stratified prequantum modules.* In the constructions explained in (1.2) above, we now replace the $(\mathbb{R}, A)$ -Lie algebra $D_{\{\cdot, \cdot\}}$ with the $(\mathbb{R}, A)$ -Lie algebra $\Omega^1(N)_{\{\cdot, \cdot\}}$ and, accordingly, we replace the extension (1.2.1) of $(\mathbb{R}, A)$ -Lie algebras with the corresponding extension

$$(1.5.1) \quad 0 \rightarrow A \rightarrow \overline{L}_{\{\cdot, \cdot\}} \rightarrow \Omega^1(N)_{\{\cdot, \cdot\}} \rightarrow 0$$

of $(\mathbb{R}, A)$ -Lie algebras. Given an $(A \otimes \mathbb{C})$ -module M , we refer to an $(A, \overline{L}_{\{\cdot, \cdot\}})$ -module structure

$$(1.5.2) \quad \chi: \overline{L}_{\{\cdot, \cdot\}} \rightarrow \text{End}_{\mathbb{R}}(M)$$

on M as a *geometric prequantum module* structure or, more simply, as a *prequantum module* structure, provided the composite of χ with the canonical epimorphism from $\overline{L}_{\{\cdot, \cdot\}}^a$ onto $\overline{L}_{\{\cdot, \cdot\}}$ is an algebraic prequantum module structure. A pair (M, χ) consisting of an $(A \otimes \mathbb{C})$ -module M and a (geometric) prequantum module structure

will be referred to as a (*geometric*) *prequantum module* (for the stratified symplectic space $(N, C^\infty(N), \{\cdot, \cdot\})$).

In particular, let (N, σ) be a quantizable symplectic manifold. Consider the 2-form π induced by the symplectic Poisson structure and, as in (1.2) above, write the adjoint of π as $\pi^\sharp: \Omega^1(N) \rightarrow \text{Vect}(N)$. Under the circumstances of (1.1) above, let $M = \Gamma^\infty(\zeta)$; the assignments

$$\chi_\nabla(a, 0) = i a \text{Id}_M, \quad \chi_\nabla(0, \alpha) = \nabla_{\pi^\sharp(\alpha)}, \quad a \in A, \quad \alpha \in \Omega^1(N),$$

yield a geometric prequantum module structure

$$(1.5.3) \quad \chi_\nabla: \overline{L}_{\{\cdot, \cdot\}} \rightarrow \text{End}_{\mathbb{C}}(M) \subseteq \text{End}_{\mathbb{R}}(M)$$

for $(A, \{\cdot, \cdot\})$. This is just the ordinary prequantization construction in another guise. In fact, under the adjoint $\pi^\sharp$ from $\Omega^1(N)$ to $\text{Vect}(N)$, the prequantization formula (1.2.8) passes to the more usual prequantization formula (1.1.1). Occasionally we shall refer to a prequantum module structure of the kind (1.5.3) as *smooth*.

Let N be a stratified symplectic space, with stratified symplectic Poisson algebra $(C^\infty(N), \{\cdot, \cdot\})$. For each stratum Y , let $(C^\infty(Y), \{\cdot, \cdot\}^Y)$ be its ordinary *smooth* symplectic Poisson structure, and let

$$(1.5.4) \quad 0 \rightarrow C^\infty(Y) \rightarrow \overline{L}_{\{\cdot, \cdot\}^Y} \rightarrow \Omega^1(Y)_{\{\cdot, \cdot\}^Y} \rightarrow 0$$

be the corresponding extension (1.5.1) of $(\mathbb{R}, A)$ -Lie algebras where $\Omega^1(Y)$ is the (projective) $C^\infty(Y)$ -module of ordinary 1-forms on Y . We define a *stratified prequantum module* for N to consist of

- a (geometric) prequantum module (M, χ) for $(C^\infty(N), \{\cdot, \cdot\})$ having the property that, for any stratum Y , the canonical linear map of complex vector spaces from $M_{\overline{Y}} = C^\infty(\overline{Y}) \otimes_{C^\infty(N)} M$ to $M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M$ is injective, together with,
- for each stratum Y , a prequantum module structure χ_Y for $(C^\infty(Y), \{\cdot, \cdot\}^Y)$ on the induced module $M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M$ in such a way that the canonical linear map of complex vector spaces from M to M_Y is a morphism of prequantum modules from (M, χ) to (M_Y, χ_Y) .

Here ‘morphism of prequantum modules from (M, χ) to (M_Y, χ_Y) ’ means that (i) the canonical linear map of complex vector spaces from M to M_Y , (ii) the adjoints $\chi^\sharp$ and $\chi_Y^\sharp$ of the structure maps, and (iii) the morphism from $\overline{L}_{\{\cdot, \cdot\}}$ to $\overline{L}_{\{\cdot, \cdot\}^Y}$ induced by the restriction map, make commutative the diagram

$$(1.5.5) \quad \begin{array}{ccc} \overline{L}_{\{\cdot, \cdot\}} \otimes M & \xrightarrow{\chi^\sharp} & M \\ \downarrow & & \downarrow \\ \overline{L}_{\{\cdot, \cdot\}^Y} \otimes M_Y & \xrightarrow{\chi_Y^\sharp} & M_Y. \end{array}$$

Occasionally we shall denote a stratified prequantum module structure by $(\chi, \{\chi_Y\})$ where Y runs through the strata, or sometimes more simply just by χ , with an abuse of notation. For intelligibility, we note that, on a stratum Y , the induced module $M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M$ will often come down to the space of sections of an ordinary smooth complex (perhaps V-) line bundle and, perhaps with a grain of salt, a prequantum module structure χ_Y for $(C^\infty(Y), \{\cdot, \cdot\}^Y)$ on M_Y will then come down to ordinary prequantization, cf. (1.5.3).

Theorem 1.5.6. *Let $(N, C^\infty(N), \{\cdot, \cdot\})$ be a stratified symplectic space, let Y be a stratum of N , let $(\bar{Y}, C^\infty(\bar{Y}), \{\cdot, \cdot\}^{\bar{Y}})$ be the induced stratified symplectic structure on the closure $\bar{Y}$ of Y , cf. (1.4), and let (M, χ) be a stratified prequantum module for $(N, C^\infty(N), \{\cdot, \cdot\})$. The induced $C^\infty(\bar{Y})$ -module $M_{\bar{Y}} = C^\infty(\bar{Y}) \otimes_{C^\infty(N)} M$ inherits a stratified prequantum module structure*

$$(1.5.7) \quad \chi_{\bar{Y}}: \bar{L}_{\{\cdot, \cdot\}^{\bar{Y}}} \rightarrow \text{End}_{\mathbb{R}}(M_{\bar{Y}})$$

for $(\bar{Y}, C^\infty(\bar{Y}), \{\cdot, \cdot\}^{\bar{Y}})$ in such a way that the canonical linear map of complex vector spaces from M to $M_{\bar{Y}}$ is a morphism of stratified prequantum modules from (M, χ) to $(M_{\bar{Y}}, \chi_{\bar{Y}})$.

Proof. The morphism (1.4.3) of Lie-Rinehart algebras plainly induces a morphism

$$(1.5.8) \quad (\phi, \phi_b): (C^\infty(N), \bar{L}_{\{\cdot, \cdot\}}) \rightarrow (C^\infty(\bar{Y}), \bar{L}_{\{\cdot, \cdot\}^{\bar{Y}}})$$

of Lie-Rinehart algebras in such a way that the restriction

$$(1.5.9) \quad (C^\infty(\bar{Y}), \bar{L}_{\{\cdot, \cdot\}^{\bar{Y}}}) \rightarrow (C^\infty(Y), \bar{L}_{\{\cdot, \cdot\}^Y})$$

to the stratum Y , combined with (1.5.8), amounts to the restriction morphism

$$(1.5.10) \quad (\phi, \phi_b): (C^\infty(N), \bar{L}_{\{\cdot, \cdot\}}) \rightarrow (C^\infty(Y), \bar{L}_{\{\cdot, \cdot\}^Y})$$

from N to the stratum Y . To isolate what we must precisely prove, write $(A, L) = (C^\infty(N), \bar{L}_{\{\cdot, \cdot\}})$, $(A', L') = (C^\infty(\bar{Y}), \bar{L}_{\{\cdot, \cdot\}^{\bar{Y}}})$, $M_{\bar{Y}} = M' = A' \otimes_A M$, and write the Poisson structures on A and A' as $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$, respectively. Let I be the ideal of functions in $C^\infty(N)$, necessarily a Poisson ideal (as we have already observed), which vanish on Y and hence on $\bar{Y}$. By construction, A' is canonically isomorphic to A/I . We must prove that the L -action $\chi: L \rightarrow \text{End}_{\mathbb{R}}(M)$ on M passes to an L' -action $\chi': L' \rightarrow \text{End}_{\mathbb{R}}(M')$ on M' . In view of standard arguments from the theory of formal differentials, the canonical surjection from D_A to $D_{A'}$ fits into the exact sequence

$$(1.5.11) \quad I/I^2 \rightarrow A' \otimes_A D_A \rightarrow D_{A'} \rightarrow 0$$

of A' -modules, in fact, $(\mathbb{R}, A')$ -Lie algebras, where the unlabelled left-hand arrow is given by the assignment to $f \in I$ of $1 \otimes df$. This exact sequence, in turn, lifts to an exact sequence

$$I/I^2 \rightarrow A' \otimes_A \bar{L}_{\{\cdot, \cdot\}}^a \rightarrow \bar{L}_{\{\cdot, \cdot\}'}^a \rightarrow 0$$

of A' -modules, even $(\mathbb{R}, A')$ -Lie algebras, and dividing out the differentials which vanish at every point (of $\bar{Y}$) we obtain an exact sequence

$$I/I^2 \rightarrow A' \otimes_A L \rightarrow L' \rightarrow 0$$

of $(\mathbb{R}, A')$ -Lie algebras. Since $M' = A' \otimes_A M$, the (A, L) -module structure on M passes to an $(A', A' \otimes_A L)$ -module structure on M' . Thus we must prove that, whenever

$f \in I$, that is, whenever f is a function in $C^\infty(N)$ which vanishes on Y , the element $1 \otimes df$ of $A' \otimes_A L$ acts trivially on M' . This may be seen as follows:

Since (M, χ) is a stratified prequantum module, in view of the definition, with reference to the stratum Y (which is an ordinary smooth manifold), the induced module $M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M$ has a prequantum module structure χ_Y for $(C^\infty(Y), \{\cdot, \cdot\}^Y)$ in such a way that the canonical linear map of complex vector spaces from M to M_Y is a morphism of prequantum modules from (M, χ) to (M_Y, χ_Y) . The $C^\infty(Y)$ -module underlying the latter may be written as

$$M_Y = C^\infty(Y) \otimes_{C^\infty(N)} M \cong C^\infty(Y) \otimes_{A'} M',$$

and the induced morphism from $M' \cong M_{\overline{Y}}$ to M_Y is compatible with the actions, with respect to the induced morphism of Lie-Rinehart algebras from $(A', A' \otimes_A L)$ to $(C^\infty(Y), \Omega^1(Y)_{\{\cdot, \cdot\}^Y})$; that is to say: these morphisms make the diagram

$$\begin{array}{ccc} (A' \otimes_A L) \otimes M' & \longrightarrow & M' \\ \downarrow & & \downarrow \\ \overline{L}_{\{\cdot, \cdot\}^Y} \otimes M_Y & \longrightarrow & M_Y \end{array}$$

commutative. Consequently the element $1 \otimes df$ of $A' \otimes_A L$ acts trivially on M_Y since it acts thereupon through the map from $A' \otimes_A L$ to $\overline{L}_{\{\cdot, \cdot\}^Y}$ where it becomes trivial, and thence $1 \otimes df$ acts trivially on $M' \cong M_{\overline{Y}}$ since, by definition, the canonical map of complex vector spaces from $M_{\overline{Y}}$ to M_Y is required to be injective. $\square$

(1.6) *Costratified prequantum spaces.* Let $(N, C^\infty(N), \{\cdot, \cdot\})$ be a stratified symplectic space. Under the circumstances of (1.5.6), when Y runs through the strata of N , we will refer to the system

$$(1.6.1) \quad (M_{\overline{Y}}, \chi_{\overline{Y}}: \overline{L}_{\{\cdot, \cdot\}^{\overline{Y}}} \rightarrow \text{End}_{\mathbb{R}}(M_{\overline{Y}}))$$

of prequantum modules, together with, for every pair of strata Y, Y' such that $Y' \subseteq \overline{Y}$, the induced morphism

$$(1.6.2) \quad (M_{\overline{Y}}, \chi_{\overline{Y}}) \rightarrow (M_{\overline{Y}'}, \chi_{\overline{Y}'})$$

of prequantum modules, as a *costratified prequantum space*. More formally: Consider the category $\mathcal{C}_N$ whose objects are the strata of N and whose morphisms are the inclusions $Y' \subseteq \overline{Y}$. We define a *costratified complex vector space* on N to be a contravariant functor from $\mathcal{C}_N$ to the category of complex vector spaces, and a *costratified prequantum space* on N to be a costratified complex vector space together with a compatible system of prequantum module structures. The discussion in (1.4) and (1.5) may be summarized by saying that a stratified prequantum module (M, χ) for $(N, C^\infty(N), \{\cdot, \cdot\})$ determines a costratified prequantum space on N : For every stratum Y , let $(\overline{Y}, C^\infty(\overline{Y}), \{\cdot, \cdot\}^{\overline{Y}})$ be the induced stratified symplectic structure on the closure $\overline{Y}$ of Y given in (1.4), and let $(C^\infty(\overline{Y}), \overline{L}_{\{\cdot, \cdot\}^{\overline{Y}}})$ be the corresponding Lie-Rinehart algebra. Then the assignment to a stratum Y of the induced stratified

prequantum module $(M_{\overline{Y}}, \chi_{\overline{Y}})$ is a functor on $\mathcal{C}_N$ in an obvious fashion; in particular, whenever Y and Y' are two strata such that $Y' \subseteq \overline{Y}$, restriction yields a morphism

$$(M_{\overline{Y}}, \chi_{\overline{Y}}) \rightarrow (M_{\overline{Y'}}, \chi_{\overline{Y'}})$$

of stratified prequantum modules. The system which encompasses *all* these restriction morphisms is the corresponding costratified prequantum space.

2. Prequantum modules and reduction

Consider an ordinary smooth symplectic manifold N , acted upon by a compact Lie group G in a hamiltonian fashion with momentum mapping $\mu: N \rightarrow \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G . Suppose in addition that the symplectic manifold N is quantizable, let $\zeta: \Lambda \rightarrow N$ be a prequantum bundle, and suppose that the hamiltonian G -action on N lifts to an action on ζ preserving the connection. When G is connected, this additional assumption is (well known to be) redundant (since the action is hamiltonian) and it will suffice to replace G with a suitable covering group if need be so that the hamiltonian action on N lifts to an action of the covering group on ζ preserving the connection, cf. e. g. [22]; we then work with this covering group which we continue to denote by G . Thus, with this preparation out of the way, G acts on ζ . Let $M = \Gamma^\infty(\zeta)$ (the space of smooth complex valued sections of ζ), endowed with the smooth prequantum module structure (1.5.3), which we now write as $\chi: \overline{L}_{\{\cdot, \cdot\}} \rightarrow \text{End}_{\mathbb{R}}(M)$. In this paper we will only explore the case of zero value of the momentum mapping. The case of non-zero value would require much more effort and, in particular, the connection preserving prequantum lift of the action might be inappropriate.

The reduced space $N^{\text{red}} = \mu^{-1}(0)/G$ is well known to be stratified by orbit types. As usual, the superscript “ $-G$ ” will refer to G -invariants. Let I be the ideal of smooth functions on N which vanish on the zero locus $\mu^{-1}(0)$, and let $C^\infty(N^{\text{red}}) = (C^\infty(N))^G / I^G$, the algebra of smooth G -invariant functions, divided out by the ideal of smooth G -invariant functions which vanish on the zero locus. This is an algebra of continuous functions on N^{red} in an obvious fashion such that, restricted to any stratum, these functions are smooth; in other words, $C^\infty(N^{\text{red}})$ is a smooth structure on N^{red} . As observed in ARMS-CUSHMAN-GOTAY [1], cf. the proof of Theorem 1 in [1], the Noether theorem entails that the ideal I^G of G -invariant functions which vanish on $\mu^{-1}(0)$ is a Poisson ideal in the Poisson algebra $(C^\infty(N))^G$ of smooth G -invariant functions, that is, the symplectic Poisson structure on $C^\infty(N)$ descends to a Poisson structure $\{\cdot, \cdot\}^{\text{red}}$ on $C^\infty(N^{\text{red}})$.

Consider the sheaf or complex V-line bundle (also called orbi bundle)

$$\zeta^{\text{red}}: \Lambda^{\text{red}} = (\Lambda|_{\mu^{-1}(0)})/G \rightarrow N^{\text{red}}.$$

As a sheaf (in the category of spaces and ordinary continuous functions) it may be written as the direct image $q_* (\zeta|_{\mu^{-1}(0)})$ where $q: \mu^{-1}(0) \rightarrow N^{\text{red}}$ refers to the projection. On each stratum, cf. e. g. [8, 30], this V-line bundle restricts to an ordinary smooth complex V-line bundle. The sheaf (or V-line bundle) ζ gives rise to a prequantization construction for the reduced Poisson algebra, in the following fashion: We will write the space of sections of ζ that vanish on the zero locus of μ as IM . We mention in passing that a partition of unity argument shows that any

section of ζ that vanishes on the zero locus may be written as a sum $\sum \beta_j \zeta_j$ where the ζ_j 's are just sections of ζ and where the β_j 's are functions in I , that is, functions which vanish on the zero locus $\mu^{-1}(0)$. This justifies the notation. We shall not exploit this observation, though. Let $M^{\text{red}} = M^G / (IM)^G$, that is, the space M^G of ordinary smooth G -invariant sections of ζ , modulo the subspace $(IM)^G$ of smooth G -invariant sections that vanish on the zero locus $\mu^{-1}(0)$. By construction, M^{red} may canonically be identified with a space of continuous sections of ζ^{red} which, on each stratum, restrict to a smooth section, and M^{red} inherits a $C^\infty(N^{\text{red}}, \mathbb{C})$ -module structure. Accordingly, we will occasionally denote M^{red} by $\Gamma^\infty(\zeta^{\text{red}})$ when M^{red} is viewed being endowed with this $C^\infty(N^{\text{red}}, \mathbb{C})$ -module structure. As a side remark we note that, since associating to an ordinary complex vector bundle on N^{red} its space of continuous sections is an equivalence of categories from complex vector bundles to projective modules over the algebra $C(N^{\text{red}}, \mathbb{C})$ of continuous functions on N^{red} , when M^{red} is a projective $C^\infty(N^{\text{red}}, \mathbb{C})$ -module the reduced V-line bundle ζ^{red} is an ordinary line bundle. The converse is presumably true as well; it would rely on the corresponding equivalence of categories spelled out for $C^\infty(N^{\text{red}}, \mathbb{C})$ -modules but details have not been worked out yet.

Theorem 2.1. *Given an ordinary G -equivariant prequantum bundle ζ on the smooth hamiltonian G -space (N, μ) , the prequantum module structure χ of $M = \Gamma^\infty(\zeta)$ determines a stratified prequantum module structure χ^{red} on M^{red} (made explicit in (2.9) below) for the stratified symplectic space $(N^{\text{red}}, C^\infty(N^{\text{red}}), \{\cdot, \cdot\}^{\text{red}})$.*

Proof. Let $A = C^\infty(N)$, write $\{\cdot, \cdot\}^G$ for the induced Poisson structure on A^G , and let ∇ be the operator of covariant derivative determined by the connection on the prequantum bundle ζ . The corresponding extension (1.2.1) of $(\mathbb{R}, A^G)$ -Lie algebras may be written as

$$(2.2) \quad 0 \rightarrow A^G \rightarrow \bar{L}_{\{\cdot, \cdot\}^G}^a \rightarrow D_{\{\cdot, \cdot\}^G} \rightarrow 0$$

and, by naturality, the inclusion $(A^G, \{\cdot, \cdot\}^G) \rightarrow (A, \{\cdot, \cdot\})$ of Poisson algebras induces a commutative diagram

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A^G & \longrightarrow & \bar{L}_{\{\cdot, \cdot\}^G}^a & \longrightarrow & D_{\{\cdot, \cdot\}^G} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & \bar{L}_{\{\cdot, \cdot\}}^a & \longrightarrow & D_{\{\cdot, \cdot\}} \longrightarrow 0 \end{array}$$

of extensions of Lie-Rinehart algebras; notice that here the algebra variable changes whence we cannot refer merely to $(\mathbb{R}, A)$ -Lie algebras. The composite of χ (more precisely, of the corresponding algebraic prequantum module structure) with the induced map from $\bar{L}_{\{\cdot, \cdot\}^G}^a$ to $\bar{L}_{\{\cdot, \cdot\}}^a$ yields an $(A^G, \bar{L}_{\{\cdot, \cdot\}^G}^a)$ -module structure on M .

By symmetry, the $\bar{L}_{\{\cdot, \cdot\}^G}^a$ -action on M preserves M^G since, given a G -equivariant section η of ζ and a G -equivariant vector field X on N , the section $\nabla_X \eta$ of ζ is also G -equivariant. Hence M^G is a submodule of M , for the $(A^G, \bar{L}_{\{\cdot, \cdot\}^G}^a)$ -module structure. We write $\chi^G: \bar{L}_{\{\cdot, \cdot\}^G}^a \rightarrow \text{End}_{\mathbb{R}}(M^G)$ for the resulting algebraic prequantum module structure for $(A^G, \{\cdot, \cdot\}^G)$ on M^G . We now assert that this $(A^G, \bar{L}_{\{\cdot, \cdot\}^G}^a)$ -module structure χ^G preserves $(IM)^G \subseteq M^G$. Since we already know that χ^G preserves M^G it will suffice to see that it preserves IM .

In order to justify the last claim, let $\Lambda^\times \subseteq \Lambda$ be the subspace of non-zero vectors, so that ζ , restricted to $\Lambda^\times$, is the corresponding principal $\mathbb{C}^\times$ -bundle which, abusing the notation ζ , we denote by $\zeta: \Lambda^\times \rightarrow N$ as well. The identity

$$h(\zeta(z)) = h^\sharp(z)z, \quad z \in \Lambda^\times,$$

is well known to establish an isomorphism between the space M of sections h of ζ and the space of functions $h^\sharp: \Lambda^\times \rightarrow \mathbb{C}$ which are *equivariant* in the sense that

$$h^\sharp(cz) = c^{-1}h^\sharp(z), \quad z \in \Lambda^\times, \quad c \in \mathbb{C}^\times.$$

For any vector field X on N , the covariant derivative of a section h in the direction of X is the section $\nabla_X h$ of ζ given by

$$\nabla_X h(\zeta(z)) = (X^\sharp h^\sharp)(z)z, \quad z \in \Lambda^\times,$$

where $X^\sharp$ is the horizontal lift of X (with reference to the connection corresponding to the prequantum bundle structure) to a vector field on $\Lambda^\times$; see e. g. [33] for details. Let $f \in (C^\infty(N))^G$. By the Noether theorem, the momentum mapping $\mu: N \rightarrow \mathfrak{g}^*$ is constant along the flow lines of the hamiltonian vector field X_f ($= \{\cdot, f\}$) of f as well as along the flow lines of the vector field $-X_f = \{f, \cdot\}$ which comes into play in the prequantization formula (1.1.1). Since each flow line of $X_f^\sharp$ in $\Lambda^\times$ is the (unique) horizontal lift of a flow line of X_f in N , the composite

$$\mu^\sharp = \mu \circ \zeta: \Lambda^\times \rightarrow \mathfrak{g}^*$$

is constant along the flow lines of $X_f^\sharp$ in $\Lambda^\times$.

Let $h^\sharp: \Lambda^\times \rightarrow \mathbb{C}$ be an equivariant function which vanishes on $(\mu^\sharp)^{-1}(0)$. Thus $h^\sharp$ corresponds to a section h of ζ which vanishes on $\mu^{-1}(0)$, i. e. $h \in IM$. Let $q \in (\mu^\sharp)^{-1}(0)$, and let $f \in (C^\infty(N))^G$. Consider a flow line

$$J \rightarrow \Lambda^\times, \quad t \mapsto \exp(tX_f^\sharp)q,$$

of $X_f^\sharp$ having q as its starting point where J is a suitable open interval containing 0. Since $\mu^\sharp$ is constant along the flow lines of $X_f^\sharp$, $\mu^\sharp(\exp(tX_f^\sharp)q) = 0$ for every $t \in J$. Consequently

$$(X_f^\sharp(h^\sharp))(q) = \frac{d}{dt} \left(h^\sharp(\exp(tX_f^\sharp)q) \right) \Big|_{t=0} = 0,$$

that is, the vanishing of the section h on $\mu^{-1}(0)$ implies that of $\nabla_{X_f} h$ as well. This observation entails that the $(A^G, \overline{L}_{\{\cdot, \cdot\}}^a)_G$ -module structure χ^G preserves $IM \subseteq M$ since f is an arbitrary G -invariant function. For on the Lie-Rinehart generators $(0, \alpha)$ of $\overline{L}_{\{\cdot, \cdot\}} = A \oplus \Omega^1(N)$ where $\alpha \in \Omega^1(N)$, cf. (1.2.2) for the corresponding “algebraic” construction, the prequantum module structure (1.5.3) written there as χ_∇ , is given by $\chi_\nabla(0, \alpha) = \nabla_{\pi^\sharp(\alpha)}$, and it suffices to take here as generators differentials α of the kind $\alpha = dh$ where h runs through (smooth) functions on N . Likewise, as

an $(\mathbb{R}, A^G)$ -Lie algebra, $\overline{L}_{\{\cdot, \cdot\}^G}^a$ is generated by differentials $\alpha = df$ of G -invariant functions f on N , more precisely, by the elements

$$(0, df) \in A^G \oplus D_{\{\cdot, \cdot\}^G} = \overline{L}_{\{\cdot, \cdot\}^G}^a.$$

Consequently the $(A^G, \overline{L}_{\{\cdot, \cdot\}^G}^a)$ -module structure χ^G preserves $IM \subseteq M$ as asserted. Since it also preserves $M^G \subseteq M$, it preserves $(IM)^G \subseteq M^G$ and therefore induces an $(A^G, \overline{L}_{\{\cdot, \cdot\}^G}^a)$ -module structure

$$(2.4) \quad \overline{L}_{\{\cdot, \cdot\}^G}^a \longrightarrow \text{End}_{\mathbb{R}}(M^{\text{red}})$$

on $M^{\text{red}} = M^G/(IM)^G$.

The obvious epimorphism from $D_{\{\cdot, \cdot\}^G}$ onto $D_{\{\cdot, \cdot\}^{\text{red}}}$ and that from $\overline{L}_{\{\cdot, \cdot\}^G}^a$ onto $\overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a$ together give rise to *induced* $(\mathbb{R}, A^{\text{red}})$ -Lie algebra structures on $A^{\text{red}} \otimes_{A^G} D_{\{\cdot, \cdot\}^G}$ and $A^{\text{red}} \otimes_{A^G} \overline{L}_{\{\cdot, \cdot\}^G}^a$, respectively; see Section 1 in [13] for details on induced structures. Application of the functor $A^{\text{red}} \otimes_{A^G} -$ to (2.2) then yields the extension

$$(2.5) \quad 0 \rightarrow A^{\text{red}} \rightarrow A^{\text{red}} \otimes_{A^G} \overline{L}_{\{\cdot, \cdot\}^G}^a \rightarrow A^{\text{red}} \otimes_{A^G} D_{\{\cdot, \cdot\}^G} \rightarrow 0$$

of $(\mathbb{R}, A^{\text{red}})$ -Lie algebras. It is clear that (2.4) factors through the obvious epimorphism from $\overline{L}_{\{\cdot, \cdot\}^G}^a$ onto $A^{\text{red}} \otimes_{A^G} \overline{L}_{\{\cdot, \cdot\}^G}^a$.

We now assert that (2.4) factors even through the epimorphism from $\overline{L}_{\{\cdot, \cdot\}^G}^a$ onto $\overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a$ induced by the epimorphism of Poisson algebras from $(A^G, \{\cdot, \cdot\}^G)$ to $(A^{\text{red}}, \{\cdot, \cdot\}^{\text{red}})$. Indeed, by naturality, the inclusion of Poisson algebras from $(A^G, \{\cdot, \cdot\}^G)$ into $(A, \{\cdot, \cdot\})$ induces the commutative diagram

$$(2.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & A^G & \longrightarrow & \overline{L}_{\{\cdot, \cdot\}^G}^a & \longrightarrow & D_{\{\cdot, \cdot\}^G} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{\text{red}} & \longrightarrow & A^{\text{red}} \otimes_{A^G} \overline{L}_{\{\cdot, \cdot\}^G}^a & \longrightarrow & A^{\text{red}} \otimes_{A^G} D_{\{\cdot, \cdot\}^G} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{\text{red}} & \longrightarrow & \overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a & \longrightarrow & D_{\{\cdot, \cdot\}^{\text{red}}} \longrightarrow 0. \end{array}$$

By the general theory of Kähler differentials, the obvious morphism of A^{red} -modules from $A^{\text{red}} \otimes_{A^G} D_{A^G}$ to $D_{A^{\text{red}}}$ fits into an exact sequence

$$(2.7) \quad I^G/(I^G)^2 \longrightarrow A^{\text{red}} \otimes_{A^G} D_{A^G} \longrightarrow D_{A^{\text{red}}} \longrightarrow 0$$

of A^{red} -modules where the first arrow is given by the association

$$a \bmod (I^G)^2 \longmapsto 1 \otimes da \in A^{\text{red}} \otimes_{A^G} D_{A^G},$$

cf. (1.5.11) above. Now we claim that, for $f \in I^G$ and $h \in M^G$, we have

$$df(h) = \nabla_{\pi^\sharp(df)}(h) \in (IM)^G$$

where df is viewed as the element $(0, df)$ of $\overline{L}_{\{\cdot, \cdot\}}^a = A^G \oplus D_{\{\cdot, \cdot\}}^G$. To justify this claim we recall first that, as already noted above, for any $q \in \mu^{-1}(0)$,

$$\mu(\exp(tX_f)q) = \mu(q) = 0$$

whence the flow line of X_f starting at q lies in $\mu^{-1}(0)$, in fact, in a stratum thereof. Given a smooth G -invariant function k on N , by the Noether theorem, the momentum mapping μ is constant along the trajectories of X_k . Hence, given $q \in \mu^{-1}(0)$, since $\mu(\exp(tX_k)q) = \mu(q)$, we have

$$\{f, k\}(q) = -(X_f(k))(q) = (X_k(f))(q) = \frac{d}{dt} (f(\exp(tX_k)q)) \big|_{t=0} = 0$$

Hence

$$(X_f(k))(q) = \frac{d}{dt} (k(\exp(tX_f)q)) \big|_{t=0} = 0.$$

Our aim is to prove that, likewise, for $q^\sharp \in \zeta^{-1}(q)$,

$$(X_f^\sharp(h^\sharp))(q^\sharp) = \frac{d}{dt} (h^\sharp(\exp(tX_f^\sharp)q^\sharp)) \big|_{t=0} = 0$$

where $h^\sharp$ is the smooth G -invariant function on $\Lambda^\times$ which corresponds to the smooth G -equivariant section h of ζ as explained above. To this end, we refine the reasoning in the following fashion: Let f be any smooth function on N which is zero on $\mu^{-1}(0)$, at this stage not necessarily G -invariant and, as before, let $q \in \mu^{-1}(0)$. Then, for any $Y \in \ker(d\mu_q: T_q N \rightarrow \mathfrak{g}^*)$, we have

$$\omega(X_f|_q, Y) = Yf = (Y, df(q)) = 0.$$

Since the annihilator of $\ker(d\mu_q)$ coincides with the tangent space $T_q Gq \subseteq T_q N$ of the G -orbit $Gq \subseteq N$, we conclude that $X_f|_q \in T_q Gq$. Hence, given any smooth G -invariant function k on N ,

$$(X_f(k))(q) = X_f|_q(k) = 0.$$

Since the connection on ζ is G -equivariant, the horizontal lift $X_f^\sharp|_{q^\sharp}$ of $X_f|_q$ lies in $T_{q^\sharp} Gq^\sharp \subseteq T_{q^\sharp} \Lambda^\times$. Consequently

$$(X_f^\sharp(h^\sharp))(q^\sharp) = X_f^\sharp|_{q^\sharp}(h^\sharp) = 0,$$

for $h^\sharp$ is G -invariant. Since $q \in \mu^{-1}(0)$ is arbitrary, we conclude that

$$\nabla_{X_f} h = \nabla_{\pi^\sharp(df)}(h)$$

vanishes on $\mu^{-1}(0)$, that is, that $\nabla_{\pi^\sharp(df)}$ lies in IM . By symmetry, if, in addition, f is G -invariant, $\nabla_{\pi^\sharp(df)}$ is G -equivariant whence $\nabla_{\pi^\sharp(df)}(h)$ lies in $(IM)^G$ as asserted.

Consequently (2.4) factors through the epimorphism from $\overline{L}_{\{\cdot, \cdot\}^G}^a$ onto $\overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a$ and hence induces an $(A^{\text{red}}, \overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a)$ -module structure

$$(2.8) \quad \overline{L}_{\{\cdot, \cdot\}^{\text{red}}}^a \longrightarrow \text{End}_{\mathbb{R}}(M^{\text{red}})$$

on M^{red} which is, indeed, that of an algebraic prequantum module for $(A^{\text{red}}, \{\cdot, \cdot\}^{\text{red}})$. Since the formal differentials which are zero at every point of N^{red} act trivially on M^{red} , (2.8) factors through a geometric prequantum module structure

$$(2.9) \quad \chi^{\text{red}}: \overline{L}_{\{\cdot, \cdot\}^{\text{red}}} \longrightarrow \text{End}_{\mathbb{R}}(M^{\text{red}}).$$

For each stratum $N_{(H)}^{\text{red}}$, restriction of the reduced V-line bundle ζ^{red} to that stratum yields a V-line bundle $\zeta_{(H)}$; since the stratum arises by ordinary symplectic reduction [32], the space of smooth sections $M_{(H)} = \Gamma^{\infty}(\zeta_{(H)})$ of $\zeta_{(H)}$ inherits a prequantum module structure

$$\chi_{(H)}: \overline{L}_{\{\cdot, \cdot\}_{(H)}} \longrightarrow \text{End}_{\mathbb{R}}(M_{(H)})$$

for $(A_{(H)}, \{\cdot, \cdot\}_{(H)})$. By naturality, the restriction mapping is compatible with these structures, that is, the restriction mappings and the adjoints $\chi_{\text{red}}^{\#}$ and $\chi_{(H)}^{\#}$ of the corresponding prequantum module structures make the requisite diagram

$$\begin{array}{ccc} \overline{L}_{\{\cdot, \cdot\}^{\text{red}}} \otimes M^{\text{red}} & \xrightarrow{\chi_{\text{red}}^{\#}} & M^{\text{red}} \\ \downarrow & & \downarrow \\ \overline{L}_{\{\cdot, \cdot\}_{(H)}} \otimes M_{(H)} & \xrightarrow{\chi_{(H)}^{\#}} & M_{(H)} \end{array}$$

commutative. In other words, the data determine a stratified prequantum module structure $(\chi^{\text{red}}, \{\chi_{(H)}\})$ for the stratified symplectic space $(N^{\text{red}}, C^{\infty}(N^{\text{red}}), \{\cdot, \cdot\}^{\text{red}})$. This completes the proof of Theorem 2.1. $\square$

3. Quantization

Let $(N, C^{\infty}(N), \{\cdot, \cdot\})$ be an arbitrary stratified symplectic space, and let $P \subseteq \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ be a stratified complex polarization (see Section 2 of [18]). We do not recall the general definition of a stratified complex polarization but, for intelligibility, we will briefly explain the special case of *complex analytic stratified Kähler polarization*. The results of the present paper will involve only this special case; we will build up the general theory in terms of general stratified complex polarizations, though. Thus suppose that the stratified space N , endowed with the smooth structure $C^{\infty}(N)$, is *fine*, that is, for an arbitrary locally finite open covering $\mathcal{U}$ of N , there is a partition of unity subordinate to $\mathcal{U}$, and that N is endowed with a complex analytic structure such that the following hold: (i) each stratum is complex analytic, that is, the stratification of N (as a stratified symplectic space) is a refinement of the complex analytic stratification; (ii) each (germ of) holomorphic function belongs to the smooth structure $C^{\infty}(N, \mathbb{C})$; and (iii) for every pair of (germs

of) holomorphic functions f and h , the Poisson bracket $\{f, h\}$ is zero. Then the $C^\infty(N, \mathbb{C})$ -submodule P of $\Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ generated by differentials of the kind udf where f is holomorphic and u a bump function in $C^\infty(N, \mathbb{C})$ (the notion of bump function being interpreted in terms of the *fineness* of the structure) is a complex analytic stratified Kähler polarization for N . See Theorem 2.5 in [18] for details.

We shall say that a function $f \in C^\infty(N)$ is *compatible with the polarization* P or, equivalently, *quantizable in the polarization* P provided, for every $\alpha \in P$, $[df, \alpha] \in P$. When N is a smooth symplectic manifold and $\{\cdot, \cdot\}$ its ordinary symplectic Poisson structure, this notion of compatibility with the polarization boils down to the usual notion of classical observable which is compatible with or, equivalently, quantizable in, the given complex polarization. See also p. 217 of [14].

The extension of $(\mathbb{C}, C^\infty(N, \mathbb{C}))$ -algebras determined by the complexified Poisson 2-form $\pi \otimes \mathbb{C}$ arises from the corresponding extension (1.5.1) by complexification and has the form

$$(3.1) \quad 0 \rightarrow C^\infty(N, \mathbb{C}) \rightarrow \overline{L}_{\{\cdot, \cdot\}}^{\mathbb{C}} \rightarrow \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}} \rightarrow 0.$$

Recall that, cf. (1.2.2), as a $C^\infty(N, \mathbb{C})$ -module, $\overline{L}_{\{\cdot, \cdot\}}^{\mathbb{C}}$ is the direct sum of $C^\infty(N, \mathbb{C})$ and $\Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ whence there is an obvious section κ for (3.1). Now a function $f \in C^\infty(N)$ is compatible with the polarization P if and only if, for every $\alpha \in P$,

$$[(f, df), (0, \alpha)] \in \kappa(P).$$

More formally, we may proceed as follows: Let $\Omega^1(N, \mathbb{C})_P \subseteq \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ be the Lie subalgebra which consists of all $\alpha \in \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ such that, for every $\beta \in P$, $[\alpha, \beta] \in \Omega^1(N, \mathbb{C})_P$; in other words, $\Omega^1(N, \mathbb{C})_P$ is the *normalizer* (in the sense of Lie algebras) of P in $\Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$. For intelligibility, we note that, under the adjoint $\pi^\#$ from $\Omega^1(N, \mathbb{C})$ to the complexified vector fields $\text{Vect}(N, \mathbb{C})$, the Lie algebra $\Omega^1(N, \mathbb{C})_P$ passes to the Lie subalgebra $\text{Vect}(N, \mathbb{C})_P$ of $\text{Vect}(N, \mathbb{C})$ which consists of complexified vector fields on N that are compatible with the polarization P ; here $\text{Vect}(N, \mathbb{C})$ refers to the $(\mathbb{C}, C^\infty(N, \mathbb{C}))$ -Lie algebra of complexified vector fields on N .

Let $C^P(N, \mathbb{C}) \subseteq C^\infty(N, \mathbb{C})$ be the subalgebra of P -invariant elements; in general, non-trivial functions in $C^P(N, \mathbb{C})$ will exist at most locally, and we should talk about the *sheaf of germs of* P -invariant functions. When N is a complex analytic stratified Kähler space, with stratified Kähler polarization P , $C^P(N, \mathbb{C})$ amounts to the sheaf of germs of holomorphic functions. A special case thereof is that of a smooth Kähler manifold where P arises from the ordinary holomorphic polarization. In the general case, the Lie algebras $\overline{L}_P^{\mathbb{C}}$ and $\Omega^1(N, \mathbb{C})_P$ inherit $(\mathbb{C}, C^P(N, \mathbb{C}))$ -Lie algebra structures in an obvious fashion, the extension (3.1) restricts to an extension

$$0 \rightarrow C^\infty(N, \mathbb{C}) \rightarrow \overline{L}_P^{\mathbb{C}} \rightarrow \Omega^1(N, \mathbb{C})_P \rightarrow 0$$

of $(\mathbb{C}, C^\infty(N, \mathbb{C}))$ -algebras, and $\overline{L}_P^{\mathbb{C}}$ may be viewed as a sub $(\mathbb{C}, C^P(N, \mathbb{C}))$ -Lie algebra of $\overline{L}_{\{\cdot, \cdot\}}^{\mathbb{C}}$. For later reference, we spell out the following, whose proof is straightforward and left to the reader.

Proposition 3.2. *A function $f \in C^\infty(N)$ is quantizable in the polarization P if and only $(f, df) \in \overline{L}_{\{\cdot, \cdot\}}^\mathbb{C}$ lies in $\overline{L}_P^\mathbb{C}$, that is to say, if and only if f lies in the pre-image of $\overline{L}_P^\mathbb{C}$ under the canonical map ι from $C^\infty(N)$ to $\overline{L}_{\{\cdot, \cdot\}}^\mathbb{C}$. $\square$*

Let (M, χ) be a stratified prequantum module for $(C^\infty(N), \{\cdot, \cdot\})$. The composite of the injection of P into $\Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ with κ , combined with the prequantum module structure χ , yields a $(C^\infty(N, \mathbb{C}), P)$ -module structure on M ; we denote the subspace of invariants by M^P , cf. what is said in our paper [14]. Likewise, on each stratum Y , we have the subspace of invariants $M_Y^{P_Y}$, and the restriction map induces a linear map $M^P \rightarrow M_Y^{P_Y}$ of complex vector spaces. We refer to the system consisting of M^P and the restriction maps to the $M_Y^{P_Y}$'s as the *stratified quantum module* determined by the stratified polarization P . Given a stratified quantum module, the prequantization formula (1.2.8) induces a representation of the elements of $C^\infty(N)$ which are quantizable in the stratified polarization P by $\mathbb{C}$ -linear operators on the stratified quantum module, and this representation satisfies the conditions (1.2.6) and (1.2.7).

As a side remark we note that, more generally, with respect to the Lie-Rinehart algebra $(C^\infty(N, \mathbb{C}), P)$, the Lie-Rinehart complex $(\text{Alt}_{C^\infty(N, \mathbb{C})}(P, M), d)$ as well as, for every stratum Y of N , the Lie-Rinehart complexes $(\text{Alt}_{C^\infty(Y, \mathbb{C})}(P_Y, M_Y), d)$, determine a system consisting of the Lie-Rinehart cohomology groups $H^*(P, M)$ together with the restriction maps to the Lie-Rinehart cohomology groups $H^*(P_Y, M_Y)$. The system consisting of M^P and the restriction maps to the $M_Y^{P_Y}$'s explained earlier (where Y runs through the strata) boils down to the corresponding zero'th cohomology groups. The prequantization formula (1.2.8) now induces a representation of the elements of $C^\infty(N)$ which are quantizable in the stratified polarization P by $\mathbb{C}$ -linear operators on $H^*(P, M)$ as well as on the $H^*(P_Y, M_Y)$'s, these representations satisfy the conditions (1.2.6) and (1.2.7) as well, and the entire system carries the appropriate costratified structure. Since we shall not exploit this kind of costratified structure in the rest of the paper, we refrain from spelling out details. An illustration for a situation with a single stratum where ordinary Hodge cohomology groups come into play will be given shortly.

In particular, suppose that $(N, C^\infty(N), \{\cdot, \cdot\}, P)$ is a *complex analytic* stratified Kähler space, cf. [18], and let (M, χ) be a stratified prequantum module for $(C^\infty(N), \{\cdot, \cdot\})$. We shall refer to (M, χ) as a *complex analytic* stratified prequantum module provided M is the space of sections of a complex V-line bundle ζ on N in such a way that P endows ζ via χ with a complex analytic structure, that is to say, M^P amounts to the sheaf of germs of holomorphic sections of ζ .

Next we show that stratified Kähler quantization is compatible with passing to the closure of a stratum. Thus, let $(N, C^\infty(N), \{\cdot, \cdot\}, P)$ be a stratified Kähler space, let Y be a stratum of N , and let $(\overline{Y}, C^\infty(\overline{Y}), \{\cdot, \cdot\}^{\overline{Y}})$ be the induced stratified symplectic structure on the closure $\overline{Y}$ of Y , cf. (1.4).

Proposition 3.3. *The stratified Kähler polarization $P \subseteq \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ induces a stratified Kähler polarization $P_{\overline{Y}} \subseteq \Omega^1(\overline{Y}, \mathbb{C})_{\{\cdot, \cdot\}^{\overline{Y}}}$ for $(\overline{Y}, C^\infty(\overline{Y}), \{\cdot, \cdot\}^{\overline{Y}})$. When P is complex analytic, so is $P_{\overline{Y}}$.*

Proof. Let I be the ideal of functions in $C^\infty(N)$ which vanish on Y (and hence on

$\bar{Y}$). The canonical projection from $C^\infty(N)$ to $C^\infty(\bar{Y})$ induces an exact sequence

$$I/I^2 \rightarrow C^\infty(\bar{Y}) \otimes_{C^\infty(N)} \Omega^1(N, \mathbb{C}) \rightarrow \Omega^1(\bar{Y}, \mathbb{C}) \rightarrow 0.$$

The image $P_{\bar{Y}} \subseteq \Omega^1(\bar{Y}, \mathbb{C})$ of the induced $C^\infty(\bar{Y})$ -submodule $C^\infty(\bar{Y}) \otimes_{C^\infty(N)} P$ of $C^\infty(\bar{Y}) \otimes_{C^\infty(N)} \Omega^1(N, \mathbb{C})$ is a stratified Kähler polarization for $(\bar{Y}, C^\infty(\bar{Y}), \{\cdot, \cdot\}^{\bar{Y}})$. When P is complex analytic, so is $P_{\bar{Y}}$. $\square$

Let (M, χ) be a stratified prequantum module for $(N, C^\infty(N), \{\cdot, \cdot\})$, for any stratum Y of N , let

$$\chi_{\bar{Y}}: \bar{L}_{\{\cdot, \cdot\}^{\bar{Y}}} \rightarrow \text{End}_{\mathbb{R}}(M_{\bar{Y}})$$

be the induced stratified prequantum module structure for $(\bar{Y}, C^\infty(\bar{Y}), \{\cdot, \cdot\}^{\bar{Y}})$, cf. (1.5.7), and let $P_{\bar{Y}} \subseteq \Omega^1(\bar{Y}, \mathbb{C})_{\{\cdot, \cdot\}^{\bar{Y}}}$ be the induced stratified Kähler polarization.

Theorem 3.4. *For any stratum Y , the morphism of stratified prequantum modules from (M, χ) to $(M_{\bar{Y}}, \chi_{\bar{Y}})$ passes to a morphism of stratified quantum modules from (M^P, χ) to $((M_{\bar{Y}})^{P_{\bar{Y}}}, \chi_{\bar{Y}})$. In particular, for every pair of strata Y, Y' such that $Y' \subseteq \bar{Y}$, the induced morphism of stratified prequantum modules from $(M_{\bar{Y}}, \chi_{\bar{Y}})$ to $(M_{\bar{Y}'}, \chi_{\bar{Y}'})$ passes to a morphism of stratified quantum modules from $((M_{\bar{Y}})^{P_{\bar{Y}}}, \chi_{\bar{Y}})$ to $((M_{\bar{Y}'})^{P_{\bar{Y}'}}), \chi_{\bar{Y}'})$.*

Proof. This is straightforward and left to the reader. $\square$

We will refer to the costratified complex vector space which arises from the costratified prequantum space coming from a stratified prequantum module on a stratified symplectic space (cf. (1.6) above) by taking invariants, as in Theorem 3.4, with reference to a stratified polarization, as a *costratified quantum space*. The linear maps between the constituents of a costratified quantum space are not required to be compatible with Hilbert space structures, though, whatever these structures may be.

Corollary 3.5. *Stratified Kähler quantization on a (quantizable) complex analytic stratified Kähler space $(N, C^\infty(N), \{\cdot, \cdot\}, P)$ yields a costratified quantum space, defined on the category $\mathcal{C}_N$. $\square$*

Finally we will show how stratified quantum modules and hence costratified quantum spaces arise in mathematical nature: Consider a smooth quantizable (positive) Kähler manifold N , viewed as a stratified symplectic space with a single stratum, let $F \subseteq T^{\mathbb{C}}N$ be its Kähler polarization, and let P be the (complex analytic) stratified Kähler polarization (in our sense) arising as the pre-image in $\Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}}$ of the space $\Gamma^\infty F$ of smooth sections of F , with reference to the induced isomorphism

$$\pi_{\{\cdot, \cdot\}}^\# \otimes \mathbb{C}: \Omega^1(N, \mathbb{C})_{\{\cdot, \cdot\}} \rightarrow \text{Vect}(N, \mathbb{C}),$$

cf. (2.1) in [18]. Let $\zeta: E \rightarrow N$ be a prequantum bundle, and let (M, χ) be the corresponding smooth prequantum module $(\Gamma^\infty(\zeta), \chi_\nabla)$, ∇ being the corresponding hermitian connection, cf. (1.5.2) above; we view (M, χ) as a stratified prequantum module (with a single stratum). The line bundle ζ is well known to inherit a holomorphic structure whence (M, χ) is a complex analytic stratified prequantum

module: the quantum module M^P equals that of F -polarized sections of ζ in the usual sense and these are precisely the holomorphic sections of ζ . In this case, the Lie-Rinehart cohomology groups $H^*(P, M)$ are just the Hodge cohomology groups of N with values in the holomorphic line bundle ζ . More generally, when N is a complex analytic stratified Kähler space, the Lie-Rinehart cohomology $H^*(P, M)$ is related with the cohomology of the sheaf of germs of ζ . In the smooth case, the prequantization formula (1.2.8), applied to M^P and quantizable observables, then amounts to geometric quantization in the usual sense; instead of stratified quantum module we shall then simply say *quantum module*.

Let G be a compact Lie group, and suppose that (i) $G^\mathbb{C}$ acts holomorphically on N in such a way that the restriction to G is hamiltonian and that (ii) the Kähler structure is G -invariant; let $\mu: N \rightarrow \mathfrak{g}^*$ be a corresponding momentum mapping. Suppose that the G -action lifts to an action on ζ preserving the connection. We have already observed that, for connected G , given the G -action on N , the assumption that G act on ζ is redundant and it will suffice to replace G by an appropriate covering group if necessary. The prequantum module M inherits a G -action preserving the polarization P and hence the quantum module M^P , that is, the space of holomorphic sections of ζ , is a complex representation space for G . The quantum module M^P is the corresponding *unreduced* quantum state space, except that there is no Hilbert space structure present yet, and *reduction after quantization*, for the *quantum state spaces*, amounts to taking the space $(M^P)^G$ of G -invariant holomorphic sections.

On the other hand, by Proposition 4.2 of [18], the (positive) Kähler polarization induces a (positive) complex analytic stratified Kähler polarization P^{red} on the reduced space N^{red} , with its stratified symplectic Poisson algebra $(C^\infty(N^{\text{red}}), \{\cdot, \cdot\}^{\text{red}})$. By Theorem 2.1, the prequantum module (M, χ) passes to a stratified prequantum module

$$(M^{\text{red}}, \chi^{\text{red}}: \bar{L}_{\{\cdot, \cdot\}^{\text{red}}} \longrightarrow \text{End}_{\mathbb{R}}(M^{\text{red}}))$$

for the stratified symplectic space $(N^{\text{red}}, C^\infty(N^{\text{red}}), \{\cdot, \cdot\}^{\text{red}})$. *Quantization after reduction*, for the *quantum state spaces*, now amounts to taking the corresponding *reduced* quantum module or *reduced quantum state space* $(M^{\text{red}})^{P^{\text{red}}}$, that is, the space of P^{red} -invariants in $M^{\text{red}} = \Gamma^\infty(\zeta^{\text{red}})$.

The projection map from M^G to $M^{\text{red}} = M^G/(IM)^G$ plainly restricts to a linear map

$$\rho: (M^P)^G \rightarrow (M^{\text{red}})^{P^{\text{red}}}$$

of complex vector spaces, referred to as *state space comparison map* in the introduction and, as far as the comparison of G -invariant unreduced and reduced quantum observables is concerned, the statement that *Kähler quantization commutes with reduction* amounts to the following.

Theorem 3.6. *Let N be a (positive) Kähler manifold with a holomorphic $G^\mathbb{C}$ -action whose restriction to G preserves the Kähler structure and is hamiltonian. Let f be a G -invariant smooth function on N which is quantizable (i. e. preserves the Kähler polarization P). Then its class $[f] \in C^\infty(N^{\text{red}}) (= (C^\infty(N))^G/I^G)$ is quantizable (i. e. preserves the stratified Kähler polarization P^{red}) and, for every $h \in (M^P)^G$,*

$$\rho(\widehat{f}(h)) = \widehat{[f]}(\rho(h)).$$

Proof. This is seen by direct comparison of the requisite formulas (1.2.8) for the unreduced and reduced cases. We leave the details to the reader. $\square$

We now suppose that μ is an admissible momentum mapping, that is, that for every $m \in N$ the path of steepest descent through m is contained in a compact set [21] (§9). For example, a proper momentum mapping is admissible. Another example of an admissible momentum mapping is the standard momentum mapping of a unitary representation space for a compact Lie group; see Ex. 2.1 in [30]. We recall from [30] that, for admissible μ , the reduced bundle ζ^{red} inherits a holomorphic structure in the following fashion: Let $N^{\text{ss}} \subseteq N$ be the subspace of semistable points (the points m of N such that the closure of the $G^{\mathbb{C}}$ -orbit through m intersects the zero level set $\mu^{-1}(0)$); the inclusion $\mu^{-1}(0) \subseteq N^{\text{ss}}$ then induces a homeomorphism $N^{\text{red}} \rightarrow N^{\text{ss}}/G^{\mathbb{C}}$ [30] (Theorem 2.3). Likewise, when $E^{\text{ss}} \subseteq E$ denotes the pre-image of N^{ss} , the inclusion $E|_{\mu^{-1}(0)} \subseteq E^{\text{ss}}$ induces a homeomorphism $E^{\text{red}} = E|_{\mu^{-1}(0)}/G \rightarrow E^{\text{ss}}/G^{\mathbb{C}}$. According to [28], the sheaf of germs of G -equivariant holomorphic sections of $\zeta|_{N^{\text{ss}}}$ is a coherent $\mathcal{O}_{N^{\text{red}}}$ -module. This entails that the germs of G -equivariant holomorphic sections of $\zeta|_{N^{\text{ss}}}$ endow E^{red} with a complex analytic structure in such a way that $\zeta^{\text{red}}: E^{\text{red}} \rightarrow N^{\text{red}}$ is complex analytic [30] (Proposition 2.11).

The statement “Kähler quantization commutes with reduction” is then completed by the following two observations which relate the quantum state spaces.

Proposition 3.7. [30] *When μ is admissible and when N^{red} has a top stratum (i. e. an open dense stratum), for example when μ is proper, the reduced stratified prequantum module $(M^{\text{red}}, \chi^{\text{red}})$ is complex analytic. More precisely: The inclusion $\Gamma^{\text{hol}}(\zeta^{\text{red}}) \rightarrow \Gamma^{\infty}(\zeta^{\text{red}}) = M^{\text{red}}$ of complex vector spaces identifies $\Gamma^{\text{hol}}(\zeta^{\text{red}})$ with the space $(M^{\text{red}})^{P^{\text{red}}}$ of P^{red} -polarized elements of M^{red} .*

Proof. A P^{red} -polarized element of $M^{\text{red}} = \Gamma^{\infty}(\zeta^{\text{red}})$ is a continuous section of ζ^{red} which is holomorphic on each stratum, in particular, on the top stratum. Since, as a complex analytic space, the reduced space N^{red} is normal, we conclude that a P^{red} -polarized element of M^{red} is indeed a holomorphic section of ζ^{red} . $\square$

Theorem 3.8. [30] *Under the circumstances of Theorem 3.6, when the momentum mapping μ is proper, in particular when N is compact, the map ρ is an isomorphism of complex vector spaces.*

Proof. This is an immediate consequence of Theorem 2.15 in [30]. In fact, the map ρ is induced by the inclusion $\Gamma^{\text{hol}}(\zeta^{\text{red}}) \rightarrow \Gamma^{\infty}(\zeta^{\text{red}}) = M^{\text{red}}$ and the inclusion $N^{\text{ss}} \subseteq N$. $\square$

A version of Theorem 3.8 has been established in (4.15) of [25]; cf. also [31] and the literature there, as well as [27] and [36] for generalizations to higher dimensional sheaf cohomology and [24] and the literature there for a generalization in a different direction.

REMARK 3.9. The statements of Theorems 3.6 and 3.8 are logically independent; in particular the statement of Theorem 3.6 makes sense whether or not ρ is an isomorphism, and its proof does not rely on ρ being an isomorphism. This raises the issue whether there may exist states on the reduced level which do *not* arise from

(invariant) unreduced states. In fact, when the state space comparison map is not an isomorphism, interesting things can happen; see [7].

REMARK 3.10. We do not make any claim as to whether or not the state space comparison map is unitary. In the physical context, this issue is, however, vital. An extensive discussion of the unitarity issue for the cotangent bundle case or, in physicist's terminology, for the Schrödinger representation, may be found in [6] together with a thorough treatment of various examples (for the cotangent bundle case) where the state space comparison map is unitary as well as of examples where this is not the case. In [8] and the subsequent literature referenced in [31], the question whether or not the state space comparison map is unitary is not addressed. Certain cases where unitarity has been established (under circumstances essentially different from ours) may be found in [3] and [10]. In the example treated in the next section, all Hilbert space structures will derive from the same Hilbert space and the unitarity issue is thereby circumvented.

4. Holomorphic nilpotent orbits and singular Fock spaces

For illustration, we will explore a class of examples involving holomorphic nilpotent orbits; these arise from the standard simple Lie algebras of hermitian type [18]. On the unreduced level, the corresponding quantum phase spaces are variants of ordinary Fock space; reduction then carries the underlying classical phase space to the closure of a holomorphic nilpotent orbit. Accordingly, we may view the costratified quantum phase space arising from Kähler quantization over the closure of a holomorphic nilpotent orbit (cf. (4.6) below) as a *singular* Fock space. The representations of compact Lie groups which will show up below are, of course, entirely classical. What is *new* in our approach is the construction of *representations by Kähler quantization on a Kähler space with singularities*.

(4.1) *Fock- and related spaces.* Consider $W = \mathbb{C}^m$, with its standard complex and Kähler structures, and let $A = C^\infty(W)$. By means of complex coordinates $z_1, \dots, z_m$ for W —these are linear functions on W , i. e. lie in the complex dual W^* of W —the ordinary smooth symplectic Poisson structure $\{\cdot, \cdot\}$ on the complexification $A \otimes \mathbb{C} = C^\infty(W, \mathbb{C})$ may be described by the formulas

$$(4.1.1) \quad \{z_j, \bar{z}_k\} = -2i\delta_{j,k}, \quad 1 \leq j, k \leq m.$$

To explain briefly the Kähler quantization on W in our framework, denote by $\zeta: W \times \mathbb{C} \rightarrow W$ the trivial line bundle, with its standard hermitian structure $\langle \cdot, \cdot \rangle$ which assigns the value $\langle \sigma, \sigma' \rangle = \phi \bar{\phi}'$ to two sections $\sigma = \phi \cdot 1$ and $\sigma' = \phi' \cdot 1$ of ζ . Recall that a 1-form $\vartheta: \Omega^1(W) \rightarrow A \otimes \mathbb{C}$ is called a (complex) *Poisson potential* (for $\{\cdot, \cdot\}$) provided $d\vartheta = \pi_{\{\cdot, \cdot\}}$ in the cochain complex computing Poisson cohomology [13, 14]. The assignments

$$(4.1.2) \quad \vartheta(dz_j) = \frac{1}{2}z_j, \quad \vartheta(d\bar{z}_j) = \frac{1}{2}\bar{z}_j, \quad 1 \leq j \leq m,$$

yield a Poisson potential ϑ for the Poisson bracket $\{\cdot, \cdot\}$ on $A \otimes \mathbb{C}$ (p. 220 of [14]); taking M to be the space $\Gamma^\infty(\zeta)$ of smooth complex sections of ζ so that M is essentially a copy of $C^\infty(W, \mathbb{C})$, and letting

$$(4.1.3) \quad \chi(a, \alpha)(h) = -i\vartheta(\alpha)h + \alpha(h) + iah, \quad h \in C^\infty(W, \mathbb{C}), a \in A, \alpha \in \Omega^1(W)_{\{\cdot, \cdot\}},$$

we obtain a prequantum module structure

$$(4.1.4) \quad \chi: \overline{L}_{\{\cdot, \cdot\}} \longrightarrow \text{End}_{\mathbb{R}}(M)$$

for $(A, \{\cdot, \cdot\})$, cf. (1.5) above. The prequantum module (M, χ) is plainly geometric and arises from a symplectic structure, cf. (1.5.3). Indeed, in the symplectic language, the symplectic potential which corresponds to ϑ is the ordinary 1-form $\frac{i}{4} \sum (z_j d\bar{z}_j - \bar{z}_j dz_j)$ which, when z_j is written as $q_j + ip_j$ ($1 \leq j \leq m$), has the form $\frac{1}{2} \sum (q_j dp_j - p_j dq_j)$. The corresponding operator of covariant derivative amounts to the ordinary hermitian connection on ζ , that is, for every $\alpha \in \overline{L}_{\{\cdot, \cdot\}}$ and every pair of smooth complex sections σ and σ' of ζ , $\langle \chi(\alpha)\sigma, \sigma' \rangle + \langle \sigma, \chi(\alpha)\sigma' \rangle$ equals $\alpha \langle \sigma, \sigma' \rangle$, and χ is characterized by this property. In view of Proposition 2.3 of [18], the holomorphic polarization P (or Kähler polarization in our sense, cf. Section 3 above,) is generated by the holomorphic differentials $dz_1, \dots, dz_m$ and, smooth complex sections σ being identified with smooth complex valued functions ψ on W , the resulting quantum module M^P consists of smooth complex functions ψ on W satisfying the requirement

$$0 = \chi(0, dz_j)(\psi) = -i\vartheta(dz_j)\psi + \{z_j, \psi\}, \quad 1 \leq j \leq m,$$

that is,

$$0 = i\{z_j, \psi\} + \vartheta(dz_j)\psi = 2\frac{\partial\psi}{\partial\bar{z}_j} + \frac{1}{2}z_j\psi, \quad 1 \leq j \leq m.$$

This implies the standard fact that M^P consists of functions ψ in $\mathbf{z}$ and $\bar{\mathbf{z}}$ which may be written in the form

$$(4.1.5) \quad \psi(\mathbf{z}, \bar{\mathbf{z}}) = \phi(\mathbf{z})e^{-\frac{z\bar{z}}{4}},$$

for an entire holomorphic function ϕ in $\mathbf{z} = (z_1, \dots, z_m)$, that is to say, as a module over the algebra of entire holomorphic functions on W , M^P is free, generated by the function ψ_0 given by the expression

$$(4.1.6) \quad \psi_0(\mathbf{z}, \bar{\mathbf{z}}) = e^{-\frac{z\bar{z}}{4}},$$

and $\langle \psi_0, \psi_0 \rangle = e^{-\frac{z\bar{z}}{2}}$. In the physical interpretation, ψ_0 represents the *ground state*. The inner product $\psi \cdot \psi'$ of two such elements ψ and ψ' of $M^P = \Gamma^{\text{hol}}(\zeta)$, where $\psi(\mathbf{z}, \bar{\mathbf{z}}) = \phi(\mathbf{z})e^{-\frac{z\bar{z}}{4}}$ and $\psi'(\mathbf{z}, \bar{\mathbf{z}}) = \phi'(\mathbf{z})e^{-\frac{z\bar{z}}{4}}$, is given by the standard formula

$$(4.1.7) \quad \psi \cdot \psi' = \int \phi \overline{\phi'} e^{-\frac{z\bar{z}}{2}} \varepsilon_m, \quad \varepsilon_m = \frac{\omega^m}{(2\pi)^m m!},$$

where ω refers to the symplectic form, and the physical Hilbert space, the *bosonic Fock* space $\mathcal{F}$, is the completion of the complex vector space of square integrable functions of the kind (4.1.5). For $k \geq 0$, we will write $\mathcal{F}_k$ for the subspace of $\mathcal{F}$ which consists of functions of this kind having ϕ a homogeneous degree k polynomial. It is well known that, on each $\mathcal{F}_k$ ($k \geq 0$), the integral (4.1.7) converges. For our purposes, the integral (4.1.7) will provide the requisite Hilbert space structures.

(4.2) *Unreduced observables.* A classical observable f , that is, a function on W , is directly quantizable (in the holomorphic polarization P) provided $\{z_k, \{z_j, f\}\}$ vanishes

for every $1 \leq j, k \leq m$ (p. 219 of [14]). In particular, every classical observable f that is at most linear in the $\bar{z}_j$'s is quantizable. The quantization $f \mapsto \widehat{f}$ of a quantizable classical observable f is then given by the formula (1.2.8) which, rewritten in terms of the Poisson potential ϑ given by (4.1.2), amounts to

$$(4.2.1) \quad \widehat{f}(\psi) = -i\{f, \psi\} + (f - \vartheta(df))\psi,$$

cf. [14] (2.6.7). For example, the energy function $f_E(\mathbf{z}, \bar{\mathbf{z}}) = \frac{1}{2}\mathbf{z}\bar{\mathbf{z}}$ is quantizable in this polarization and satisfies $\vartheta(df_E) = f_E$; thus, in view of (4.1.1), its quantization is given by

$$(4.2.2) \quad \widehat{f_E}(\psi) = -i\{f_E, \psi\} = -\frac{i}{2} \sum (z_j\{\bar{z}_j, \psi\} + \bar{z}_j\{z_j, \psi\}) = \sum \left(z_j \frac{\partial \psi}{\partial z_j} - \bar{z}_j \frac{\partial \psi}{\partial \bar{z}_j} \right)$$

whence, with the notation $E = \sum z_j \frac{\partial}{\partial z_j}$ for the ordinary EULER operator, for $\psi = \phi\psi_0$ in M^P , cf. (4.1.5), we have

$$(4.2.3) \quad \widehat{f_E}(\psi) = E(\phi)\psi_0.$$

Thus, when ϕ is a homogeneous degree k polynomial, $\widehat{f_E}(\psi) = k\psi$, the operator $\widehat{f_E}$ has the non-negative integers as its spectrum and, for $k \geq 0$, $\mathcal{F}_k$ is the eigenspace associated to k . This is of course known to be physically incorrect, the requisite additional term arising from the metaplectic correction. We will address this issue elsewhere.

(4.3) *Symmetries.* The momentum mapping (having the value zero at the origin of W) for the action of the maximal compact subgroup $U(m)$ of $\mathrm{Sp}(m, \mathbb{R}) \cong \mathrm{Sp}(W)$ ($U(m)$ being the maximal compact subgroup which fixes the complex structure of W) is well known to be given by

$$(4.3.1) \quad \mu: W \rightarrow \mathfrak{u}(m)^*, \quad \mu^X(\mathbf{z}) = \frac{i}{2} \sum x_{j,k} \bar{z}_j z_k, \quad X = [x_{j,k}] \in \mathfrak{u}(m).$$

In particular, for $X = -i\mathrm{Id} \in \mathfrak{u}(m)$, μ^X equals the energy function f_E . For general $X \in \mathfrak{u}(m)$, since μ^X involves the $\bar{z}_j$'s only linearly, the function $f = \mu^X$, viewed as an infinitesimal symmetry, is quantizable and satisfies $\vartheta(df) = f$. The formula (1.2.9) (which refers to infinitesimal symmetries) then comes down to

$$(4.3.2) \quad \widetilde{f}(\psi) = \{f, \psi\}$$

and yields the standard $\mathfrak{u}(m)$ -representation on $\mathbb{C}[W] = S_{\mathbb{C}}[W^*]$ ($= \mathbb{C}[z_1, \dots, z_m]$) (by skew-symmetric operators); this representation integrates to the standard $U(m)$ -representation on $S_{\mathbb{C}}[W^*]$.

Let H be a closed subgroup of $U(m)$; thus H is a compact subgroup of $\mathrm{Sp}(W) = \mathrm{Sp}(m, \mathbb{R})$, and restricting the $U(m)$ -representation yields a representation of H on each $\mathcal{F}_k$ and hence on $\mathcal{F}$. Let G be a subgroup of $\mathrm{Sp}(W)$ such that G and H constitute a *real (reductive) dual pair* in $\mathrm{Sp}(W)$ [12], and let $\mathfrak{g}$ be the Lie algebra of G , realized as a subalgebra of $\mathfrak{sp}(W)$ in the obvious fashion. We view W as the *unreduced phase*

space of a classical system with symmetries given by the representation of H on W . Elements of the Lie algebra $\mathfrak{sp}(W)$ may then be viewed as classical observables and the elements of $\mathfrak{g}$ as classical H -invariant observables.

(4.4) *Reduction after quantization.* The elements of $\mathfrak{k} = \text{Lie}(K)$ ($= \mathfrak{g} \cap \mathfrak{u}(m)$) may be viewed as quantizable classical observables. Restriction yields representations of H and $\mathfrak{k}$ on $\mathcal{F}$ and on each $\mathcal{F}_k$ ($k \geq 0$) in such a way that the H - and $\mathfrak{k}$ -representations centralize each other; here and below the Fock space $\mathcal{F}$ and its homogeneous components depend on the parameter s but we do not indicate this dependence in notation. We view the H -representation as a symmetry and the $\mathfrak{k}$ -representation as a quantization of classical observables. The corresponding formulas for $\mathfrak{h}$ and $\mathfrak{k}$ then result from (1.2.8) and (1.2.9) and, cf. (4.2.2), have the form

$$(4.4.1) \quad \widehat{f}(\psi) = -i\{f, \psi\} \text{ for } \mathfrak{k}, \quad \widetilde{f}(\psi) = \{f, \psi\} \text{ for } \mathfrak{h}.$$

Reduction after quantization then amounts to passing to the $\mathfrak{k}$ -representation on the space $\mathcal{F}^H$ of H -invariants given by (4.4.1); this $\mathfrak{k}$ -representation decomposes into representations on the homogeneous degree k constituents $\mathcal{F}_k^H$.

(4.5) *Quantization after reduction.* Let $\mu_H: W \rightarrow \mathfrak{h}^*$ denote the H -momentum mapping having the value zero at the origin; this momentum mapping is given by the composite of (4.3.1) with the projection onto $\mathfrak{h}^*$. By [18] (Proposition 4.2), the reduced space W^{red} inherits a normal (complex analytic stratified) Kähler structure.

We now recall the following three basic pairs where we write $K = G \cap \text{U}(m)$ and where the symmetric constituent $\mathfrak{p}$ of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is spelled out explicitly, for later reference:

$$(4.5.1) \quad (G, H, K) = (\text{Sp}(\ell, \mathbb{R}), \text{O}(s, \mathbb{R}), \text{U}(\ell)), \quad m = 2s\ell, s \leq \ell, \quad \mathfrak{p} \cong S_{\mathbb{C}}^2[\mathbb{C}^\ell];$$

$$(4.5.2) \quad (G, H, K) = (\text{U}(p, q), \text{U}(s), \text{U}(p) \times \text{U}(q)), \quad m = s(p + q), s \leq q \leq p, \quad \mathfrak{p} \cong M_{q,p}(\mathbb{C});$$

$$(4.5.3) \quad (G, H, K) = (\text{O}^*(2n), \text{Sp}(s), \text{U}(n)), \quad m = 4ns, s \leq [\frac{n}{2}], \quad \mathfrak{p} \cong \Lambda_{\mathbb{C}}^2[\mathbb{C}^n] \cong \mathfrak{o}(n, \mathbb{C});$$

here $\text{Sp}(s) = \text{U}(s, \mathbb{H})$, the unitary group over the quaternions $\mathbb{H}$.

See Section 5 of [18] for details and notation. The pairs (4.5.1), (4.5.2), (4.5.3) correspond precisely to, respectively, (5.2)(i), (iii), (iv) in [12]. We note that the positive integer $\ell, q, [\frac{n}{2}]$, is the real rank of, respectively, $\text{Sp}(\ell, \mathbb{R})$, $\text{U}(p, q)$, $\text{O}^*(2n)$, and allowing the parameter s to exceed the real rank of G will not produce any new examples below. In case (4.5.1), the reduced space may be interpreted as the classical phase space of ℓ particles in $\mathbb{R}^s$ with total angular momentum zero.

The general reductive dual pair (G, H) with H compact arises from taking products of finitely many copies of the basic pairs. To simplify the exposition, we will now assume that (G, H) is any of the three basic pairs. By Theorem 5.3 of [18], when $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via the half-trace pairing (for the moment any positive multiple of the Killing form would do), the G -momentum mapping $\mu_G: W \rightarrow \mathfrak{g}^*$ induces an embedding of the H -reduced space $W^{\text{red}} = \mu_H^{-1}(0)/H$ into $\mathfrak{g}^*$, and this embedding yields a normal Kähler space isomorphism from W^{red} onto the normal Kähler space $(\overline{\mathcal{O}_s}, C^\infty(\overline{\mathcal{O}_s}), \{\cdot, \cdot\}, P_s^{\text{red}})$ whose underlying space is the closure of the holomorphic nilpotent orbit $\mathcal{O}_s$ in $\mathfrak{g}$; here P_s^{red} denotes the (complex analytic) stratified Kähler polarization on $\overline{\mathcal{O}_s}$ explained in [18].

The holomorphic nilpotent orbits $\mathcal{O}_0, \dots, \mathcal{O}_r$ are linearly ordered in such a way that

$$(4.5.4) \quad \{0\} = \mathcal{O}_0 \subseteq \overline{\mathcal{O}_1} \subseteq \dots \subseteq \overline{\mathcal{O}_r}$$

[18] (3.3.10); here r denotes the real rank of $\mathfrak{g}$. The orbit $\mathcal{O}_1$ is the *minimal* nilpotent orbit (for G) which, in the literature, plays a major role. The top orbit $\mathcal{O}_r$ is referred to as the *principal holomorphic* nilpotent orbit in [18]. To explain briefly the stratification and the complex analytic structures recall that, with reference to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the symmetric constituent $\mathfrak{p}$ inherits a complex structure such that, after complexification so that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{p}_+$ refers to the holomorphic constituent, the canonical map from $\mathfrak{p}$ to $\mathfrak{p}_+$ is an isomorphism of complex vector spaces; the orthogonal projection to $\mathfrak{p}$, restricted to $\overline{\mathcal{O}_r}$, is a homeomorphism, and in this way $\overline{\mathcal{O}_r}$ is endowed with an affine complex structure. The G -momentum mapping $\mu_G: W \rightarrow \mathfrak{g}^*$, combined with the isomorphism onto $\mathfrak{g}$ induced by the half-trace pairing followed by the projection to $\mathfrak{p}$, amounts to the (complex) Hilbert map of invariant theory—it is at this stage that the significance of the choice of half-trace pairing on $\mathfrak{g}$ comes to the fore; in this fashion, the induced map $\mathbb{C}[\mathfrak{p}] \rightarrow \mathbb{C}[W] = S_{\mathbb{C}}[W^*]$ yields an isomorphism from the complex affine coordinate ring $\mathbb{C}[\mathfrak{p}]$ of $\mathfrak{p}$ onto the H -invariants $S_{\mathbb{C}}[W^*]^H$. This identifies the complex affine coordinate ring $\mathbb{C}[\overline{\mathcal{O}_r}]$ of $\overline{\mathcal{O}_r}$ with $S_{\mathbb{C}}[W^*]^H$. For $1 \leq s < r$, complex analytically, the strata $\mathcal{O}_s$ are the $K^{\mathbb{C}}$ -orbits of $\mathfrak{p} \cong \overline{\mathcal{O}_r}$, and $\overline{\mathcal{O}_s}$ is the corresponding affine subvariety of $\overline{\mathcal{O}_r} \cong \mathfrak{p}$.

The algebra $C^\infty(\overline{\mathcal{O}_s})$ is that of Whitney smooth functions on $\overline{\mathcal{O}_s}$, with reference to the embedding into $\mathfrak{g}^*$, and is *not* an algebra of ordinary smooth functions; the Poisson structure comes from this embedding as well. By Theorem 2.1, the reduction procedure described in Section 2, applied to ζ , yields a stratified prequantum module $(M_s^{\text{red}}, \chi_s^{\text{red}})$ for $(C^\infty(\overline{\mathcal{O}_s}), \{\cdot, \cdot\})$; we write this stratified prequantum module as $(M_s^{\text{red}}, \chi_s^{\text{red}})$. In the case at hand, ζ descends to an ordinary complex line bundle $\zeta_s = \zeta_s^{\text{red}}$ on $\overline{\mathcal{O}_s}$ which is, in fact, still trivial, and M_s^{red} is just a free $C^\infty(\overline{\mathcal{O}_s})$ -module of rank 1.

The function ψ_0 (cf. (4.1.6)) is invariant under $U(m)$ and hence under H and the quantum module $(M_s^{\text{red}})^{P_s^{\text{red}}}$ is the free $\mathbb{C}[\overline{\mathcal{O}_s}]$ -module generated by the (class of the) function ψ_0 . Even though the momentum mapping is not proper, the statement of Theorem 3.7 is still true, that is, the Hilbert map of invariant theory induces an isomorphism from $\mathcal{F}^H = \mathbb{C}[W^*]^H \langle \psi_0 \rangle$ onto $(M_s^{\text{red}})^{P_s^{\text{red}}} = \mathbb{C}[\overline{\mathcal{O}_s}] \langle \psi_0 \rangle$. In physics, ψ_0 amounts to the reduced ground state. Furthermore, by construction, the formulas (4.4.1) descend; hence, for every quantizable $f \in C^\infty(\overline{\mathcal{O}_s})$ and every $\psi \in (M_s^{\text{red}})^{P_s^{\text{red}}}$, (1.2.8) and (1.2.9) amount to

$$(4.5.5) \quad \begin{aligned} \widehat{f}(\psi) &= -i\{f, \psi\}^{\text{red}} \text{ (observables),} \\ \widetilde{f}(\psi) &= \{f, \psi\}^{\text{red}} \text{ (infinitesimal symmetries).} \end{aligned}$$

These descriptions of the prequantum- and quantum modules involve only the reduced data and make no reference to the unreduced data.

The group G acts on $\overline{\mathcal{O}_s}$ via the adjoint action, and the subgroup K of G is that of transformations preserving the complex analytic stratified Kähler structure. The inclusion $\overline{\mathcal{O}_s} \subseteq \mathfrak{g}$, combined with the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the half-trace pairing, is a stratified symplectic space momentum mapping (cf. Section 4 of [18] for this notion) for the G -action on $\overline{\mathcal{O}_s}$ which, combined with the projection to $\mathfrak{k}^*$, provides a stratified symplectic space momentum mapping

$$(4.5.6) \quad \mu: \overline{\mathcal{O}_s} \rightarrow \mathfrak{k}^*$$

for the corresponding K -action. Up to the identification of $\mathfrak{k}$ with its dual (via the half-trace pairing), this momentum mapping amounts to the orthogonal projection from $\overline{\mathcal{O}}_s \subseteq \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to $\mathfrak{k}$. Via this momentum mapping, the elements of the Lie algebra $\mathfrak{k}$ constitute a Lie subalgebra lying in the quantizable elements of $C^\infty(\overline{\mathcal{O}}_s)$, and the formula (4.5.5) yields in particular a $\mathfrak{k}$ -representation on $(M_s^{\text{red}})^{P_s^{\text{red}}} = \mathbb{C}[\overline{\mathcal{O}}_s]\langle\psi_0\rangle$.

The composite of the momentum mapping (4.5.6) with the infinitesimal generator $-Z \in \mathfrak{k}$ of the central circle subgroup S^1 of K , viewed as a linear form on $\mathfrak{k}^*$, provides a stratified symplectic space momentum mapping

$$(4.5.7) \quad \mu_{-Z}: \overline{\mathcal{O}}_s \rightarrow \text{Lie}(S^1)^* \cong \mathbf{R}$$

for the hamiltonian S^1 -action on $\overline{\mathcal{O}}_s$ obtained from letting the circle group S^1 act by the inversion map from S^1 onto the central circle subgroup of K ; we note that $Z = 2z$, z being the H -element of the Lie algebra $(\mathfrak{g}, z)$ of hermitian type (cf. Section 3 of [18] for the notion of H -element). This momentum mapping is the *reduced energy* function $[f_E]$ (whence the minus sign) and reveals certain peculiar features of the reduced system: The function $[f_E]$ is an element of $C^\infty(\overline{\mathcal{O}}_s)$ but *not* an ordinary smooth function, not even for $s = r$; for $s = r$, $\overline{\mathcal{O}}_r \cong \mathfrak{p}$ is a complex affine space and hence a smooth manifold but the algebra of functions $C^\infty(\overline{\mathcal{O}}_r)$ underlying the stratified symplectic structure is *strictly larger* than the algebra of ordinary smooth functions on $\mathfrak{p}$. Moreover, $[f_E]$ is *not* homogeneous quadratic, and the reduced energy operator $\widehat{[f_E]}$ (given by the formula (4.5.5)) has only *even* (non-negative) eigenvalues; indeed, we can as well compute this operator from the formula (4.2.3), noticing that only *even* entire holomorphic functions will come into play.

(4.6) *The costratified quantum space structure.* Let $1 \leq s \leq r$. Whenever $s' < s$, restriction yields a morphism of stratified quantum modules from $(M_s^{\text{red}})^{P_s^{\text{red}}}$ to $(M_{s'}^{\text{red}})^{P_{s'}^{\text{red}}}$; the complex vector spaces M_s^{red} and $M_{s'}^{\text{red}}$ are just the free $C^\infty(\overline{\mathcal{O}}_s, \mathbb{C})$ - and $C^\infty(\overline{\mathcal{O}}_{s'}, \mathbb{C})$ -modules, respectively, in a single generator whence, $\overline{\mathcal{O}}_{s'}$ being complex analytically an affine subvariety of $\overline{\mathcal{O}}_s$, the restriction morphism amounts to the canonical surjection $\mathbb{C}[\overline{\mathcal{O}}_s] \rightarrow \mathbb{C}[\overline{\mathcal{O}}_{s'}]$ from the affine complex coordinate ring of $\overline{\mathcal{O}}_s$ to that of $\overline{\mathcal{O}}_{s'}$. Thus the resulting costratified quantum space for $\overline{\mathcal{O}}_s$ arises from the system

$$\mathbb{C}\langle\psi_0\rangle \leftarrow \mathbb{C}[\overline{\mathcal{O}}_1]\langle\psi_0\rangle \leftarrow \dots \leftarrow \mathbb{C}[\overline{\mathcal{O}}_s]\langle\psi_0\rangle$$

by Hilbert space completion where the notation ψ_0 for the basis elements is slightly abused. Here each arrow is actually a morphism of representations for the corresponding quantizable observables, in particular, a morphism of $\mathfrak{k}$ -representations. Plainly, this structure integrates to a costratified K -representation, i. e. corresponding system of K -representations. We view the resulting costratified quantum phase space for $\overline{\mathcal{O}}_s$ as a *singular* Fock space.

(4.7) *Quantization commutes with reduction.* By Theorems 3.5 and 3.7, it makes no difference whether we compute the value of an unreduced quantum observable in an unreduced quantum state or the value of the corresponding reduced quantum observable in the corresponding reduced quantum state. In particular, this remark applies to the elements of $\mathfrak{k}$; viewed as quantizable reduced observables via the momentum mapping (4.5.6), they lie in the reduced Poisson algebra $C^\infty(W^{\text{red}}) \cong C^\infty(\overline{\mathcal{O}}_s)$. Reduction after quantization yields the $\mathfrak{k}$ -representation on the invariants $\mathcal{F}^H = \mathbb{C}[W]^H\langle\psi_0\rangle$

given by (4.4.1), quantization after reduction yields the $\mathfrak{k}$ -representation on $\mathbb{C}[\overline{\mathcal{O}_s}]\langle\psi_0\rangle$ given by (4.5.5), and the Hilbert map of invariant theory provides an isomorphism of $\mathfrak{k}$ -representations between the two.

(4.8) *The classical unreduced constant harmonic oscillator energy phase space.* Let $k \geq 1$ be a positive integer. The energy function f_E on W (given by $f_E(\mathbf{z}, \overline{\mathbf{z}}) = \frac{1}{2}\mathbf{z}\overline{\mathbf{z}}$) is the momentum mapping for the hamiltonian S^1 -action on W obtained from letting $\alpha \in S^1$ act on W via multiplication by α^{-1} . With respect to the energy value k , the reduced space is a copy of $\mathbb{CP}^{m-1}$, endowed with the symplectic form $k\omega$ where ω denotes the symplectic form which is the negative of the imaginary part of the Fubini-Study metric on $\mathbb{CP}^{m-1}$. The k 'th power $\mathcal{O}(k) = (\mathcal{O}(1))^{\otimes k}$ of the ordinary hyperplane bundle $\mathcal{O}(1)$, endowed with its hermitian connection, is a prequantum bundle for $(\mathbb{CP}^{m-1}, k\omega)$, and Kähler quantization yields the (finite dimensional) quantum phase space $S_{\mathbb{C}}^k[W^*] = \mathcal{F}_k$. Indeed, the line bundle $\mathcal{O}(k)$ actually arises as a special case of the reduction procedure for (ordinary smooth) prequantum bundles spelled out in Theorem 2.1. Thus Kähler quantization on $\mathbb{CP}^{m-1}$ recovers $W = \mathbb{C}^m$ in the sense that Kähler quantization on $(P[W], k\omega)$ picks out the homogeneous degree k component of $S_{\mathbb{C}}[W^*]$; cf. p. 96 and p. 190 of [37]. In other words, the symmetric algebra $S_{\mathbb{C}}[W^*] = \mathbb{C}[z_1, \dots, z_m]$ (viewed as a graded algebra) being the homogeneous coordinate ring of $\mathbb{CP}^{m-1}$, this homogeneous coordinate ring amounts to $\bigoplus_{k \geq 0} \Gamma^{\text{hol}}(\mathcal{O}(k))$. Corollary 4.11.2 below will spell out a similar relationship in a singular situation. See also the discussion in [29] for related issues.

(4.9) *Symmetries.* The induced $U(m)$ -action on $(\mathbb{CP}^{m-1}, k\omega)$ ($k \geq 1$) is (well known to be) hamiltonian, the requisite momentum mapping being induced from (4.3.1), and Kähler quantization yields the homogeneous degree k constituent $\mathcal{F}_k = S_{\mathbb{C}}^k[W^*]$, an irreducible summand of the $U(m)$ -representation on $S_{\mathbb{C}}[W^*] = \mathbb{C}[z_1, \dots, z_m]$. Thus this representation is seen as arising by first reducing with respect to the energy function and quantizing thereafter and, as an illustration of the principle that quantization commutes with reduction, this representation arises as well as the Euler operator eigenspace associated to k , cf. (4.2) above, the Euler operator being the quantized harmonic oscillator hamiltonian.

(4.10) *Reduction after quantization.* Let $k \geq 1$. Reduction after quantization now amounts to passing to the resulting $\mathfrak{k}$ -representation on the H -invariant subspace $\mathcal{F}_k^H$.

(4.11) *Quantization after reduction.* In view of Proposition 4.2 and Theorem 10.1 in [18], symplectic reduction applied to $(\mathbb{CP}^{m-1}, k\omega)$ with reference to the induced H -action yields the normal compact complex analytic stratified Kähler space

$$Q_{s,k} = (Q_s, C^\infty(Q_s), \{\cdot, \cdot\}_k^{\text{red}}, P_s^{\text{red}});$$

henceforth, when we wish to indicate all the structure including the stratified symplectic Poisson bracket, we will use the notation $Q_{s,k}$. In case (4.5.1), this space may be interpreted as the classical phase space of ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy k . As a complex analytic space, Q_s is a projective variety and, with an abuse of notation, P_s^{red} refers to the resulting complex analytic stratified Kähler polarization on Q_s .

With reference to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, cf. (4.5) above, by Theorem 10.1 in [18], when r refers to the *real* rank of $\mathfrak{g}$, as a complex analytic space, Q_r is the complex projective space $P(\mathfrak{p})$ and thus amounts to ordinary complex projective

space $\mathbb{CP}^d$ where $d = \frac{\ell(\ell+1)}{2} - 1$, $d = pq - 1$, $d = \frac{n(n-1)}{2} - 1$, according to, respectively, the cases (4.5.1), (4.5.2), (4.5.3). For $1 \leq s \leq r$, complex analytically, Q_s arises from projectivization of the closure $\overline{\mathcal{O}_s}$ of the holomorphic nilpotent orbit $\mathcal{O}_s$, the strata of Q_s are the $K^\mathbb{C}$ -orbits in Q_s , and their closures constitute an ascending chain $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_s$ of projective varieties. However, the stratified symplectic Poisson structure $(C^\infty(Q_r), \{\cdot, \cdot\}_k^{\text{red}})$ on $Q_r (= \mathbb{CP}^d)$ differs from the standard Poisson structure coming from (a multiple of) the (Fubini-Study) symplectic structure in an essential fashion.

The reduction procedure for prequantum modules given in Section 2 above, applied to the prequantum module M arising from the space of smooth sections of $\zeta = \mathcal{O}(k)$, yields a prequantum module $M_{s,k}^{\text{red}}$ for $(C^\infty(Q_s), \{\cdot, \cdot\}_k^{\text{red}})$. As a module over $C^\infty(Q_s)$, $M_{s,k}^{\text{red}}$ consists of continuous sections of the bundle $\zeta_{s,k}^{\text{red}}$ on Q_s arising from reduction applied to the k 'th power $\mathcal{O}(k)$ of the hyperplane bundle which are smooth on each stratum, and $\zeta_{s,k}^{\text{red}}$ inherits a holomorphic structure. Theorem 3.8, or a direct argument, entails that the canonical map ρ (coming into play in Theorems 3.6 and 3.8) identifies the space of holomorphic sections of $\zeta_{s,k}^{\text{red}}$ with that of H -invariant holomorphic sections of $\mathcal{O}(k)$.

For $1 \leq s < r$, we write $\iota_{Q_s}: Q_s \rightarrow Q_r = \mathbb{CP}^d$ for the embedding, cf. Theorem 10.1 in [18]. This embedding determines a homogeneous coordinate ring $S[Q_s]$ for Q_s .

Proposition 4.11.1. *Suppose that $r \geq 2$ (so that $d \geq 1$), and let $1 \leq s \leq r$. For $k \geq 1$, reduction carries the holomorphic line bundle $\mathcal{O}(2k)$ over $\mathbb{CP}^{m-1}$ to the holomorphic line bundle $\mathcal{O}_{Q_s}(k) = \iota_{Q_s}^* \mathcal{O}_{\mathbb{CP}^d}(k)$ over the reduced space Q_s and $\mathcal{O}(2k-1)$ to a sheaf (or complex V -line bundle) having no non-zero holomorphic section. Thus the space of polarized elements, that is, that of P^{red} -invariant ones, of $M_{s,2k}^{\text{red}}$ is the space of holomorphic sections $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ of $\mathcal{O}_{Q_s}(k)$ and that of polarized elements of $M_{s,2k-1}^{\text{red}}$ is zero.*

Proof. In view of the naturality of the constructions, it suffices to establish the first statement for the case $s = r$. By invariant theory, the space of H -invariant holomorphic sections of the line bundle $\mathcal{O}(k)$ over $\mathbb{CP}^{m-1}$ is zero for k odd and for k even, the dimension of the space of H -invariant holomorphic sections of $\mathcal{O}(k)$ coincides with the dimension of the space of holomorphic sections of the line bundle $\mathcal{O}_{Q_r}(\frac{k}{2})$ over $Q_r = \mathbb{CP}^d$. This implies the first assertion. The second one makes explicit the present construction of quantum module. $\square$

Thus, for $k \geq 1$, the compact normal Kähler space $Q_{s,2k}$ is quantizable, having the space of $C^\infty(Q_{s,2k})$ -sections of the holomorphic line bundle $\mathcal{O}_{Q_s}(k)$ as its stratified prequantum module. Inspection establishes the following.

Corollary 4.11.2. *For $1 \leq s < r$, the restriction homomorphism from $\Gamma^{\text{hol}}(Q_r, \mathcal{O}_{Q_r}(1))$ to $\Gamma^{\text{hol}}(Q_s, \mathcal{O}_{Q_s}(1))$ is an isomorphism of complex vector spaces. Consequently the canonical map from the homogeneous coordinate ring $S[Q_s]$ of Q_s to $\bigoplus_{k \geq 0} \Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ is an isomorphism.* $\square$

The second statement of this Corollary says that Kähler quantization on Q_s recovers $\overline{\mathcal{O}_s}$. It also entails the (well known) fact that Q_s is projectively normal, cf. Ex. 5.14 on p. 126 of [11] and the discussion in (10.7) of [18].

For $k \geq 1$, the Kähler quantization procedure developed in Section 3, applied to the complex analytic stratified Kähler space $Q_{s,2k}$ ($1 \leq s \leq r$), yields the *costratified quantum space*

$$\Gamma^{\text{hol}}(\mathcal{O}_{Q_1}(k)) \leftarrow \dots \leftarrow \Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k)).$$

Each vector space $\Gamma^{\text{hol}}(\mathcal{O}_{Q_{s'}}(k))$ ($1 \leq s' \leq s$) is a representation space for the quantizable observables in $C^\infty(Q_s)$, in particular, a $\mathfrak{k}$ -representation, and each arrow is a morphism of representations; these arrows are just restriction maps. This structure globalizes to a costratified K -representation.

(4.12) *Quantization commutes with reduction.* For $1 \leq s \leq r$, the vector space $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ ($k \geq 1$) coincides with the finite dimensional space $\mathcal{F}_k^H$ of H -invariants, and the $\mathfrak{k}$ -representation on $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ coincides with the representation (4.4.2) of $\mathfrak{k}$ on $\mathcal{F}_k^H$. The compactness of Q_s and hence its singular structure, as made precise in Theorem 10.1 in [18], are *crucial* at this stage. *Had we carried out Kähler quantization on the top stratum of Q_s only, which is in fact a smooth (non-compact) Kähler manifold, we would have obtained an infinite dimensional quantum phase space instead of the finite dimensional vector space $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$.* Thus forgetting the lower strata amounts to a *loss of information* and entails *inconsistent results*.

(4.13) *The K -symmetries on the closures of holomorphic nilpotent orbits and their projectivizations.* Let $(\mathfrak{g}, z)$ be an arbitrary simple Lie algebra of hermitian type, let r be its real rank and, as before, let $\{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_r\}$ the holomorphic nilpotent orbits, ordered in such a way that $\{0\} = \mathcal{O}_0 \subseteq \overline{\mathcal{O}_1} \subseteq \dots \subseteq \overline{\mathcal{O}_s}$. Let $1 \leq s \leq r$ and $k \geq 1$. The compact normal Kähler space $Q_{s,k}$ arises as well from the closure $\overline{\mathcal{O}_s} \subseteq \mathfrak{g}$ of the holomorphic nilpotent orbit $\mathcal{O}_s$ of $\mathfrak{g}$ by stratified symplectic reduction, with reference to the momentum mapping (4.5.7) and energy value k . The K -action and stratified momentum mapping (4.5.6) descend to a K -action on Q_s and stratified momentum mapping

$$\mu_k: Q_{s,k} \rightarrow \mathfrak{k}^*.$$

By construction, the image $\mu_k(Q_{s,k})$ lies in the hyperplane $-Z = k$. A version of the statement of the *Kirillov conjecture*, but for *normal Kähler spaces* (rather than smooth Kähler manifolds) now takes the following form: *Those irreducible K -representations which correspond to the coadjoint orbits in the image $\mu_{2k}(Q_{s,2k}) \subseteq \mathfrak{k}^*$ are precisely the representations which occur in the stratified Kähler quantization $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ of $Q_{s,2k}$. Consequently the irreducible representations occurring in the projective coordinate ring $S[Q_s]$ or, equivalently, in the affine coordinate ring $\mathbb{C}[\overline{\mathcal{O}_s}]$, correspond bijectively to the integral coadjoint orbits in the image $\mu(\overline{\mathcal{O}_s}) \subseteq \mathfrak{k}^*$ of the momentum mapping (4.5.6).* We will show elsewhere that, in fact, under the present circumstances, the statement of the *convexity theorem* obtains, that is, the intersection $\mu_{2k}(Q_{s,2k}) \cap t_+^*$ with a Weyl chamber t_+^* is a convex polytope which meets exactly the coadjoint orbits corresponding to the irreducible K -representations in $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$.

These claims may be justified by means of the following observation which also provides further insight: With a notation introduced in Section 3 of [18], the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{n}_r^- \oplus \mathfrak{l}_r \oplus \mathfrak{n}_r^+$ where r refers to the real rank of $\mathfrak{g}$; cf. [18] (3.3.4(r)). Let $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{l}_r$, let $L_r \subseteq G$ be the subgroup having Lie algebra $\mathfrak{l}_r$, and let $K_0 = K \cap L_r$. The adjoint K -action on $\mathfrak{g}$ restricts to an action of K_0 on $\mathfrak{n}_r^-$ and $\mathfrak{n}_r^+$ in an obvious fashion. For example, when $\mathfrak{g} = \mathfrak{sp}(\ell, \mathbb{R})$,

$r = \ell$, $\mathfrak{l}_r \cong \mathfrak{gl}(\ell, \mathbb{R})$, $\mathfrak{n}_r^- \cong S_{\mathbb{R}}^2[\mathbb{R}^\ell] \cong \mathfrak{n}_r^+$, the space of real symmetric $(\ell \times \ell)$ -matrices, $G = \mathrm{Sp}(\ell, \mathbb{R})$, $L_r \cong \mathrm{GL}(\ell, \mathbb{R})$, $K = \mathrm{U}(\ell)$, $K_0 = \mathrm{O}(\ell, \mathbb{R})$, and the action thereof on $S_{\mathbb{R}}^2[\mathbb{R}^\ell]$ is the ordinary one. In the general case, the vector bundle $K \times_{K_0} \mathfrak{n}_r^+ \rightarrow K/K_0$ may be identified with the cotangent bundle $T^*(K/K_0) \rightarrow K/K_0$ of the compact homogeneous space K/K_0 , in the following fashion: The total space $T^*(K/K_0)$ may be written as $K \times_{K_0} \mathfrak{k}_0^\perp$ where $\mathfrak{k}_0^\perp \subseteq \mathfrak{k}^*$ is the annihilator of $\mathfrak{k}_0$ in $\mathfrak{k}^*$. However, the orthogonal projection from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to $\mathfrak{k}$, restricted to $\mathfrak{n}_r^+$, is an isomorphism from $\mathfrak{n}_r^+$ onto a subspace of $\mathfrak{k}$ which, under the (negative of the) Killing form, is the orthogonal complement of $\mathfrak{k}_0$. Now, the assignment to $(x, Y) \in K \times \mathfrak{n}_r^+$ of $\mathrm{Ad}(x)Y \in \mathfrak{g}$ induces a K -equivariant map Φ from $T^*(K/K_0)$ to $\mathfrak{g}$ whose image is the union of all pseudoholomorphic nilpotent orbits in $\mathfrak{g}$, and the composite of Φ with the orthogonal projection from $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ to $\mathfrak{k}$, followed by the isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$ induced by the half-trace pairing, is the ordinary K -momentum mapping for the standard hamiltonian K -action on $T^*(K/K_0)$. The restriction of Φ to $K \times_{K_0} O_r^+$ where $O_r^+ \subseteq \mathfrak{n}_r^+$ is a suitable “positive definite” part of $\mathfrak{n}_r^+$ is a diffeomorphism onto the top stratum $\mathcal{O}_r$ of $\overline{\mathcal{O}_r}$ and thus exhibits $\mathcal{O}_r$ as a fiber bundle over K/K_0 having O_r^+ as its fiber. Likewise, when O_r is the space of positive semidefinite elements of $\mathfrak{n}_r^+$, the restriction of Φ to $K \times_{K_0} O_r$ is a surjection onto the closure $\overline{\mathcal{O}_r}$ of $\mathcal{O}_r$. For example, when $\mathfrak{g} = \mathfrak{sp}(\ell, \mathbb{R})$ so that $\mathfrak{n}_r^+$ is the space of real symmetric $(\ell \times \ell)$ -matrices with the ordinary $\mathrm{O}(\ell, \mathbb{R})$ -action, O_r^+ is that of ordinary positive definite ones.

REMARK 4.14. As a complex analytic space, Q_r is the complex projective space $\mathbb{P}[\mathfrak{p}]$ on the complex vector space $\mathfrak{p}$ and, by construction, $\mathfrak{p}$ comes with a K -representation. Similarly as in (4.3) above, with $\mathfrak{p}$ instead of W , a choice of K -invariant hermitian form on $\mathfrak{p}$ then determines a momentum mapping from $\mathfrak{p}$ to $\mathfrak{k}^*$ and hence, $\mathbb{P}[\mathfrak{p}]$ being endowed with a multiple of the Fubini-Study symplectic form, a momentum mapping $m: \mathbb{P}[\mathfrak{p}] \rightarrow \mathfrak{k}^*$. The latter is a special case of certain momentum mappings explored in [2], [26] and elsewhere. In particular, the Guillemin-Sternberg convexity result obtains [9], and ordinary Kähler quantization on the smooth Kähler manifolds $\mathbb{P}[\mathfrak{p}]$ (when the symplectic structure runs through all positive multiples of the Fubini-Study symplectic structure) yields the same unitary K -representations as our stratified Kähler quantization on the $Q_{r,2k}$ ’s ($k \geq 0$). However, our hamiltonian K -space structures on the $Q_{r,2k}$ ’s differ from the ordinary hamiltonian K -space structures on the $\mathbb{P}[\mathfrak{p}]$ ’s in an essential fashion: For any $k \geq 1$, the real structure $(C^\infty(Q_{r,2k}), \{\cdot, \cdot\})$ is a stratified symplectic structure which involves continuous functions which are not necessarily smooth and has the nice feature that it restricts to a stratified symplectic structure on any stratum $Q_{s,2k}$, and the momentum mapping $\mu_{r,2k}$ is *not* the map m .

(4.15) *Explicit descriptions.* For $k \geq 0$, $\Gamma^{\mathrm{hol}}(\mathcal{O}_{Q_r}(k)) = S_{\mathbb{C}}^k[\mathfrak{p}^*]$ (the space of homogeneous degree k polynomial functions on $\mathfrak{p}$), and the decomposition of $S_{\mathbb{C}}^k[\mathfrak{p}^*]$ into its irreducible K -representations is, of course, well known and classical. We now reproduce suitable highest weight vectors:

Case (4.5.1): Following the procedure on p. 563 of [12] where this is done for $K^{\mathbb{C}} = \mathrm{GL}(\ell, \mathbb{C})$, introduce coordinates on $\mathbb{C}^\ell$. These give rise to coordinates $\{x_{i,j} = x_{j,i}; 1 \leq i, j \leq \ell\}$ on $\mathfrak{p} = S_{\mathbb{C}}^2[\mathbb{C}^\ell]$, and the determinants

$$\delta_1 = x_{1,1}, \quad \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{1,2} & x_{2,2} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{vmatrix}, \quad \text{etc.}$$

are highest weight vectors for certain $U(\ell)$ -representations.

Case (4.5.2): With the obvious notation $x_{i,j}$ for coordinates on $\mathfrak{p} = M_{q,p}(\mathbb{C})$, the determinants

$$\delta_1 = x_{1,1}, \quad \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{vmatrix}, \quad \text{etc.}$$

are highest weight vectors for certain $(U(p) \times U(q))$ -representations, cf. e. g. [38] and [12] p. 567 (where this is explained for the complexified group $GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$).

Case (4.5.3): Introduce coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$ and let

$$\delta_1 = x_1 \wedge x_2, \quad \delta_2 = x_1 \wedge x_2 \wedge x_3 \wedge x_4, \quad \delta_3 = x_1 \wedge x_2 \dots \wedge x_5 \wedge x_6, \quad \text{etc.}$$

These are highest weight vectors for certain $U(n)$ -representations.

For $1 \leq s \leq r$ and $k \geq 1$, the K -representation $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ is the sum of the irreducible representations having as highest weight vectors the monomials

$$\delta_1^\alpha \delta_2^\beta \dots \delta_s^\gamma, \quad \alpha + 2\beta + \dots + s\gamma = k,$$

and the morphism from $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$ to $\Gamma^{\text{hol}}(\mathcal{O}_{Q_{s-1}}(k))$ is an isomorphism on the span of those irreducible representations which do not involve δ_s and has the span of the remaining ones as its kernel. The statement referred to above as a version of the Kirillov conjecture can now be made more explicit in the following fashion: *Those irreducible K -representations which correspond to the coadjoint orbits in the image $\mu_{2k}(O_{s'} \setminus O_{s'-1}) \subseteq \mathfrak{k}^*$ of the stratum $O_{s'} \setminus O_{s'-1}$ ($1 \leq s' \leq s$) are precisely the irreducible representations having as highest weight vectors the monomials $\delta_1^\alpha \delta_2^\beta \dots \delta_{s'}^\gamma$ ($\alpha + 2\beta + \dots + s'\gamma = k$) involving $\delta_{s'}$ explicitly, i. e. with $\gamma \geq 1$.*

REMARK 4.16. These observations may be interpreted by saying that, for the compact hamiltonian K -spaces $(Q_{s,2k}, \mu_{2k})$, Kähler quantization commutes with reduction even though the underlying spaces have singularities.

REMARK 4.17. Translating back this information to the spaces $\overline{\mathcal{O}_s}$ ($s \leq r$), we conclude: With reference to the stratified K -momentum mapping (4.5.6), *those irreducible K -representations which correspond to the coadjoint orbits in the image $\mu(\mathcal{O}_{s'}) \subseteq \mathfrak{k}^*$ of the stratum $\mathcal{O}_{s'}$ ($1 \leq s' \leq s$) of $\overline{\mathcal{O}_s}$ are precisely the irreducible representations having as highest weight vectors the monomials $\delta_1^\alpha \delta_2^\beta \dots \delta_{s'}^\gamma$ involving $\delta_{s'}$ explicitly, i. e. with $\gamma \geq 1$.* Thus for the non-compact hamiltonian K -spaces $\overline{\mathcal{O}_s}$, Kähler quantization commutes with reduction even though the underlying spaces have singularities.

REFERENCES

1. J. M. Arms, R. Cushman, and M. J. Gotay, *A universal reduction procedure for Hamiltonian group actions*, in: The geometry of Hamiltonian systems, T. Ratiu, ed., MSRI Publ. **20** (1991), Springer Verlag, Berlin · Heidelberg · New York · Tokyo, 33–51.

2. M. Brion, *Sur l'image de l'application moment*, in: “Séminaire d'algèbre Paul Dubreuil et Marie-Paule Malliavin”, Paris 1986, Lecture Notes in Math. **1296** (1987), Springer Verlag, Berlin · Heidelberg · New York, 177–192.
3. B. Driver and B. Hall, *Yang-Mills theory and the Bargmann-Segal transform*, Comm. Math. Phys. **201** (1999), 249–290.
4. C. Emmrich and H. Roemer, *Orbifolds as configuration spaces of systems with gauge symmetries*, Comm. Math. Phys. **129** (1990), 69–94.
5. M. J. Gotay, *Poisson reduction and quantization for the $n + 1$ photon*, J. of Math. Phys. **25** (1984), 2154–2159.
6. M. J. Gotay, *Constraints, reduction, and quantization*, J. of Math. Phys. **27** (1986), 2051–2066.
7. M. J. Gotay, *Negative energy states in quantum gravity*, Class. Quant. Grav. **3** (1986), 487–491.
8. V. W. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), 515–538.
9. V. W. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
10. B. C. Hall, *Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type*, Comm. in Math. Phys. **226** (2002), 233–268, [quant-ph/0012015](https://arxiv.org/abs/math-ph/0012015).
11. R. Hartshorne, *Algebraic Geometry*, Graduate texts in Mathematics No. 52, Springer Verlag, Berlin · Göttingen · Heidelberg, 1977.
12. R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.
13. J. Huebschmann, *Poisson cohomology and quantization*, J. für die reine und angewandte Mathematik **408** (1990), 57–113.
14. J. Huebschmann, *On the quantization of Poisson algebras*, Symplectic Geometry and Mathematical Physics, Actes du colloque en l'honneur de Jean-Marie Souriau, P. Donato, C. Duval, J. Elhadad, G.M. Tuynman, eds.; Progress in Mathematics, Vol. 99 (1991), Birkhäuser Verlag, Boston · Basel · Berlin, 204–233.
15. J. Huebschmann, *Poisson geometry of certain moduli spaces*, Lectures delivered at the “14th Winter School”, Srni, Czeque Republic, January 1994, Rendiconti del Circolo Matematico di Palermo, Serie II **39** (1996), 15–35.
16. J. Huebschmann, *On the Poisson geometry of certain moduli spaces*, in: Proceedings of an international workshop on “Lie theory and its applications in physics”, Clausthal, 1995, H. D. Doebner, V. K. Dobrev, J. Hilgert, eds. (1996), World Scientific, Singapore · New Jersey · London · Hong Kong, 89–101.
17. J. Huebschmann, *Singularities and Poisson geometry of certain representation spaces*, in: Quantization of Singular Symplectic Quotients, N. P. Landsman, M. Pflaum, M. Schlichenmaier, eds., Workshop, Oberwolfach, August 1999, Progress in Mathematics, Vol. 198 (2001), Birkhäuser Verlag, Boston · Basel · Berlin, 119–135, [math.DG/0012184](https://arxiv.org/abs/math.DG/0012184).
18. J. Huebschmann, *Kähler spaces, nilpotent orbits, and singular reduction*, [math.DG/0104213](https://arxiv.org/abs/math.DG/0104213), Memoirs of the AMS (to appear).
19. J. Huebschmann, *Singular Poisson-Kähler geometry of Severi varieties and their ambient spaces*, [math.DG/0206143](https://arxiv.org/abs/math.DG/0206143).

20. J. Huebschmann, *Lie-Rinehart algebras, descent, and quantization*, Fields Institute Communications (to appear), [math.SG/0303016](#).
21. F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Princeton University Press, Princeton, New Jersey, 1984.
22. B. Kostant, *Quantization and unitary representations*, In: Lectures in Modern Analysis and Applications, III, ed. C. T. Taam, Lecture Notes in Math. **170** (1970), Springer Verlag, Berlin · Heidelberg · New York, 87–207.
23. I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Advanced Studies in Contemporary Mathematics, vol. 1, Gordon and Breach Science Publishers, New York, London, Paris, Montreux, Tokyo, 1986.
24. E. Meinrenken, *Symplectic surgery and the Spin^c -Dirac operator*, Advances in Math. **134** (1998), 240–277.
25. M. S. Narasimhan and T. R. Ramadas, *Factorization of generalized theta functions*, Inventiones **114** (1993), 565–623.
26. L. Ness, *A stratification of the null cone via the moment map*, Amer. J. of Math. **106** (1984), 1231–1329.
27. T. R. Ramadas, *Factorization of generalised theta functions II: The Verlinde formula*, Topology **35** (1996), 641–654.
28. M. Roberts, *A note on coherent G -sheaves*, Math. Ann. **275** (1986), 573–582.
29. M. Schlichenmaier, *Singular projective varieties and quantization*, in: Quantization of Singular Symplectic Quotients, N. P. Landsman, M. Pflaum, M. Schlichenmaier, eds., Workshop, Oberwolfach, August 1999, Progress in Mathematics, Vol. 198 (2001), Birkhäuser Verlag, Boston · Basel · Berlin, 259–282.
30. R. Sjamaar, *Holomorphic slices, symplectic reduction, and multiplicities of representations*, Ann. of Math. **141** (1995), 87–129.
31. R. Sjamaar, *Symplectic reduction and Riemann-Roch formulas for multiplicities*, Bull. Amer. Math. Soc. **33** (1996), 327–338.
32. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), 375–422.
33. J. Śniatycki, *Geometric quantization and quantum mechanics*, Applied Mathematical Sciences No. 30, Springer Verlag, Berlin · Heidelberg · New York, 1980.
34. J. Śniatycki, *Constraints and quantization*, in: Nonlinear partial differential operators and quantization procedures, Clausthal 1981, eds. S. I. Anderson and H. D. Doebner, Lecture Notes in Mathematics, No. 1037 (1983), Springer Verlag, Berlin · Heidelberg · New York, 301–334.
35. J. Śniatycki and A. Weinstein, *Reduction and quantization for singular moment mappings*, Lett. Math. Phys. **7** (1983), 155–161.
36. C. Teleman, *The quantization conjecture revisited*, Ann. of Math. **152** (2000), 1–43.
37. N. M. J. Woodhouse, *Geometric quantization*, Second edition, Clarendon Press, Oxford, 1991.
38. D. Zhelobenko, *Compact Lie groups and their representations*, Transl. Math. Mono. No. 40, American Math. Soc., Providence R. I., 1973.

USTL, UFR DE MATHÉMATIQUES, CNRS-UMR 8524
F-59 655 VILLENEUVE D'ASCQ CÉDEX, FRANCE
JOHANNES.HUEBSCHMANN@MATH.UNIV-LILLE1.FR