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Alechina, N.A.; van Lambalgen, M.

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Natasha Alechina, Michiel van Lambalgen

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GENERALIZED QUANTIFICATION AS SUBSTRUCTURAL LOGIC

NATASHA ALECHINA AND MICHIEL VAN LAMBALGEN

Abstract. We show how sequent calculi for some generalized quantifiers can be obtained by generalizing the Herbrand approach to ordinary first order proof theory. Typical of the Herbrand approach, as compared to plain sequent calculus, is increased control over relations of dependence between variables. In the case of generalized quantifiers, explicit attention to relations of dependence becomes indispensible for setting up proof systems. It is shown that this can be done by turning variables into structured objects, governed by various types of structural rules. These structured variables are interpreted semantically by means of a dependence relation. This relation is an analogue of the accessibility relation in modal logic. We then isolate a class of axioms for generalized quantifiers which correspond to first-order conditions on the dependence relation.

§1. **Introduction.** Generalized quantifiers were introduced by Mostowski [1957] to capture notions which are not expressible in first order logic, such as 'finitely many,' or 'uncountably many.' Abstractly, a generalized quantifier Q on a first order model $\mathscr{A} = \langle A, ...\rangle$ is a set of subsets of A, which Mostowski required to be closed under bijections. The latter requirement may be weakened to closure under isomorphism of an appropriate type, to accomodate quantifiers such as Friedman's 'almost all,' which can be interpreted using either category or measure (cf. Steinhorn [1985a,b]). If $\langle \mathscr{A}, Q\rangle$ denotes the expanded model, we may define truth for a first order language expanded with a quantifier Q as follows:

$$\langle \mathscr{A}, Q\rangle \models Qx\varphi(x) \text{ if and only if } \{x \mid \langle \mathscr{A}, Q\rangle \models \varphi(x)\} \in Q.$$

According to this definition, generalized quantifiers bind first order variables. They generalize $\exists$ and $\forall$ in the sense that they can be interpreted as an essentially arbitrary set of subsets of $\mathscr{A}$, instead of just $\{B \mid B \subseteq A,\ B \text{ nonempty}\}$ and $\{A\}$, respectively. In other words, the generalization envisaged by Mostowski is *semantic*.

In trying to develop a Gentzen-style proof theory for generalized quantifiers, which would contain either left- and right-introduction rules, or introduction and elimination rules, one quickly notices that the semantic characterization gives very little immediate information about syntactic behaviour.

What seems clear at the outset is that a generalized quantifier Q, even if it resembles the universal quantifier in satisfying distribution over conjunction, is nevertheless *syntactically* analogous to the existential quantifier. For instance, the

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filter quantifier 'co-countably many' satisfies $\forall x Qy(x \neq y)$, but not $Qy\forall x(x \neq y)$, which suggests that the variable bound by Q in a certain sense depends on the variable bound by $\forall$. In the case of the existential quantifier, this dependence can be characterized entirely by means of syntactic side conditions on the rules; can something similar be done for generalized quantifiers?

We shall approach this problem in the following manner.

Firstly we reformulate first order logic in such a way that all dependencies are made explicit in the notation, as indices on variables. The rules for $\exists$ are stated in terms of these indexed variables, and it is shown that we recover ordinary first order logic by means of an additional rule which sanctions substitution of an ordinary variable by an indexed variable (under certain restrictions). In other words, we consider that the usual formulations of first order logic contain a hidden structural rule, namely, the afore-mentioned substitution rule, which roughly has the effect of wiping out dependencies in certain circumstances. One could perhaps quarrel with our usage of the term 'structural rule' in this context. However, if one takes it to be characteristic of a structural rule that no logical constant occurs in its formulation, then it seems reasonable to call a substitution rule structural; indeed, in his introduction to Došen and Schroeder-Heister [1993], Došen mentions a substitution rule as an example of a structural rule (p. 5; cf. also his work on the interaction of substitution rules with the familiar propositional structural rules, Došen [1993]).

Secondly, having an explicit substitution rule at our disposal, we may now drop it, or introduce weak versions instead, *to obtain a whole new class of substructural logics*. It can be shown that several well-known generalized quantifiers, such as 'many,' 'uncountably many' and 'almost all,' have (complete) sequent systems which arise from various modified substitution rules, hence can be seen as substructural logics. The situation here is roughly analogous to propositional logic, where different logics (classical, intuitionistic or linear) share the same introduction and elimination rules for connectives, but differ with respect to structural rules. Below, we shall illustrate this phenomenon using the quantifier 'uncountably many.'

However, the main emphasis of this paper is on the interpretation of the substitution rules as properties of dependence between variables. In the case of first order logic, the dependencies implicitely given by existential quantifiers can be modelled by Skolem functions; in fact, soundness of the proof system for classical predicate logic containing indexed variables is shown by interpreting them as Skolem functions. As was observed above, in the case of generalized quantifiers also the quantifier bound by a 'universal' quantifier can be dependent; this type of dependence will be modelled by a set-valued function, or in other words, by a relation.

Typically, this is a relation between an object a and a finite set of objects B, with the intuitive interpretation that *a depends on B*. Bearing in mind that the type of dependence characteristic of first order logic can itself be characterized by first order conditions (namely, functionality), we may now ask whether properties of generalized quantifiers correspond to first order properties of this more general notion of dependence. Below, we isolate a class of formulas for which this is indeed the case; for instance, the formula $\forall x Qy(x \neq y)$ corresponds to irreflexivity of the dependence relation. We also show that not every formula has a correspondent in this sense; in particular, the formula expressing extensionality has no correspondent.

This is a consequence of a more general result on non-existence of correspondents, the proof of which uses properties of the constructible hierarchy L. A slight adaptation of the argument even proves that the property of having a correspondent is undecidable.

§2. **Substitution as a structural rule.** It is no exaggeration to say that the basic problem in setting up a (Gentzen style) proof system for first order logic, is the proper management of dependencies between variables. Classical analysis texts used the fortuitious terminology of dependent and independent variables: in a functional expression '$y = f(x)$,' y and x are called the dependent and independent variable, respectively. Analogously, in a first order formula we may call a variable bound by $\exists$ dependent, and a variable bound by $\forall$ independent. An explicit way of dealing with dependence is provided by Herbrand's approach to proof theory, where dependence is modelled by Skolem functions; the increased control that this gives can be exploited to prove decidability results, e.g., for prenex formulas beginning with a prefix of the form $\exists \cdots \exists\forall\exists \cdots \exists$.

Some proof systems, especially those containing an existential instantiation rule, make the dependencies explicit by using instantial terms indexed by the parameters (typically obtained by universal instantiation) on which the term depends. A nice overview of such systems is contained in Fine [1985] (cf. also de Queiroz and Gabbay [1995]).

It is claimed here that introducing dependence explicitly into first order logic, as indices on variables, allows one to formulate distinctions not otherwise expressible; in particular, generalized quantifiers will be seen to arise from various forms of substitution of indexed variables. An analogy with the propositional structural rules may be helpful here. If in sequents $\Gamma \Rightarrow \Delta$, Γ and Δ are sets, there is little room for formulating distinctions in the way Γ and Δ are given. Substructural logics can arise only when the Γ and Δ have been given more structure, be they multisets or sequences. Similarly, we can formulate first order classical logic in such a way that the different roles of variables are made explicit in the notation; suitable structural rules will then ensure that for first order logic, these distinctions do not really matter. There also exists a formal parallel between the two cases; it will be seen below that, on the one hand, the structural rules necessary to retrieve first order logic are rather like contraction and weakening, and, on the other hand, that contraction at the index level interacts with contraction at the propositional level. As a consequence, absence of these structural rules leads to a decidable system, because the number of contractions used in a derivation can now be predicted.

2.1. Syntax. We shall define a language for first order logic where the variable bound by $\exists$ carries all (and only) the variables on which that variable depends. These variables will be called *indexed variables*, denoted by $x_{\bar{z}}$. In the case of classical first order logic, indexed variables are interpreted using Skolem functions, but in the context of generalized quantification, they will receive a modified interpretation. Since variables bound by existential quantifiers are always considered to be dependent, we allow $\exists$ only in expressions of the form $\exists x_{\bar{z}}$. On the other hand, variables bound by $\forall$ are taken to be independent, so in principle $\forall$ should only occur in the form $\forall x$, where x is a non-indexed variable. However, for convenience we depart from this idea in two ways:

(i) With respect to a classical semantics, $\neg\forall x\neg$ denotes an existential quantifier. In accordance with our basic motivation to make all dependencies explicit, it would be reasonable to disallow a formula of the form $\neg\forall x\neg\varphi$. Nevertheless, we shall continue to use such implicit dependencies (which in classical logic can be made explicit).

(ii) It is also convenient to allow expressions such as $\forall x_{\bar{z}}$, in anticipation of the substructural logics where the universal quantifier may also depend on other quantifiers. Of course, the structural rules adopted for classical first order logic should be such that $\forall x_{\bar{z}}\varphi$ and $\forall x\varphi$ are equivalent.

DEFINITION 1. Let *Var* be the set of variables of a first order language $\mathscr{L}$. Define a new set of variables *IVar* by recursion, as follows:

$IVar_0 = Var$,
$IVar_{n+1} = IVar_n \cup \{x_{\bar{z}}, x'_{\bar{z}}, x''_{\bar{z}}, \dots \mid \bar{z} \subseteq IVar_n \text{ is a finite set}\}$,
$IVar = \bigcup_n IVar_n$.

Let $\mathscr{L}(IVar)$ denote $\mathscr{L} \cup IVar$. We shall refer to variables in $IVar \setminus Var$ as indexed variables. Observe that $x_\emptyset$ (where $\emptyset$ denotes the empty set) is an indexed variable, not an ordinary variable. Occasionally we shall use s, t as (meta) variables over elements of *IVar*. ⊣

DEFINITION 2. $Form(\mathscr{L}(IVar))$, the set of formulas of $\mathscr{L}(IVar)$, is defined as follows:

(i) If A is an n-ary relation symbol, and $t_1, \dots, t_n \in IVar$, then $A(t_1, \dots, t_n) \in Form(\mathscr{L}(Ivar))$; the set of atomic formulas of $\mathscr{L}(IVar)$ is denoted as $AtForm(\mathscr{L}(IVar))$.
(ii) If $\varphi, \psi \in Form(\mathscr{L}(IVar))$, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\neg\psi$.
(iii) If $\psi \in Form(\mathscr{L}(IVar))$ and $x \in Var$, then $\forall x\psi \in Form(\mathscr{L}(IVar))$.
(iv) If $\psi \in Form(\mathscr{L}(IVar))$ and $t \in IVar \setminus Var$ such that no variable occurring in t is bound by a quantifier in ψ, then $\exists t\psi, \forall t\psi \in Form(\mathscr{L}(IVar))$. ⊣

The condition on t in (iv) is necessary to exclude oddities like $\exists x_t \exists t\varphi(x_t, t)$; if we read formulas from left to right, x_t is already 'partially bound' in the formula $\exists t\varphi(x_t, t)$, so it does not seem to make sense to quantify over x_t.

The intuition behind this definition is that the index of an existentially quantified variable contains *exactly* those (possibly indexed) variables on which the (quantifier binding the) variable depends.

DEFINITION 3. $FV(\psi)$, the set of free variables occurring in a formula ψ, is defined inductively as follows, starting from the set $FV(t)$ of free variables occurring in $t \in IVar$:

(i) $FV(x) = \{x\}$, $x \in Var$.
(ii) $FV(x_{\bar{z}}) = \{x_{\bar{z}}\} \cup FV(z_1) \cup \dots \cup FV(z_k)$ if $\bar{z} = z_1 \dots z_k$.
(iii) $FV(A(t_1, \dots, t_n)) = FV(t_1) \cup \dots \cup FV(t_n)$.
(iv) $FV(\neg\psi) = FV(\psi)$,
$FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \rightarrow \psi) = FV(\varphi) \cup FV(\psi)$.
(v) If $\bar{z} = z_1 \cdots z_k$,
$FV(\forall x_{\bar{z}}\psi) = FV(\exists x_{\bar{z}}\psi) = FV(\psi) \cup FV(z_1) \cup \dots \cup FV(z_k) - \{x_{\bar{z}}\}$.
(vi) $FV(\forall x\psi) = FV(\psi) - \{x\}$. ⊣

We may identify formulas of first order predicate logic with a subset of $Form(\mathscr{L}(IVar))$:

DEFINITION 4. Define a translation * from $Form(\mathscr{L}) \cup AtForm(\mathscr{L}(IVar))$ to $Form(\mathscr{L}(IVar))$ as follows:

(i) $A(t_1, \dots, t_n)^* = A(t_1, \dots, t_n)$, where the t_i may denote indexed variables.

(ii) * commutes with $\neg, \wedge, \vee, \rightarrow, \forall$.

(iii) $(\exists y \psi(y, \bar{z}))^* = \exists x_{\bar{z}} \psi(x_{\bar{z}}, \bar{z})^*$, where $\bar{z}$ contains exactly the free (possibly indexed) variables in $\exists y \psi$. ⊣

Observe that the slightly unusual formulation of (i) is necessary since applying * to a sentence of first order logic will yield atomic formulas in which indexed variables occur.

The range of $Form(\mathscr{L})$ under * will provisionally be called the set of linear formulas (an extension of the notion of linearity will be given later). Equivalently, one may define linear formulas as follows:

DEFINITION 5. (1) A quantifier $\exists t$ (where $t \in IVar \setminus Var$) *occurs linearly* in $\exists t \varphi(t)$ if $\varphi(t) = \varphi(t, \bar{y})$ and $t = x_{\bar{y}}$.

(2) A formula of $\mathscr{L}(IVar)$ is *linear* if (i) universal quantifiers occur only in the form $\forall x$ (where $x \in Var$), (ii) existential quantifiers occur linearly, and (iii) no indexed variable occurs free. ⊣

An example of a non-linear formula is $\forall y \exists x_y \forall z \exists x_z \varphi$; the semantics for indexed variables described in Section 8 transforms this into the branching quantifier

$$\begin{pmatrix} \forall y \exists x \\ \forall z \exists x' \end{pmatrix} \varphi.$$

Note that even in the case of first order logic, we allow nested dependencies, which means that we also allow an existential quantifier to depend on *existential* quantifiers which precede it. This expressiveness is not necessary for classical first order logic (where we may take the existentially quantified variable to depend only on the *universal* quantifiers which precede it), but it is essential for generalized quantification. Some such extension is also necessary for intuitionistic versions of Herbrand's theorem, due to the fact that $\neg \forall x \varphi$ is not equivalent to $\exists x \neg \varphi$; in the construction of Mints [1969], a Skolem function eliminating a positive quantifier Q also carries arguments for each $\forall$ in the scope of a $\neg$.

2.2. Semantics: Informal introduction. We shall now describe a semantics for the fragment of $\mathscr{L}(IVar)$ consisting of linear formulas. The advantage of linear formulas is that negation is well-behaved, a feature that is exploited in clauses 4 and 6 of the definition below. In later sections, this fragment will be successively enlarged to include for example branching generalized quantification; at this stage, however, negation becomes troublesome (cf. Section 8.1). Of course, providing a dependence semantics for predicate logic is belabouring the obvious, but we need this groundwork for the extensions to come.

It might seem straightforward to provide a semantics for the language of indexed variables; copy the usual definition with assignments $f : IVar \longrightarrow A$ instead of assignments $f : Var \longrightarrow A$. There is, however, a hidden subtlety here. Recall that the intuitive interpretation of an indexed variable is that it contains *exactly* those variables on which an existential quantifier depends.

Consider a formula $\forall y \exists x_\emptyset \varphi(y, x_\emptyset)$. Since the index of the variable bound by $\exists$ is empty, this quantifier should not depend on anything; in particular,

$$\forall y \exists x_\emptyset \varphi(y, x_\emptyset) \equiv \exists x_\emptyset \forall y \varphi(y, x_\emptyset)$$

should be valid.

Clearly, using the standard truth definition with assignments $f : IVar \longrightarrow A$ does not give this.

Rather, we use assignments $f : Var \longrightarrow A$ and interpret indexed variables by means of compositions of Skolem functions.

Let $\mathscr{F}$ be a set of new function symbols F, countably many for each arity n. We shall represent indexed variables by these Skolem functions, as follows. Let $\sharp_0$ be a mapping 1-1 associating a function symbol $F \in \mathscr{F}$ with each $t \in IVar \setminus Var$, such that, if $t = x_{y_1,\dots,y_n}$, then F has arity n. We may use $\sharp_0$ to define a mapping $\sharp$ from $IVar$ to compositions of functions from $\mathscr{F}$:

$\sharp(x) = x \; (x \in Var)$
if $t = x_{y_1,\dots,y_n}$, $\sharp(t) = \sharp_0(t)(\sharp(y_1), \dots, \sharp(y_n))$.

$\sharp$ naturally leads to the set *ITerm* of indexed terms, obtained as follows:

$ITerm_0 = Var$
$ITerm_{n+1} = \{x_{s_1,\dots,s_k}, x'_{s_1,\dots,s_k}, \dots | s_i \in IVar_n \text{ or } s_i \in ITerm_n\} \cup$
$\cup \{F(s_1, \dots, s_k) | F \in \mathscr{F}, s_i \in IVar_n \text{ or } s_i \in ITerm_n\}$
$ITerm = \bigcup_n ITerm_n$.

$\sharp$ has an obvious extension to *ITerm*; we shall sometimes abuse notation by assuming that $\sharp$ is also defined on *ITerm*.

For the definition of satisfaction it will be convenient to use implication as a defined connective.

DEFINITION 6. Let $\mathscr{A}$ be a first order model. Satisfiability on $\mathscr{A}$ for the set of linear formulas is given by a map $\sharp$ from $Form(\mathscr{L}(ITerm))$ into a second-order language on $\mathscr{L}(ITerm)$:

1. $\sharp(\neg)A(t_1, \dots, t_n) = (\neg)A(\sharp(t_1), \dots, \sharp(t_n))$
2. $\sharp\neg\neg\varphi = \sharp\varphi$
3. $\sharp(\varphi \wedge \psi) = \sharp\varphi \wedge \sharp\psi$, $\sharp(\varphi \vee \psi) = \sharp\varphi \vee \sharp\psi$
4. $\sharp\neg(\varphi \wedge \psi) = \sharp\neg\varphi \vee \sharp\neg\psi$, $\sharp\neg(\varphi \vee \psi) = \sharp\neg\varphi \wedge \sharp\neg\psi$
5. $\sharp\exists t\varphi(t) = \exists\sharp_0(t)\sharp\varphi(t)$, $\sharp\neg\exists t\varphi(t) = \forall x\sharp\neg\varphi(x)$ (where x is new)
6. $\sharp\forall x\varphi(x) = \forall x\sharp\varphi(x)$, $\sharp\neg\forall x\varphi(x) = \exists\sharp_0(x_{\bar{y}})\sharp\neg\varphi(x_{\bar{y}}, \bar{y})$ (where $x_{\bar{y}}$ is new and $\bar{y}$ are exactly the free variables of $\forall x\varphi(x)$).

We say that a linear formula φ is satisfied in $\mathscr{A}$ under an assignment f if $\sharp\varphi$ is satisfied in $\mathscr{A}$ under f in the second-order sense. Observe that, by AC and diagonalisation, $\sharp\varphi$ can be transformed into the usual Skolem form. ⊣

The following example will give the reader a glimpse of the extension of Definition 6 to be provided in Section 8.

EXAMPLE 1. $\mathscr{A} \models \forall y \exists x_y \forall z \exists x_z \varphi(y, x_y, z, x_z)$ iff $\mathscr{A} \models \exists F_1 \exists F_2 \varphi(y, F_1(y), z, F_2(z))$, hence the quantifierprefix $\forall y \exists x_y \forall z \exists x_z$ is a branching quantifier.

2.3. Proof theory. Our procedure will now be as follows. We give a sequent calculus **LIV** for formulas containing indexed variables with the property that **LK** proves $\Gamma \Rightarrow \Delta$ if and only if **LIV** proves $\Gamma^* \Rightarrow \Delta^*$ for sequences of first order formulas Γ, Δ (where **LK** is the ordinary Gentzen calculus).

DEFINITION 7. The sequent calculus **LIV** for linear formulas of $\mathscr{L}(IVar)$ is given by the following rules. The propositional structural rules and the rules for $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $=$ are as usual. The rules for $\forall$ come in two forms: the usual rules for $\forall y$ (with a proviso given below), where $y \in Var$, and special rules $\forall_i l, r$ for indexed variables.

Axioms $A \Rightarrow A$ (note that A may contain indexed variables)

Rules

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \exists x_{\bar{z}} \psi(x_{\bar{z}}, \bar{z}), \Delta}\ \exists r \qquad \frac{\Gamma, \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}{\Gamma, \exists x_{\bar{z}} \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}\ \exists l$$

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \forall x_{\bar{z}} \psi(x_{\bar{z}}, \bar{z}), \Delta}\ \forall_i r \qquad \frac{\Gamma, \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}{\Gamma, \forall x_{\bar{z}} \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}\ \forall_i l$$

$$\frac{\Gamma \Rightarrow \psi(t), \Delta}{\Gamma \Rightarrow \psi(s), \Delta}\ SUBr \qquad \frac{\Gamma, \psi(t) \Rightarrow \Delta}{\Gamma, \psi(s) \Rightarrow \Delta}\ SUBl.$$

RESTRICTIONS. In $\exists l$ and $\forall_i r$ we require that $x_{\bar{z}}$ does not occur free in Γ and Δ, also not in indices. In $SUBr$ and $SUBl$, s and t are elements of $IVar$; the restriction on $SUBr, l$ is that t does not occur free in Γ and Δ. The usual restriction on $\forall r$ should be interpreted to mean also that the eigenvariable does not occur in an index (cf. Definition 3). ⊣

CONVENTION 1. The restrictions on substitution notwithstanding, it is useful to have a supply of variables, carrying the same index, which are intersubstitutable. Hence we shall assume that given a variable $x_{\bar{z}}$, there are countably many $x'_{\bar{z}}$ such that the following substitution rule holds (with the obvious restrictions):

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}), \Delta}{\Gamma \Rightarrow \psi(x'_{\bar{z}}), \Delta}.$$

Similarly

$$\frac{\Gamma \Rightarrow \psi(y), \Delta}{\Gamma \Rightarrow \psi(y'), \Delta}$$

$(y, y' \in Var)$.

MOTIVATION. The left and right rules for $\exists$ make explicit the idea implicit in first order logic, that an object instantiating x in $\exists x \psi(x, \bar{z})$ somehow depends on the parameters $\bar{z}$. The substitution rules ensure that in cases where the dependency is only apparent, that is, when the sequent puts no constraint on the value of an indexed variable, the dependency may be erased; we may then replace an indexed variable by an ordinary variable.

Examples of derivations

1. $\neg\forall x\neg\psi(x,\bar{z})\Rightarrow\exists x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})$.

   PROOF.
$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow\neg\psi(x_{\bar{z}},\bar{z}),\psi(x_{\bar{z}},\bar{z})\quad[\text{derivable}]}{\Rightarrow\neg\psi(x_{\bar{z}},\bar{z}),\exists x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})}\ [\exists r]}{\Rightarrow\neg\psi(x,\bar{z}),\exists x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})}\ [SUBr]}{\Rightarrow\forall x\neg\psi(x,\bar{z}),\exists x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})}\ [\forall r]}{\neg\forall x\neg\psi(x,\bar{z})\Rightarrow\exists x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})}\ [\neg r].$$

2. As is to be expected, one can also prove, using $SUBr$, $\forall x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})\Rightarrow\forall x\psi(x,\bar{z})$ and $\forall x\psi(x,\bar{z})\Rightarrow\forall x_{\bar{z}}\psi(x_{\bar{z}},\bar{z})$.

   The following derived rules show that $\forall$, $\exists$ have all the properties of ordinary first order quantifiers.

3. $$\frac{\Gamma,\psi(t,\bar{z})\Rightarrow\Delta}{\Gamma,\forall x\psi(x,\bar{z})\Rightarrow\Delta}\ \forall l\ [\text{derived}].$$

   PROOF.
$$\dfrac{\dfrac{\dfrac{\psi(x,\bar{z})\Rightarrow\psi(x,\bar{z})}{\forall x\psi(x,\bar{z})\Rightarrow\psi(x,\bar{z})}\ [\forall l]}{\forall x\psi(x,\bar{z})\Rightarrow\psi(t,\bar{z})}\ [SUBr]\qquad\Gamma,\ \psi(t,\bar{z})\Rightarrow\Delta}{\Gamma,\ \forall x\psi(x,\bar{z})\Rightarrow\Delta}\ [CUT]$$

4. $$\frac{\Gamma\Rightarrow\psi(t,\bar{z}),\Delta}{\Gamma\Rightarrow\forall x\psi(x,\bar{z}),\Delta}\ \forall r\ [\text{derived}]$$
   (given that t is not free in Γ, Δ).

   PROOF. Immediately from $SUBr$.

5. $$\frac{\Gamma\Rightarrow\psi(t,\bar{y})\Delta}{\Gamma\Rightarrow\exists x_{\bar{y}}\psi(x,\bar{y}),\Delta}\ \exists r\ [\text{derived}].$$

   PROOF.
$$\dfrac{\dfrac{\dfrac{\Gamma\Rightarrow\psi(t,\bar{y}),\Delta}{\Gamma,\neg\psi(t,\bar{y})\Rightarrow\Delta}}{\Gamma,\forall x\neg\psi(x,\bar{y})\Rightarrow\Delta}\qquad\Rightarrow\forall x\neg\psi(x,\bar{y}),\exists x_{\bar{y}}\psi(x_{\bar{y}},\bar{y})}{\Gamma\Rightarrow\Delta,\exists x_{y}\psi(x_{\bar{y}},\bar{y})}\ [CUT]$$

6. $$\frac{\Gamma,\psi(t,\bar{y})\Rightarrow\Delta}{\Gamma,\exists x_{\bar{y}}\psi(x,\bar{y}),\Delta}\ \exists l\ [\text{derived}]$$
   (given that t is free in Γ, Δ).

   PROOF. Immediately from $SUBl$.

THEOREM 1. ***LK*** *proves* $\Gamma\Rightarrow\Delta$ *if and only if* ***LIV*** *proves* $\Gamma^*\Rightarrow\Delta^*$ *for sequences of first order formulas* Γ, Δ.

PROOF. (i) Take a proof of $\Gamma\Rightarrow\Delta$; and replace instances of $\exists l, r$ by $\exists l, r$[derived]. (ii) Let $\mathscr{A}$ be a first order model, and let $\mathscr{A}'$ be the full second-order model on $\mathscr{A}$. It suffices to show that the rules of **LIV** are sound for $\mathscr{A}'$. This is a routine verification; e.g., $SUBr$ is sound because the assumptions guarantee that $\sharp_0(t)$ does not occur in Γ or Δ, so that it can take any value whatsoever. ⊣

Clearly the resulting derivation of $\Gamma^*\Rightarrow\Delta^*$ uses many applications of CUT; to obtain a proof system for the logic of indexed variables which enjoys cut elimination, we have to add the rules $\forall r, l$ [derived] and $\exists r, l$ [derived] to **LIV** (note that $SUBl$

can be derived from *SUBr*, but we have to add it for the same reason). This is connected to the fact that **LK** does not have a sub*term* property; in **LIV**, CUT is necessary to wipe out information on the indexed variables used in a proof. This situation can be described alternatively as follows. We could have used $\exists r$ rules of the form

$$\frac{\Gamma, \Rightarrow \psi(t)\Delta}{\Gamma \Rightarrow \exists t\psi(t), \Delta}$$

and formulate the right contraction rule as Herbrand did: occurrences of a formula which differ only in their bound (possibly indexed) variables can be identified. Here, *contraction* is used to wipe out information stored in the variables[1].

2.4. Logic without *SUB* and generalized quantifiers. Suppose we drop the substitution rules from the system introduced above. What could be the meaning of this move? For a start, it is clear that the derivations just given, showing that $\exists x_{\bar{z}}$ in $\exists x_{\bar{z}}\psi(x_{\bar{z}}, \bar{z})$ is the ordinary existential quantifier, do not go through anymore. Dually, in the absence of the substitution rules, $\forall x$ gives rise to infinitely many non-equivalent quantifiers $\forall x_{\bar{z}}$ which denote arbitrary choice *depending on parameters* $\bar{z}$.

Now clearly one can define a logic where one has both the usual $\exists$, $\forall$ *and* quantifiers whose interpretations depend on parameters; for this it suffices to set aside a collection of indexed variables for which the substitution rules hold, even though they are not universally valid. Strictly speaking the language $\mathscr{L}(IVar)$ does not allow this possibility; let $\mathscr{L}(IVar, \exists)$ be the extended language with the appropriate formation rule.

The main claim of this paper is that the logic of indexed variables without substitution rules is a natural environment to study the proof theory of generalized quantifiers. That is, for the purposes of proof theory it is advantageous to view a generalized quantifier Q in a formula $Qx\psi(x, \bar{z})$ as a quantifier whose interpretation depends on $\bar{z}$. Clearly the semantic characterization given by Mostowski does not support such a view at all!

Hence we shall now define a basic generalized quantifier language as a sublanguage of $\mathscr{L}(IVar, \exists)$. We assume it to be known how to add a generalized quantifier to a first order language $\mathscr{L}$; since we shall emphasize the analogies between generalized quantifiers and modal operators, we write $\Box_x$ for Qx and $\Diamond_x$ for the dual $\neg\Box_x\neg$. The resulting language is denoted by $\mathscr{L}(\Box)$.

Definition 8. $+$ is a translation from $Form(\mathscr{L}(\Box)) \cup AtForm(\mathscr{L}(IVar))$ into $Form(\mathscr{L}(IVar, \exists))$ given by

(i) $+$ is the identity on atomic formulas
(ii) $+$ commutes with the connectives and ordinary $\exists, \forall$
(iii) $(\Box_x\varphi(x, \bar{z}))^+ = \forall x_{\bar{z}}\varphi(x_{\bar{z}}, \bar{z})^+$. ⊣

Definition 9. A formula of $Form(\mathscr{L}(IVar, \exists))$ is *linear* if it is in the image of $Form(\mathscr{L}(\Box))$ under $+$. ⊣

[1]In linear logic, absence of contraction and weakening does not have an effect on the properties of $\forall$ and $\exists$. Strictly speaking this is true only if we identify *a priori* formulas which differ only in bound variables. If we do not make this identification, we obtain a trivial proliferation of quantifiers; as we hope to show, this proliferation becomes less trivial when the variables have internal structure.

Until Section 8 we shall be concerned only with linear formulas. Observe that the rules for the quantifiers $\forall x_{\bar{z}}$ and their duals can now be written more simply as

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \Diamond_x \psi(x, \bar{z}), \Delta}\ \Diamond r \qquad \frac{\Gamma, \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}{\Gamma, \Diamond_x \psi(x, \bar{z}) \Rightarrow \Delta}\ \Diamond l$$

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \Box_x \psi(x, \bar{z}), \Delta}\ \Box r \qquad \frac{\Gamma, \psi(x_{\bar{z}}, \bar{z}) \Rightarrow \Delta}{\Gamma, \Box_x \psi(x, \bar{z}) \Rightarrow \Delta}\ \Box l$$

with the same restrictions as above.

DEFINITION 10. *Minimal logic* L_{min} is the subsystem of **LIV** consisting of the propositional rules, the rules for $=$ plus $\Box r, l$ and $\Diamond r, l$ (without rules for the first order quantifiers). ⊣

Observe that L_{min} does not allow us to prove that $\Box$ is extensional; we only have

$$\forall x(\varphi(x, \bar{z}) \leftrightarrow \psi(x, \bar{z})) \rightarrow (\Box_x(\varphi(x, \bar{z}) \leftrightarrow \Box_x \psi(x, \bar{z}))$$

for the formulas with the same free variables.

THEOREM 2 (Alechina [1994,1995]). *Minimal logic without equality is decidable.* ⊣

The theorem follows from the fact that every sequent derivable in minimal logic has a cut-free derivation, such that the number of contractions is bounded by a primitive recursive function of the length of the sequent. This does not hold any more if dependencies can be erased; in other words, full predicate logic and undecidability arise from the erasure of dependencies.[2]

2.5. Proof theory for generalized quantifiers. We next illustrate how weak forms of *SUBr* can be used to derive axioms for generalized quantifiers.

EXAMPLE 2. A non-principal filter quantifier is axiomatized by

(0) $\Box_x \varphi \leftrightarrow \Box_y \varphi$, when y is free for x in φ
(1) $\Diamond_x(x = x)$
(2) $\forall y \Box_x(y \neq x)$
(3) $\Box_x \varphi \wedge \Box_x \psi \rightarrow \Box_x(\varphi \wedge \psi)$
(4) $\Box_x \varphi \wedge \forall x(\varphi \rightarrow \psi) \rightarrow \Box_x \psi$.

We show how to derive these principles from rules governing indexed variables. (0) is automatic given the definition of $\Box_x \varphi$.

(1) follows from an identity axiom:

$$\frac{\Rightarrow x_{\emptyset} = x_{\emptyset}}{\Rightarrow \Diamond_x(x = x)}\ \Diamond_r.$$

(2) requires a new axiom, called *NonId*:

$$\frac{\Rightarrow y \neq x_y \quad [NonId]}{\Rightarrow \Box_x(y \neq x) \quad \Box_r}$$

[2]This theorem, for the minimal logic with identity, also follows from a more general result in Andréka and Németi [1994] which is proved model-theoretically.

$$\Rightarrow \forall y \Box_x (y \neq x) \quad \forall r.$$

Note that *SUBr* is inconsistent with *NonId*. [3]

SUBl and *SUBr* could be used to derive (3) and (4), but since we have already assumed *NonId*, we have to replace *SUBr* by a weaker statement. The following is exactly what we need:

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{u}\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \psi(x_{\bar{v}\bar{z}}, \bar{z}), \Delta} \; SUB_{ext}$$

with the customary restrictions on $x_{\bar{u}\bar{z}}$. The sequent calculus for a free filter quantifier is denoted **LFF**. The rule is called SUB_{ext} because it is responsible for the extensionality of the quantifier $\Box_x$. It says that only dependence on variables actually occurring in a formula really matters; other dependencies can be added or erased at will.

Immediate consequences of SUB_{ext} are the following derived rules:

$$\frac{\Gamma \Rightarrow \varphi(x_{\bar{y}\bar{z}}, \bar{y}),\ \Delta}{\Gamma \Rightarrow \Box_x \varphi(x, \bar{y}),\ \Delta} \; \Box r\text{[derived]}$$

$$\frac{\Gamma,\ \varphi(x_{\bar{y}\bar{z}}, \bar{y}) \Rightarrow \Delta}{\Gamma,\ \Box_x \varphi(x, \bar{y}) \Rightarrow \Delta} \; \Box l\text{[derived]}$$

with the customary restrictions on $x_{\bar{y}\bar{z}}$. The proof of $\Box l$ [derived] uses CUT, as in Section 2.3:

PROOF.

$$\dfrac{\dfrac{\dfrac{\varphi(x_{\bar{y}}, \bar{y}) \Rightarrow \varphi(x_{\bar{y}}, \bar{y})}{\Box_x \varphi(x, \bar{y}) \Rightarrow \varphi(x_{\bar{y}}, \bar{y})}}{\Box_x \varphi(x, \bar{y}) \Rightarrow \varphi(x_{\bar{y}\bar{z}}, \bar{y})} SUB_{ext} \quad \Gamma, \varphi(x_{\bar{y}\bar{z}}, \bar{y}) \Rightarrow \Delta}{\Gamma, \Box_x \varphi(x, \bar{y}) \Rightarrow \Delta} \; CUT.$$

As an example of the use of SUB_{ext} we show how to derive the monotonicity property (4); the reader may try to derive (3) for herself.

PROOF OF (4).

$$\dfrac{\dfrac{\dfrac{\varphi(x_{\bar{y}\bar{z}}, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \psi(x, \bar{z})) \Rightarrow \psi(x_{\bar{y}\bar{z}}, \bar{z}) \;\text{[derivable]}}{\Box_x \varphi(x, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \psi(x, \bar{z})) \Rightarrow \psi(x_{\bar{y}\bar{z}}, \bar{z})} \;\Box l\text{[derived]}}{\Box_x \varphi(x, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \psi(x, \bar{z})) \Rightarrow \psi(x_{\bar{z}}, \bar{z})} \; SUB_{ext}}{\Box_x \varphi(x, \bar{y}) \wedge \forall x(\varphi(x, \bar{y}) \to \psi(x, \bar{z})) \Rightarrow \Box_x \psi(x, \bar{z})} \; \Box r.$$

COMMENT. SUB_{ext} can be decomposed in the following two rules:

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{u}\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}$$

[3] Hence adding *NonId* differentiates $\forall x$ and $\forall x_{\bar{z}}$. Quantifiers of this type have been considered previously; for instance, Hintikka proposed to interpret the quantifier combination $\forall y \forall x$ as "for all y and for all x different from y", which in our notation would be written as $\forall y \forall x_y$, satisfying $\forall y \forall x_y (y \neq x_y)$.

and

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \psi(x_{\bar{v}\bar{z}}, \bar{z}), \Delta.}$$

The first rule can be seen as an analogue of contraction, eliminating spurious dependencies, whereas the second rule is rather like weakening, which allows us to introduce spurious dependency.

EXAMPLE 3. The quantifier 'co-countably many' (Keisler [1970]) is axiomatized by properties (0–4) of the previous example together with an axiom which says that the union of countably many countable sets is countable :

(5) $\forall y \Box_x \varphi \wedge \Box_y \forall x \varphi \rightarrow \Box_x \forall y \varphi$.

It is well-known that the proof theory of this quantifier presents considerable difficulties. Since interpolation fails for 'co-countably many,' it was conjectured that there is no good sequent calculus for the full axioms (see e.g., Barwise's introduction to Barwise and Feferman [1985]). We show here that there does exist a sequent calculus for 'co-countably many,' formulated in the language of indexed variables. The rule which corresponds to (5) is a substitution rule which licenses substitution of an indexed variable under certain restrictions. However, to derive (5) from this substitution rule we need to apply CUT to an identity statement. While this does not yet prove that CUT is not eliminable, it makes it extremely unlikely. In any case, we shall see that = plays a very important role here.

To derive (5) we need the following substitution rules. We keep SUB_{ext}, but now this rule is fortified by a new rule SUB_{cc}:

$$\frac{\Gamma' \Rightarrow y \neq x'_{\bar{z}}, \Delta' \quad \Gamma \Rightarrow \varphi(x_{\bar{z}y}, y, \bar{z}), \Delta}{\Gamma, \Gamma' \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z}), \Delta, \Delta'} \; SUB_{cc}.$$

Here, we assume that $x'_{\bar{z}}$ does not occur free in Γ' and Δ' and $x_{\bar{z}y}$ does not occur free in Γ, Γ', Δ and Δ'.

The resulting system is called **LCC**.

Clearly SUB_{cc} follows from $SUBr$ and it also says that in certain circumstances a dependency can be erased.

One may question whether SUB_{cc} is a proper structural rule. Observe that SUB_{cc} is equivalent to the following rule:[4]

$$\frac{\Gamma, \forall x'_{\bar{z}}(y \neq x'_{\bar{z}}) \Rightarrow \varphi(x_{y\bar{z}}, y, \bar{z}), \Delta}{\Gamma, \forall x'_{\bar{z}}(y \neq x'_{\bar{z}}) \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z}), \Delta} \; SUB'_{cc},$$

with the obvious restriction on $x_{y\bar{z}}$ and $x_{\bar{z}}$. We have preferred SUB_{cc} over SUB'_{cc} because the latter rule involves a non-linear formula, namely $\forall x'_{\bar{z}}(y \neq x'_{\bar{z}})$. This formula $\forall x'_{\bar{z}}(y \neq x'_{\bar{z}})$ can be interpreted as saying that y is not accessible from $\bar{z}$. Indeed, in analogy with linear logic, we view the presence of this specific context as something like an exponential, which can be used to formulate a restricted form of the structural rules. More precisely, whereas the exponentials of linear logic are concerned with the question, *how many* copies of an object are available, in this case the storage operators are concerned with the *location* of an object.

[4]To derive SUB_{cc} from SUB'_{cc}, use Cut and Weakening; the other direction is trivial.

We now show how to derive (5) from SUB_{cc} and contradiction. Observe that it suffices to derive the following two sequents:

(I) $\Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow y \neq x'_{\bar{z}}, \varphi(x_{\bar{z}y}, y, \bar{z})$
(II) $\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}y}, y, \bar{z})$.

For in this case we may argue as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{I}) \qquad\qquad (\mathrm{II})}{\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z}), \varphi(x_{\bar{z}}, y, \bar{z})}\ SUB_{cc}}{\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z})}\ \textit{Contradiction}}{\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \forall y \varphi(x_{\bar{z}}, y, \bar{z})}\ \forall r}{\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \Box_x \forall y \varphi(x, y, \bar{z})}\ \Box r.$$

The derivation of (I) is based on properties of =:

$$\dfrac{\dfrac{\dfrac{\varphi(x_{\bar{z}}, x'_{\bar{z}}, \bar{z}), y = x'_{\bar{z}} \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z}) \quad [\text{=-axioms}]}{\forall x \varphi(x, x'_{\bar{z}}, \bar{z}), y = x'_{\bar{z}} \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z})}\ \forall l\ [\text{derived}]}{\Box_y \forall x \varphi(x, y, \bar{z}), y = x'_{\bar{z}} \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z})}\ \Box l}{\Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow y \neq x'_{\bar{z}}, \varphi(x_{\bar{z}}, y, \bar{z})}\ \neg r.$$

Now we derive (II):

$$\dfrac{\dfrac{\varphi(x_{\bar{z}y}, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}y}, y, \bar{z}) \quad [\text{derivable}]}{\Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}y}, y, \bar{z})}\ \Box l\ [\text{derived}]}{\forall y \Box_x \varphi(x, y, \bar{z}), \Box_y \forall x \varphi(x, y, \bar{z}) \Rightarrow \varphi(x_{\bar{z}y}, y, \bar{z})}\ \forall l.$$

We see that the derivation essentially involves =, although = need not occur in the conclusion of the derivation. A further discussion of this proof system is given in Section 8.3. ⊣

2.6. Soundness, and what the reader may expect next. The obvious question to be asked is: granted that Axiom (5) is derivable from a curious substitution rule, how do we know that **LCC** is conservative over Keisler's logic? This requires giving a semantics to the logic of indexed variables for which the substitution rules comprising **LCC** are sound.

It turns out that this semantics is given by first order models expanded with a dependence relation R, holding between objects and finite sets of objects, in such a way that substitution rules correspond to specific properties of R. (Here, a variable $x_{\bar{z}}$ is interpreted as ranging over $\{x \mid R(x, \bar{z})\}$.) For example, we shall see that SUB_{cc} corresponds to the following universal condition on R:

$$\neg R(x, y\bar{z}) \wedge \neg R(y, \bar{z}) \rightarrow \neg R(x, \bar{z}).$$

The precise meaning of this notion of correspondence will be given in the next section.

From this example we see that there exists a connection between axioms for generalized quantifiers, substitution rules for indexed variables, and properties of the dependence relation. Actually, to find the substitution rules that allow one to derive a particular quantifier axiom, it is very helpful to have at one's disposal an R-correspondent for this axiom, so the fundamental problem in this area seems to be the connection between quantifier axioms and properties of the dependence relation. To put it succinctly, we claim that quantifier axioms implicitly determine patterns of dependence between variables, and we are interested in the question whether these patterns can be made explicit.

Accordingly, our next topic will be an investigation of the relation between quantifier axioms and properties of the relation R. Here, the analogy between generalized quantifiers and modal operators comes to the fore. We shall treat such questions as: when does a quantifier axiom correspond to a statement in the language $\{R, =\}$? which quantifier axioms do not have a correspondent? is the property of having a correspondent decidable?

§3. Models with a dependence relation. Now that a generalized quantifier $\Box_x$ in a formula $\Box_x \varphi(x, \bar{z})$ is interpreted as $\forall x_{\bar{z}}$, i.e., as a quantifier over variables *dependent on* $\bar{z}$, we can introduce a new kind of model for generalized quantifiers.

To do this, we introduce an analogue of the expansion of a language by Skolem functions to model dependencies. It turns out that that dependencies characteristic for universal quantifiers are modelled by *set-valued* functions, that is, by relations.

DEFINITION 11. Let $R(x, \bar{y})$ be a relation of indefinite arity. The intuitive interpretation of $R(x, \bar{y})$ is "x depends on $\bar{y}$." Since this is a relation between objects and finite sets of objects, we shall tacitly assume axioms on R which guarantee that $\bar{y}$ can be thought of as a set. In other words, the second argument of R is invariant under permutations and repetitions. The *standard translation st* : $\mathscr{L}(\Box) \longrightarrow \mathscr{L}(R)$ is defined inductively as follows:

$st(P(x_1, \dots, x_n)) = P(x_1, \dots, x_n)$;
$st(\neg\psi) = \neg st(\psi)$;
$st(\psi_1 \wedge \psi_2) = st(\psi_1) \wedge st(\psi_2)$;
$st(\forall x\psi) = \forall x st(\psi)$;
$st(\Box_x \psi(x, \bar{y})) = \forall x(R(x, \bar{y}) \rightarrow st(\psi(x, \bar{y})))$. ⊣

We see that, in the formula $\Box_x \psi(x, \bar{y})$ the bound variable x depends only on $\bar{y}$, not on ψ (this is different from Skolem functions).

As in the case of Skolem functions, we would like to prove that every theory T has a conservative extension to a theory T' such that

$$(\flat) \qquad T' \vdash \Box_x \varphi(x, y_1, \dots, y_n) \equiv \forall x(R(x, y_1, \dots, y_n) \rightarrow \varphi(x, y_1, \dots, y_n)).$$

For theories which are consistent with the minimal logic we can indeed prove that a conservative extension T' with $(\flat)$ exists. First we define the required model extension.

DEFINITION 12. A relational model for $\mathscr{L}(\Box)$ is a triple $M = \langle D, R, V\rangle$, where D is a non-empty domain, V interprets predicate symbols, and R is a relation between elements and finite sets of elements of D, called the dependence relation.

We assume that $\{d \mid R(d, \bar{b})\}$ is non-empty for every finite subset $\bar{b}$ of D. The notion of a formula being satisfied in a model under a variable assignment is standard; the clause for $\Box_x$ reads as follows:

$$M \models^{\alpha} \Box_x \varphi(x, \bar{y}) \Leftrightarrow \forall d \in D(R(d, \alpha(\bar{y})) \to M \models^{\alpha} \varphi(d, \bar{y})).$$

A canonical relational model is a relational model where the dependence relation satisfies

$$R(x, \bar{y}) = \bigwedge_{\varphi(x,\bar{y}) \in \mathscr{L}(\Box)} \Box_x \varphi(x, \bar{y}) \to \varphi(x, \bar{y}).$$

If there is no danger of confusion, a relational model will be written in the form $\langle \mathscr{A}, R \rangle$, where $\mathscr{A}$ is a first order model. ⊣

Definition 12 does not refer to indexed variables; Skolem-type dependencies are implicit in the standard translation of a formula (which is first order). However, relational models may be used to interpret indexed variables; this is helpful, indeed necessary, when proving soundness of the substitution rules appropriate for generalized quantifiers, or when defining branching compounds of generalized quantifiers. An interpretation of indexed variables can be given by extending Definition 6. First we introduce a notational convenience: if t is an indexed variable, $\mathrm{ind}(t)$ denotes the index of t, i.e., if $t = x_{\bar{y}}$, $\mathrm{ind}(t) = \bar{y}$.

DEFINITION 13. Let $\langle \mathscr{A}, R \rangle$ be a relational model. We may interpret the linear formulas on $\langle \mathscr{A}, R \rangle$ by means of a map $\sharp$ from *Form*($\mathscr{L}$(*ITerm*)) to a second-order language containing the predicate R:

1. $\sharp(\neg)A(t_1, \dots, t_n) = \bigwedge_i \sharp R(t_i, \mathrm{ind}(t_i)) \wedge (\neg)A(\sharp(t_1), \dots, \sharp(t_n))$.
2. $\neg, \wedge, \vee$ as in Definition 6.
3. $\sharp\exists t \varphi(t) = \exists \sharp_0(t)(\sharp R(t, \mathrm{ind}(t)) \wedge \sharp\varphi(t))$,
   $\sharp\neg\exists t \varphi(t) = \forall x(\sharp R(x, \mathrm{ind}(t)) \to \sharp\neg\varphi(x))$ (where x is new).
4. $\sharp\forall s \varphi(s) = \forall x(\sharp R(x, \mathrm{ind}(s)) \to \sharp\varphi(x))$ (where x is new),
   $\sharp\neg\forall s \varphi(s) = \exists x_{\mathrm{ind}(s)}(\sharp R(x_{\mathrm{ind}(s)}), \mathrm{ind}(s)) \wedge \sharp\neg\varphi(x_{\mathrm{ind}(s)}))$, (where $x_{\mathrm{ind}(s)}$ is new).

A linear formula φ is satisfied in $\langle \mathscr{A}, R \rangle$ under an assignment f if $\sharp\varphi$ is satisfied under f in the second-order sense. ⊣

The reader may verify that this notion of satisfiability is equivalent to the one given in Definition 12.

The existence of T' satisfying ($\flat$) now follows from

THEOREM 3. *Every L_{min}-consistent set of $\mathscr{L}(\Box)$ formulas has a Henkin canonical relational model.*

The proof (a standard Henkin argument) can be found in van Benthem and Alechina [1993]. ⊣

Given the standard translation, a quantifier axiom corresponds to a schema in the language $\mathscr{L}(R)$; the main question then becomes: when can this schema be replaced by a first-order condition on R?

The reader will have observed that both the set-up and the main problem are very much analogous to familiar themes in modal logic: R plays the role of the accessibility relation, and what we ask for is a Sahlqvist theorem, i.e., a characterization of a class of formulas for which there is a first order correspondent (cf. Sahlqvist [1975]).

§4. Correspondence theory for generalized quantifiers. In order to be able to talk about *axioms* (that is, formulas) and not schemata, we introduce the substitution rule

$$\frac{\vdash \Phi(P(x_1,\dots,x_n))}{\vdash \Phi(\varphi(x_1,\dots,x_n)),}$$

provided $P(\bar{x})$ and $\varphi(\bar{x})$ have precisely the same free variables. This restriction is necessary due to the fact that in the minimal logic we have only a restricted form of extensionality. As usual, if A is an axiom, then every substitutional instance (in the above sense) of A is an axiom.

DEFINITION 14. If A is a quantifier axiom, a *correspondent in the sense of completeness* is a first order condition $A^\dagger$ on R with the following two properties:

(i) any set of sentences consistent with A in L_{min} has a relational model where $A^\dagger$ holds;

(ii) A is satisfied on any relational model where $A^\dagger$ holds. ⊣

Recall that a modal logic L is called *first order complete* if there is a set Δ of first order sentences in the language $\{R, =\}$ (where R is the accessibility relation) such that

$\vdash_L \varphi$ if and only if for every Kripke model M: if $M \models \Delta$ then $M \models \varphi$;

equivalently, $\vdash_L \varphi$ if and only if φ is true on any frame which satisfies Δ. This notion can be reformulated for the generalized quantifiers as follows:

DEFINITION 15. If L is a generalized quantifier logic, then L is *first order complete* if there is a set Δ of first order sentences in the language $\mathscr{L}(R)$ so that for every quantifier formula φ, $\vdash_L \varphi$ if and only if for every relational model M: if $M \models \Delta$ then $M \models \varphi$. ⊣

Note that, if L is finitely axiomatizable then by compactness Δ can be taken to be finite.

Definition 14 can be reformulated as follows: A has a correspondent for completeness $A^\dagger$ if $L_{min} + A$ is first order complete with respect to the class of models satisfying $A^\dagger$.

One can define a stronger notion of first order completeness using the notion of a canonical model, which plays an important role in modal correspondence theory.

DEFINITION 16. If L is a generalized quantifier logic, then L is *canonical* if L is first order complete and the corresponding set Δ of first order sentences holds in every ω-saturated canonical relational model of L. ⊣

This definition is motivated by the fact that in modal logic canonical models are ω-saturated.

The correspondence theory for generalized quantifiers developed in van Benthem and Alechina [1993] is based on the notion of frame correspondence:

DEFINITION 17. If A is a quantifier axiom, a *frame correspondent* is a first order condition $A^\star$ on R such that $\langle D, R\rangle \models A^\star$ if and only if for *any* interpretation V, $\langle D, R, V\rangle \models A$. ⊣

However, this notion is not quite adequate for our purposes. It will be seen below that the two notions of having a correspondent are different; for example, there are formulas which have a frame correspondent, but do not have a correspondent for completeness. However, the following holds:

PROPOSITION 1. *If A has a correspondent for completeness $A^\dagger$ and a frame correspondent $A^\star$, then* $\vdash_{FOL} A^\dagger \rightarrow A^\star$.

PROOF. Let $\langle D, R\rangle \models A^\dagger$. Then by Definition 14 (ii), for every interpretation V, $\langle D, R, V\rangle \models A$. By Definition 17, $\langle D, R\rangle \models A^\star$. ⊣

We now formulate the completeness part of the Sahlqvist theorem, which describes a class of formulas φ defining first-order conditions on R so that for any logic L in the language of $\mathscr{L}(\Box)$ which has a canonical relational model, $L \cup \{\varphi\}$ as an axiom is complete for the class of models where R has the first order property corresponding to φ. This class is strictly smaller than the class of formulas having a frame correspondent (cf. van Benthem and Alechina [1993]). We shall call these formulas *weak* Sahlqvist formulas. For the formulation of the theorem it is convenient to assume that the language contains special formulas $\top(\bar{z})$ and $\bot(\bar{z})$, which denote a tautology, resp. a contradiction with exactly the free variables $\bar{z}$.

THEOREM 4 (Completeness part of the Sahlqvist theorem). *All "weak Sahlqvist formulas" χ of the form $\bigwedge Qz_1 \cdots Qz_n(A \rightarrow B)$, where $n \geq 0$, each Q is either $\forall$ or $\Box$, and*

1. *A is constructed from*
   a. *atomic formulas, possibly with a quantifier prefix $Qx_1 \cdots Qx_k$, where each Q is a $\Box$- or $\forall$-quantifier;*
   b. *formulas in which atomic formulas occur only negatively,*
   c. *constant formulas (where the only predicate letters are $\top, \bot$ and $=$), using $\wedge$ and $\vee$,*
2. *in B all predicate letters occur only positively*
3. *every occurrence of a predicate letter has the same free variables*

have a correspondent in the sense of completeness; moreover, this correspondent holds in every ω-saturated canonical relational model of a logic in which χ is provable.

All conditions of the theorem can be shown to be necessary, namely, if a formula does not satisfy one of them, then it need not have a correspondent for completeness. For some conditions this is proved in van Benthem and Alechina [1993], for the rest in Section 6.

The idea of the proof of the Theorem 4 (very similar to the one used in Sambin and Vaccaro [1989]) can be illustrated by the following example.

EXAMPLE 4. Let $\mathscr{C}$ be a canonical[5] model. We show that if for every P and S

$$\Box_x P(x, \bar{y}) \rightarrow \Box_x(P(x, \bar{y}) \vee S(x, \bar{z}))$$

[5]Henceforth, "canonical model" will always mean "canonical relational model."

is valid in $\mathscr{C}$, then the accessibility relation in $\mathscr{C}$ has the property $R(x, \bar{y}\bar{z}) \rightarrow R(x, \bar{y})$.

PROOF. It is easy to see (since the consequent is monotone in S) that the axiom above is equivalent to $\Box_x P(x, \bar{y}) \rightarrow \Box_x (P(x, \bar{y}) \vee \bot(x, \bar{z}))$. We translate the validity conditions using second-order quantifiers which range only over *definable* relations of $\mathscr{C}$ (this is the difference with the case of the frame correspondence). To emphasize this difference we use quantifiers $\forall\varphi$. Note that due to the restricted Substitution Rule, if P is an n-place predicate symbol, then formulas which can be substituted for P must have precisely n variable places. Using this notation the validity condition of the axiom reads as follows:

$$\forall\varphi[\forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y}) \vee \bot(x, \bar{z}))].$$

This is equivalent to $\forall\varphi[\forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y}))]$, and, in turn, to $\bigwedge_{\varphi(x,\bar{y})}\{\forall x(R(x, \bar{y}\bar{z}) \rightarrow \varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))\}$. Moving the conjunction inside (the proof that this can be done for any positive logical function of φ, is given in the Intersection Lemma; in the given case the proof is obvious), we obtain

$$\forall x(R(x, \bar{y}\bar{z}) \rightarrow \bigwedge_{\varphi(x,\bar{y})} \{\varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))\}).$$

But by canonicity in $\mathscr{C}$ $\bigwedge_{\varphi(x,\bar{y})}\{\varphi(x, \bar{y})) : \forall x(R(x, \bar{y}) \rightarrow \varphi(x, \bar{y}))\}) \equiv R(x, \bar{y})$. Substituting $R(x, \bar{y})$ instead of the infinite conjunction yields the first-order equivalent $\forall x(R(x, \bar{y}\bar{z}) \rightarrow R(x, \bar{y}))$.

It is easy to check that in every model where $\forall x(R(x, \bar{y}\bar{z}) \rightarrow R(x, \bar{y}))$ holds, the axiom is valid. ⊣

The general case is slightly more complicated because to obtain a correspondent we must sometimes move to a different canonical model, namely, to an ω-saturated canonical model. The existence of such model for every L_{min}-consistent set of sentences is proved in Section 4.1. For canonical ω-saturated models we have

INTERSECTION LEMMA. *If X is a set of formulas with the same free variables, closed with respect to $\wedge$, and B is a formula where φ occurs positively, then in an ω-saturated model*

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\}).$$

The proof of Theorem 4 consists of the same three ingredients as those in the example: translation of the validity conditions of an axiom (eventually accompanied by some syntactic transformations), application of the Intersection Lemma, and making use of the fact that for some first-order expression $\mathscr{R}$ with R as the only predicate symbol

$$\mathscr{R}(\bar{x}, \bar{y}) \equiv \bigwedge_{\varphi(\bar{x},\bar{y})} \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathscr{R}(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}.$$

The class of such expressions will be isolated in the Closure Lemma below. But first we need

DEFINITION 18. Let M be a canonical model, and A a conjunction of atomic formulas which are prefixed by universal and $\Box$-quantifiers, so that all occurrences of a predicate symbol have the same free variables. Every occurrence of a predicate symbol P in A is therefore of the form $\bar{Q}_i\bar{x}P(\bar{x},\bar{y})$, where $\bar{Q}_i$ is the quantifier prefix of the ith occurrence. P has *a good minimal substitution* in A if $M \models \bigwedge\{\varphi(\bar{x},\bar{y}) : \bigwedge_i \bar{Q}_i\bar{x}\varphi(\bar{x},\bar{y})\} \equiv p$, where p is a first-order formula built using the predicates R, $=$, $\top$ and $\bot$ only. ⊣

For example, we have seen that if the only occurrence of P in A is of the form $\Box_x P(x,\bar{y})$, then P has a good minimal substitution in A: for every canonical model M,

$$M \models \bigwedge\{\varphi(x,\bar{y}) : \Box_x\varphi(x,\bar{y})\} \equiv R(x,\bar{y}).$$

Before we formulate the Closure Lemma, we shall get rid of two degenerate cases.

(a) Henceforth we assume that *all quantifiers are non-vacuous*, i.e., if we write a formula $\Box_x\varphi$, then x occurs free in φ. This is justified by the assumption that R is always non-empty.

(b) We shall also assume that every quantifier prefix in A contains at least one $\Box$. That this is no loss of generality can be seen as follows.

Let P occur in A with a purely universal prefix, $\forall x_1 \cdots \forall x_n P(\bar{x},\bar{y})$. Then this occurrence implies all other possible occurrences of P in A, and A is equivalent to a conjunction where $\forall x_1 \cdots \forall x_n P(\bar{x},\bar{y})$ is the only occurrence of P.

$$M \models \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall x_1 \cdots \forall x_n\varphi(\bar{x},\bar{y})\} \equiv \top(\bar{x},\bar{y}),$$

so P has a good minimal substitution in A.

Now we can state the Closure Lemma which is proved in Section 4.2:

CLOSURE LEMMA. *Let A be a conjunction of atomic formulas prefixed by $\forall$ and $\Box$-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain the frame correspondent.* ⊣

4.1. ω-saturated models and the Intersection Lemma. Let $X = \{\varphi_1(x), \varphi_2(x), \dots\}$ be a finitely realizable type in a model M, that is, for every n there is an element a_n in the domain of M such that $\varphi_1(a_n), \dots, \varphi_n(a_n)$ is true in M. If M is just an ordinary Henkin model, there does not necessarily exist an element a such that for every φ in X $\varphi(a)$ is true in M. Among other things this implies that $\Diamond_x \bigwedge\{\varphi : \varphi \in X\}$ is not equivalent to $\bigwedge\{\Diamond_x\varphi : \varphi \in X\}$ in M. But in the proof we do need that

$$M \models \Diamond_x \bigwedge\{\varphi : \varphi \in X\} \equiv \bigwedge\{\Diamond_x\varphi : \varphi \in X\}$$

(an analogue of Esakia's lemma). We therefore move from the original Henkin model to a mildly saturated extension.

THEOREM 5. *Every consistent set of $\mathscr{L}(\Box)$ formulas has a model $\mathscr{A}$ which is ω-saturated and canonical, that is*

(i) *every finitely realizable type which contains finitely many parameters is realized;*

(ii) $R_{\mathscr{A}}(d,\bar{d}) =_{df} \bigwedge_{\varphi(x,\bar{d})\in\mathscr{L}(\Box)} \Box_x\varphi(x,\bar{d}) \to \varphi(d,\bar{d})$.

PROOF. From the completeness proof for the minimal logic we know that every consistent set of formulas has a canonical model $\mathscr{C}$ where

$$R_{\mathscr{C}}(x,\bar{y}) \equiv \bigwedge_{\varphi(x,\bar{y})\in\mathscr{L}(\Box)} \Box_x\varphi(x,\bar{y}) \rightarrow \varphi(x,\bar{y}).$$

By the truth definition

$$\mathscr{C} \models \Box_x\varphi(x,\bar{y}) \Leftrightarrow \forall x(R_{\mathscr{C}}(x,\bar{y}) \Rightarrow \mathscr{C} \models \varphi(x,\bar{y})).$$

Therefore there is a first-order model $\mathscr{C}^*$ (with $R = R_{\mathscr{C}}$ just an ordinary predicate) such that if $\psi \in \mathscr{L}(\Box)$

$$\mathscr{C} \models \psi \Leftrightarrow \mathscr{C}^* \models st(\psi).$$

We shall use this fact to build the saturated model which we need, because one can apply the standard procedure of constructing an ω-saturated extension of $\mathscr{C}^*$. (While extending a model for a generalized quantifier is much more difficult.)

Take an ω-saturated elementary extension $\mathscr{A}^*$ of $\mathscr{C}^*$. It is clear that

$$\mathscr{C} \models \psi \Leftrightarrow \mathscr{C}^* \models st(\psi) \Leftrightarrow \mathscr{A}^* \models st(\psi),$$

for every sentence ψ of $\mathscr{L}(\Box)$.

Every type finitely realizable in $\mathscr{C}$ is finitely realizable in $\mathscr{C}^*$ and is therefore realized in $\mathscr{A}^*$. But $\mathscr{A}^*$ is still a first-order model; to make an $\mathscr{L}(\Box)$ model $\mathscr{A}$ out of it, we could define the accessibility relation R in $\mathscr{A}$ to be the interpretation of R in $\mathscr{A}^*$, i.e., stipulate

$$\mathscr{A} \models \Box_x\varphi(x,\bar{y}) \Leftrightarrow \forall x(R(x,\bar{y}) \Rightarrow \mathscr{A} \models \varphi(x,\bar{y})).$$

However, it is not obvious that $\mathscr{A}$ is still canonical.

Instead we define the accessibility relation anew in $\mathscr{A}$. $\mathscr{A}$ will be the expansion $\langle \mathscr{A}^*, R_{\mathscr{A}}\rangle$ of $\mathscr{A}^*$, where $R_{\mathscr{A}}$ is defined on $\mathscr{A}^*$ as

$$R_{\mathscr{A}}(x,\bar{y}) = \bigwedge_{st(\varphi(x,\bar{y})):\varphi(x,\bar{y})\in\mathscr{L}(\Box)} \forall x(R(x,\bar{y}) \rightarrow st(\varphi(x,\bar{y}))) \rightarrow st(\varphi(x,\bar{y})).$$

Note that the intersection is only over the formulas $st(\varphi(x,\bar{y}))$ such that $\varphi(x,\bar{y}) \in \mathscr{L}(\Box)$.

We are done if we can show that

LEMMA 1. *$\mathscr{A} \models \varphi \Leftrightarrow \mathscr{A}^* \models st(\varphi)$ for all formulas $\varphi \in \mathscr{L}(\Box)$.*

PROOF. By induction on the complexity of φ. The only non-trivial case is $\varphi = \Box_x\psi(x,\bar{y})$.

To prove the direction from right to left, assume that $\mathscr{A}^* \models st(\Box_x\psi(x,\bar{y}))$, that is, $\mathscr{A}^* \models \forall x(R(x,\bar{y}) \rightarrow st(\psi(x,\bar{y})))$. We want to prove $\mathscr{A} \models \Box_x\psi(x,\bar{y})$, that is $\mathscr{A} \models \forall x(R_{\mathscr{A}}(x,\bar{y}) \rightarrow \psi(x,\bar{y}))$.

Let $R_{\mathscr{A}}(x,\bar{y})$ hold in $\mathscr{A}$. By the definition of $R_{\mathscr{A}}$, $\mathscr{A}^* \models \forall x(R(x,\bar{y}) \rightarrow st(\psi(x,\bar{y}))) \rightarrow st(\psi(x,\bar{y}))$. We know that $\mathscr{A}^* \models \forall x(R(x,\bar{y}) \rightarrow st(\psi(x,\bar{y})))$. Therefore $\mathscr{A}^* \models st(\psi(x,\bar{y}))$ and, by the inductive hypothesis, $\mathscr{A} \models \psi(x,\bar{y})$.

From left to right: let $\mathscr{A} \models \Box_x\psi(x,\bar{y})$, that is $\mathscr{A} \models \forall x(R_{\mathscr{A}}(x,\bar{y}) \rightarrow \psi(x,\bar{y}))$. Let $R(x,\bar{y})$ hold in $\mathscr{A}^*$. We want to show that $\mathscr{A}^* \models st(\psi(x,\bar{y}))$. It is enough to show that $R(x,\bar{y})$ implies $R_{\mathscr{A}}(x,\bar{y})$. If this is so, we obtain $\psi(x,\bar{y})$ from $R(x,\bar{y})$

and the fact that $\mathscr{A} \models \forall x(R_{\mathscr{A}}(x, \bar{y}) \rightarrow \psi(x, \bar{y}))$, and hence applying the inductive hypothesis we also get $st(\psi(x, \bar{y}))$.

Let $R(x, \bar{y})$. Take a formula $st(\chi)$ such that $\forall x(R(x, \bar{y}) \rightarrow st(\chi(x, \bar{y})))$. Then $st(\chi(x, \bar{y}))$. This way we prove that for all $st(\chi)$, $R(x, \bar{y}) \rightarrow (\forall x(R(x, \bar{y}) \rightarrow st(\chi(x, \bar{y}))) \rightarrow st(\chi(x, \bar{y})))$.

Therefore $R(x, \bar{y}) \rightarrow \bigwedge_{st(\chi)}(\forall x(R(x, \bar{y}) \rightarrow st(\chi(x, \bar{y}))) \rightarrow st(\chi(x, \bar{y})))$, which means that $R(x, \bar{y})$ implies $R_{\mathscr{A}}(x, \bar{y})$. ⊣

This completes the proof of Theorem 5. ⊣

COMMENT. $\mathscr{C}$ is an elementary extension of $\mathscr{A}$ with respect to $\mathscr{L}(\Box)$ formulas, but not necessarily with respect to $\mathscr{L}(R)$ formulas if R is interpreted as $R_{\mathscr{A}}$.

Now we are ready to prove that in $\mathscr{A}$ the Intersection Lemma holds.

LEMMA 2 (Intersection Lemma). *If X is a set of formulas with the same free variables, closed with respect to $\wedge$, and B is a formula where φ occurs positively, then in an ω-saturated model*

$$\bigwedge\{B(\varphi) : \varphi \in X\} \equiv B(\bigwedge\{\varphi : \varphi \in X\}).$$

PROOF. By induction on the complexity of B. The basis and propositional cases are trivial.

- Let $B = \forall x B_1$. $\bigwedge\{\forall x B_1(\varphi) : \varphi \in X\} = \forall x \bigwedge\{B_1(\varphi) : \varphi \in X\}$ (because $\forall$ distributes over $\bigwedge$), and by the inductive hypothesis this is equivalent to $\forall x B_1(\bigwedge\{\varphi : \varphi \in X\})$.
- Let $B = \Box_x B_1$. This case is analogous, but since $\Box$ distributes only over conjunctions of formulas with the same free variables, it is important that all formulas in X (and therefore in $\{B_1(\varphi) : \varphi \in X\}$) have the same free variables.
- Let $B = \Diamond_x B_1$. We have to show $\bigwedge\{\Diamond_x B_1(\varphi) : \varphi \in X\} = \Diamond_x B_1(\bigwedge\{\varphi : \varphi \in X\})$, and here we need the model to be ω-saturated. Since $\Diamond_x B_1(-)$ is monotone, the direction from right to left is immediate. As to the converse, assume $M \models \Diamond_x B_1 \varphi$, for all $\varphi \in X$. Choose $B_1(\varphi_1), \ldots, B_1(\varphi_n)$ with $\varphi_i \in X$. Since X is closed under conjunction, we have by assumption $M \models \Diamond_x B_1(\varphi_1 \wedge \cdots \wedge \varphi_n)$. By monotonicity of $\Diamond_x$ and B_1, $M \models \Diamond_x(B_1(\varphi_1) \wedge \cdots \wedge B_1(\varphi_n))$. This means that there is d_n such that $M \models R(d_n, \bar{e}) \wedge B_1(\varphi_1) \wedge \cdots \wedge B_1(\varphi_n)$. Because M is ω-saturated, there is an element d: $M \models R(u, \bar{e}) \wedge \bigwedge\{B_1(\varphi) : \varphi \in X\}$. Therefore, $M \models \Diamond_x \bigwedge\{B_1(\varphi) : \varphi \in X\}$, as required.
- Let $B = \exists x B_1$: the proof is analogous to the previous case. ⊣

4.2. Closure Lemma. Let A be a conjunction as in the condition of the Closure Lemma. We also assume that all quantifiers are non-vacuous and that every quantifier prefix contains at least one $\Box$-quantifier.

Let A' be the subformula of A which contains all and only occurrences of the predicate symbol P. We shall use both the $\mathscr{L}(\Box)$-form of A', namely $\bigwedge_i \bar{Q}_i \bar{x} P(\bar{x}, \bar{y})$, and its standard translation $\bigwedge_i \forall \bar{x}(\mathscr{R}_i \rightarrow P(\bar{x}, \bar{y}))$, where i runs over the occurrences of P. In the sequel we call the $\mathscr{R}_i$ *R-conditions*. The standard translation of A' is thus equivalent to $\forall \bar{x}(\bigvee_i \mathscr{R}_i \rightarrow P(\bar{x}, \bar{y}))$. $P(\bar{x}, \bar{y})$ has a good minimal substitution in A if

$$\bigvee_i \mathscr{R}_i = \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\bigvee_i \mathscr{R}_i \to \varphi(\bar{x},\bar{y}))\}.$$

EXAMPLE 5. The R-condition corresponding to $\Box_x\Box_y P(x,y)$ is $R(x) \wedge R(y,x)$.

EXAMPLE 6. Let $A' = \forall x\Box_y P(x,y) \wedge \Box_x \forall y P(x,y)$, then

$$A'^* = \forall x\forall y(R(y,x) \to P(x,y)) \wedge \forall x\forall y(R(x) \to P(x,y))$$

which is equivalent to $\forall x\forall y(R(y,x) \vee R(x) \to P(x,y))$. The good minimal substitution for P in A must be therefore $R(y,x) \vee R(x)$.

We are going to prove the existence of good minimal substitutions for all non-vacuous quantifier prefixes containing at least one $\Box$. But first we need several propositions.

PROPOSITION 2. *Let $\mathscr{R}$ be an R-condition, such that*

$$\mathscr{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathscr{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\};$$

then

$$\mathscr{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(x,\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathscr{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\},$$

and vice versa: if

$$\mathscr{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\psi(x,\bar{y}\bar{z}) : \forall\bar{x}\forall\bar{z}(\mathscr{R}(\bar{x},\bar{y}) \to \psi(\bar{x},\bar{y}\bar{z}))\},$$

then

$$\mathscr{R}(\bar{x},\bar{y}) \equiv \bigwedge\{\varphi(\bar{x},\bar{y}) : \forall\bar{x}(\mathscr{R}(\bar{x},\bar{y}) \to \varphi(\bar{x},\bar{y}))\}.$$

PROOF. The proof is routine. ⊣

To prove the next proposition, we shall use the following tautology of the minimal logic: $L_{min} \vdash \Box_x(\Box_x\theta \to \theta)$ (the proof is easy).

PROPOSITION 3. $R(z,\bar{x}\bar{y}) \equiv \bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\}$.

PROOF. The direction from left to right is trivial. For the converse direction, we have to prove $\bigwedge\{\varphi(z,\bar{x},\bar{y}) : \forall z\forall\bar{x}(R(z,\bar{x}\bar{y}) \to \varphi(z,\bar{x},\bar{y}))\} \to R(z,\bar{x}\bar{y})$, in other words,

$$\bigwedge_\varphi(\forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(z,\bar{x},\bar{y})) \to \bigwedge_\psi(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

It suffices to derive $\bigwedge_\psi \Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$ from $\bigwedge_\varphi \forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$. Take an arbitrary $\psi(z,\bar{x},\bar{y})$. We substitute this formula for θ in the tautology above:

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})).$$

We assume that the conjunction $\bigwedge_\varphi \forall\bar{x}\Box_z\varphi(z,\bar{x},\bar{y}) \to \varphi(d,\bar{e},\bar{y})$ holds. As a special case we obtain

$$\forall\bar{x}\Box_z(\Box_z\psi(z,\bar{x},\bar{y}) \to \psi(z,\bar{x},\bar{y})) \to (\Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})).$$

Since this holds for every ψ, we can derive $\bigwedge_\psi \Box_z\psi(z,\bar{e},\bar{y}) \to \psi(d,\bar{e},\bar{y})$. ⊣

PROPOSITION 4. *If $Qx_1, \dots, Qx_n$ contains at least one $\Box$-quantifier, and all quantifiers are non-vacuous, and the only occurrence of $P(\bar{x}, \bar{y})$ in A is of the form $Qx_1 \dots Qx_n P(\bar{x}, \bar{y})$, then P has a good minimal substitution in A.*

PROOF. The general form of the prefix described in the condition of this proposition, is

$$\forall(\bar{u})_1 \Box_{z_1} \forall(\bar{u})_2 \Box_{z_2} \cdots \forall(\bar{u})_k \Box_{z_k} \forall(\bar{u})_{k+1} P(\bar{x}, \bar{y}),$$

where $k > 0$ (that is, there is at least one $\Box$ in the prefix), and $\bar{u}\bar{z} = \bar{x}$. The standard translation of this formula is

$$\forall \bar{x}(R(z_1, (\bar{u})_1, \bar{y}) \wedge \cdots \wedge R(z_k, (\bar{u})_k, z_{k-1}, (\bar{u})_{k-1}, \dots, z_1, (\bar{u})_1, \bar{y}) \rightarrow P(\bar{x}, \bar{y}))$$

(since there is at least one $\Box$-quantifier in the prefix).

We have to show that

$$\mathscr{R} \equiv \bigwedge_{i=1}^{i=k} R(z_i, (\vec{uz})_{\leq i}, \bar{y}),$$

where $(\vec{uz})_{\leq i}$ are the variables bound by the quantifiers preceding $\Box_{z_i}$, is a good minimal substitution for P.

By Propositions 2 and 3 (observe that $z_i(\vec{uz})_{\leq i} \subseteq \bar{x}$),

$$R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \equiv \bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}.$$

Note that here we essentially use the fact that the z_i occur in $P(\bar{x}, \bar{y})$, that is, that $\Box$-quantifiers are non-vacuous.

It is easy to see that $\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$.

To prove the other direction, namely

$$\bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \rightarrow \bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}),$$

we argue as follows:

$$\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \subseteq$$
$$\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$$

therefore

$$\bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \rightarrow$$
$$\bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$$

and this means that

$$\bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \rightarrow R(z_i, (\vec{uz})_{\leq i}, \bar{y}).$$

Since this holds for every i,

$$\bigwedge \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \rightarrow \bigwedge_i R(z_i, (\vec{uz})_{\leq i}, \bar{y}),$$

that is, $\mathcal{R}$ is a good minimal substitution. ⊣

PROPOSITION 5. *A disjunction of good minimal substitutions is a good minimal substitution, i.e., if for every i,* $1 \leq i \leq n$,

$$\mathcal{R}_i(\bar{z}_i, \bar{y}) = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\},$$

where $\bar{z}_i \subseteq \bar{x}$, *then*

$$\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}.$$

PROOF. Since for every $\mathcal{R}_i$

$$\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \subseteq \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\},$$

$$\bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\} \rightarrow \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\},$$

that is, for every $\mathcal{R}_i$, $\mathcal{R}_i \rightarrow \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$, and this implies $\bigvee_i \mathcal{R}_i \rightarrow \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$.

Now we prove that the implication holds also in the other direction. From Proposition 2 follows that if $\mathcal{R}_i(\bar{z}_i, \bar{y}) = \bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$, $\bar{z}_i \subseteq \bar{x}$, then $\mathcal{R}_i(\bar{z}_i, \bar{y}) = \bigwedge\{\psi(\bar{z}_i, \bar{y}) : \forall \bar{z}_i(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \psi(\bar{z}_i, \bar{y}))\}$.

Now, assume that $\bigwedge\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$ holds and none of the $\mathcal{R}_i(\bar{z}_i, \bar{y})$ holds. Then as we have just seen, there are formulas $\psi_1, \ldots, \psi_n$, such that for every i, $\forall \bar{z}_i(\mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \psi_i(\bar{z}_i, \bar{y}))$ and $\neg\psi_i(\bar{z}_i, \bar{y})$. Take the disjunction of these formulas, $\bigvee_i \psi_i(\bar{z}_i, \bar{y})$. The resulting formula is also false. An equivalent formula with free variables $\bar{x}, \bar{y}$, namely, $\bigvee_i \psi_i(\bar{z}_i, \bar{y}) \vee \bot(\bar{x})$, belongs to the set $\{\varphi(\bar{x}, \bar{y}) : \forall \bar{x}(\bigvee_i \mathcal{R}_i(\bar{z}_i, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))\}$; but this formula is false, a contradiction. ⊣

LEMMA 3 (Closure Lemma). *Let A be a conjunction of atomic formulas prefixed by* ∀ *and* □*-quantifiers, so that all occurrences of a predicate symbol in A have the same free variables. Then every atomic formula in A has a good minimal substitution, and this minimal substitution is the same as the one used to obtain a frame correspondent.*

PROOF. The lemma follows from the four propositions proved above. ⊣

4.3. Syntactic transformations. We now describe the syntactic transformations which reduce the task of finding a correspondent for an axiom χ to simple applications of the Intersection and Closure Lemmas, thus finishing the proof of the Theorem 4.

Assume that a formula χ is of the form $\bigwedge_i \chi_i$, where each χ_i is of the form $Qz_1 \cdots Qz_m(A \rightarrow B)$, A and B as in the Theorem 4. Then if every conjunct in χ corresponds to a first-order condition on R in the canonical model, say $\chi_i^{\dagger}$, χ itself corresponds to a first-order condition on R (namely, a conjunction of $\chi_i^{\dagger}$). So, it suffices to prove that every χ_i corresponds to a first-order condition on R.

Writing down the validity conditions of χ_i, we obtain

$$\forall\varphi_1 \cdots \forall\varphi_n Qz_1 \cdots Qz_m(A \to B),$$

where $\forall\varphi_j$ are the quantifiers over definable relations corresponding to the predicate symbols in χ_i. If P_j is an n-place predicate symbol, then $\forall\varphi_j$ can be instantiated on any formula with precisely n variable places.

If $m > 0$, then translating the truth conditions for the quantifiers in first-order logic, we obtain

$$\forall\varphi_1 \cdots \forall\varphi_n \forall z_1 \cdots \forall z_m(\Gamma \to (st(A) \to st(B))),$$

where Γ is a conjunction of R-conditions corresponding to the $\Box$-quantifiers in the prefix (if there are such quantifiers). This is equivalent to

$$\forall z_1 \cdots \forall z_m(\Gamma \to \forall\varphi_1 \cdots \forall\varphi_n(st(A) \to st(B))).$$

Our proceedings in cases when A contains disjunctions, constant formulas (without predicate symbols other than $=$, $\top$ and $\bot$), negative formulas, are pretty obvious. For example, in case when $st(A)$ is of the form $\vee_i st(A_i)$, we apply the fact that

$$\forall\varphi_1 \cdots \forall\varphi_n(\vee_i st(A_i) \to st(B))$$

is equivalent to

$$\wedge_i \forall\varphi_1 \cdots \forall\varphi_n(st(A_i) \to st(B));$$

substituting this formula in the formula above gives

$$\wedge_i \forall z_1 \cdots \forall z_m(\Gamma \to \forall\varphi_1 \cdots \forall\varphi_n(st(A_i) \to st(B)));$$

now it again suffices to prove that each conjunct has a correspondent.

Finally, we have a formula

$$(*) \qquad \forall z_1 \cdots \forall z_m(\Gamma' \to \forall\varphi_1 \cdots \forall\varphi_n(A' \to B')),$$

where Γ' is first-order, B' is a positive first-order formula, and A' is a conjunction of formulas $st(Qx_1 \cdots Qx_k\varphi_i)$, where all quantifiers are non-vacuous.

Assume that there is only one predicate letter P in χ. Then the reasoning goes as follows: P occurs in A' in subformulas of the form $st(Qx_1 \cdots Qx_k P(\bar{x}, \bar{y}))$, where the $\bar{x}$ are bound and the $\bar{y}$ free (rename the bound variables if necessary). The condition (*) can be rewritten as $\forall z_1 \cdots \forall z_m(\Gamma' \to \bigwedge\{B'(\varphi(\bar{x}, \bar{y})) : \bigwedge_i st(\bar{Q}_i\bar{x}\varphi(\bar{x}, \bar{y}))\})$.

Applying the Intersection Lemma,

$$\forall z_1 \cdots \forall z_m(\Gamma' \to B'(\bigwedge\{\varphi(\bar{x}, \bar{y}) : \bigwedge_i st(\bar{Q}_i\bar{x}\varphi(\bar{x}, \bar{y}))\})),$$

and by the Closure Lemma (P has a good minimal substitution in A', say p), this is equivalent to $\forall z_1 \cdots \forall z_m(\Gamma' \to B'(p))$, which is a first-order statement in R.

Now we consider the general case, when there is more than one predicate symbol. Then we eliminate the second-order quantifiers one by one in the following way. Split A' in two parts, A_1 and A_2, so that A_2 contains all and only occurrences of P_n:

$$\forall z_1 \cdots \forall z_m(\Gamma' \to \forall\varphi_1 \cdots \forall\varphi_n(A' \to B'))$$

is equivalent to

$$\forall z_1 \cdots \forall z_m(\Gamma' \to \forall\varphi_1 \cdots \forall\varphi_n(A_1 \wedge A_2 \to B')),$$

and this in turn to

$$\forall z_1 \cdots \forall z_m(\Gamma' \to \forall\varphi_1 \cdots \forall\varphi_{n-1}(A_1 \to \forall\varphi_n(A_2 \to B'))).$$

We now apply the Intersection Lemma and the Closure Lemma to $\forall\varphi_n(A_2 \to B')$.

This way all second-order quantifiers which bind predicate symbols occurring both in the antecedent and in the consequent can be eliminated.

If B contains predicate symbols which are not in the antecedent, these can be replaced by a fixed contradiction having the same parameters as the original atomic formula; since B is positive, and therefore monotone, the resulting formula is equivalent to the original one. Analogously, a predicate symbol occurring only in the antecedent can be replaced by a tautology.

Assume that A contains a predicate symbol which does not have a quantifier prefix, that is, $A \to B$ can be written as $A' \wedge P(\bar{x}) \to B(P(\bar{x}))$. By assumption, the $\bar{x}$ are free in B. Since B is positive, $B(P(\bar{x}))$ can be equivalent to $B' \wedge P(\bar{x})$ or $B' \vee P(\bar{x})$.

$$A' \wedge P(\bar{x}) \to B' \vee P(\bar{x})$$

obviously corresponds to a first order condition, namely a trivial one, and

$$A' \wedge P(\bar{x}) \to B' \wedge P(\bar{x})$$

is equivalent to $A' \wedge P(\bar{x}) \to B'$, the case which we treated above.

Let $\chi^\dagger$ be the result of applying this algorithm to χ. We proved that if χ is an axiom, then in a canonical ω-saturated model $\chi^\dagger$ holds.

The proof of the converse, namely that for *every* model M, if $M \models \chi^\dagger$, then $M \models \chi$, is standard (cf. Sambin and Vaccaro [1989], van Benthem and Alechina [1993]). One can show that $\vdash \chi^\dagger \equiv \forall P_1 \cdots \forall P_n \chi$, where $P_1, \ldots, P_n$ are all predicate letters occurring in χ (the proof uses the fact that here the quantifiers range over all possible interpretations of predicate letters, including good minimal substitutions for P_i). And this implies that if $M \models \chi^\dagger$, then M makes every instance of χ true. ⊣

EXAMPLE 7. All axioms of the free filter quantifier (except extensionality) have correspondents for completeness.

PROOF. We check the axioms given in Example 2.

$\Diamond_x x = x$ has a correspondent in every model, namely $\exists x(R(x) \wedge x = x)$, which is equivalent to $\exists x R(x)$.

$\Box_x x \neq y$ has a correspondent in every model, namely, $\forall x(R(x, y) \to x \neq y)$, which is equivalent to $\forall x \neg R(x, x)$.

$\Box_x\varphi \wedge \Box_x\psi \to \Box_x(\varphi \wedge \psi)$ is equivalent to the formula treated in Example 4; its correspondent is $\forall x \forall \bar{y} \forall \bar{z}(R(x, \bar{y}\bar{z}) \to R(x, \bar{y}))$.

$\forall x(\varphi \to \psi) \to (\Box_x\varphi \to \Box_x\psi)$ is not a weak Sahlqvist formula. We shall see later that it does not have a correspondent for completeness.

EXAMPLE 8. The characteristic axiom of the "for almost all" quantifier (the Fubini property):

$$\Box_x\Box_y P(x, y, \bar{z}) \to \Box_y\Box_x P(x, y, \bar{z})$$

corresponds to the following condition on R:

$$R(y, \bar{z}) \wedge R(x, y\bar{z}) \rightarrow R(x, \bar{z}) \wedge R(y, x\bar{z}).$$

PROOF. Rewriting the validity conditions of the axiom gives

$$\forall\varphi(\forall x \forall y(R(x, \bar{z}) \wedge R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z})) \rightarrow \forall y \forall x(R(y, \bar{z}) \wedge R(x, y\bar{z}) \rightarrow \varphi(x, y, \bar{z}))$$

which is equivalent to

$$\bigwedge\{\forall y \forall x(R(y, \bar{z}) \wedge R(x, y\bar{z}) \rightarrow \varphi(x, y, \bar{z})) : \forall x \forall y(R(x, \bar{z}) \wedge R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z}))\}.$$

By the Intersection Lemma,

$$\forall y \forall x(R(y, \bar{z}) \wedge R(x, y\bar{z}) \rightarrow \bigwedge\{\varphi(x, y, \bar{z}) : \forall x \forall y(R(x, \bar{z}) \wedge R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z}))\}),$$

while by the Closure Lemma

$$\bigwedge\{\varphi(x, y, \bar{z})) : \forall x \forall y(R(x, \bar{z}) \wedge R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z}))\} \equiv R(x, \bar{z}) \wedge R(y, x\bar{z})$$

in every canonical model. Thus we obtain the correspondent

$$\forall y \forall x(R(y, \bar{z}) \wedge R(x, y\bar{z}) \rightarrow R(x, \bar{z}) \wedge R(y, x\bar{z})). \quad \dashv$$

EXAMPLE 9. The characteristic axiom of the "co-countably many" quantifier (Keisler's axiom):

$$\forall x \Box_y P(x, y, \bar{z}) \wedge \Box_x \forall y P(x, y, \bar{z}) \rightarrow \Box_y \forall x P(x, y, \bar{z}),$$

corresponds to

$$\forall x \forall y(R(y, \bar{z}) \rightarrow R(y, x\bar{z}) \vee R(x, \bar{z})).$$

PROOF. The axiom is valid iff

$$\forall\varphi(\forall x \forall y(R(y, x\bar{z}) \rightarrow \varphi(x, y, \bar{z})) \wedge \forall x \forall y(R(x, \bar{z}) \rightarrow \varphi(x, y, \bar{z})) \rightarrow \forall x \forall y(R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z})),$$

namely,

$$\forall\varphi(\forall x \forall y(R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z})) \rightarrow \forall x \forall y(R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z})).$$

This can be rewritten as

$$\bigwedge\{\forall x \forall y(R(y, \bar{z}) \rightarrow \varphi(x, y, \bar{z})) : \forall x \forall y(R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z}))\};$$

by the Intersection Lemma,

$$\forall x \forall y(R(y, \bar{z}) \rightarrow \bigwedge\{\varphi(x, y, \bar{z}) : \forall x \forall y(R(y, x\bar{z}) \vee R(x, \bar{z})) \rightarrow \varphi(x, y, \bar{z}))\},$$

and by the Closure Lemma, $\forall x \forall y(R(y, \bar{z}) \rightarrow R(y, x\bar{z}) \vee R(x, \bar{z}))$. $\dashv$

The correspondent of the Fubini property is in Horn form, whereas the correspondent of Keisler's axiom is not. In Section 8.3 we shall see that this distinction is of some importance, so it would be of interest to isolate the class of weak Sahlqvist formulas whose correspondents are in Horn form.

§5. Soundness of the substitution rules. In this section we show that if **LCC** proves φ, for $\varphi \in \mathscr{L}(\Box)$, then φ is derivable from the Keisler axioms. To show this, we proceed as follows. First we show that **LCC** is sound for relational models where R satisfies the following properties:

1. $\exists x R(x, \bar{y})$
2. $\neg R(x, x)$
3. $R(x, \bar{y}\bar{z}) \rightarrow R(x, \bar{y})$
4. $\neg R(x, y\bar{z}) \wedge \neg R(y, \bar{z}) \rightarrow \neg R(x, \bar{z})$
5. For every formula $\theta(x, \bar{y})$ in the image of the standard translation:

$$\forall x(R(x, z\bar{y}) \rightarrow \theta(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}) \rightarrow \theta(x, \bar{y})).$$

Informally, $x_{\bar{y}}$ is interpreted as ranging over $\{x : R(x, \bar{y})\}$; formally, we consider assignments $f : \mathit{IVar} \longrightarrow A$, constrained by $R(f(x_{\bar{y}}), f(\bar{y}))$. The rules $\Box r, l$ are sound because $\{x | R(x, \bar{y})]\}$ is non-empty.

$y \neq x_y$ holds because $y = x_y$ would imply $R(y, y)$, contradicting 2.

SUB_{ext}

$$\frac{\Gamma \Rightarrow \psi(x_{\bar{u}\bar{z}}, \bar{z}), \Delta}{\Gamma \Rightarrow \psi(x_{\bar{v}\bar{z}}, \bar{z}), \Delta}$$

is sound because 3 and 5 together imply

$$\forall x(R(x, \bar{y}\bar{u}) \rightarrow \theta(x, \bar{y})) \rightarrow \forall x(R(x, \bar{y}\bar{v}) \rightarrow \theta(x, \bar{y})).$$

Lastly, we consider SUB_{cc}:

$$\frac{\Gamma' \Rightarrow y \neq x'_{\bar{z}}, \Delta' \quad \Gamma \Rightarrow \varphi(x_{y\bar{z}}, y, \bar{z}), \Delta}{\Gamma, \Gamma' \Rightarrow \varphi(x_{\bar{z}}, y, \bar{z}), \Delta, \Delta'},$$

where $x'_{\bar{z}}$ does not occur free in Γ', Δ' and $x_{y\bar{z}}$ does not occur free in Γ, Γ', Δ, Δ'.

Observe that the premises of SUB_{cc} imply under our interpretation of indexed variables

$$\Gamma' \rightarrow \forall x'(R(x', \bar{z}) \rightarrow y \neq x' \vee \Delta')$$

and

$$\Gamma \rightarrow \forall x(R(x, y\bar{z}) \rightarrow \varphi(x, y, \bar{z}) \vee \Delta),$$

respectively (given that $x'_{\bar{z}}$ is not free in Γ', Δ', and $x_{\bar{z}}y)$ is not free in Γ, Δ). We want to derive the conclusion, namely

$$\Gamma, \Gamma' \rightarrow \forall x(R(x, \bar{z}) \rightarrow \varphi(x, y, \bar{z}) \vee \Delta \vee \Delta').$$

Assume that Γ, Γ' and $R(x, \bar{z})$ hold. If one of Δ, Δ' holds, we are done. Assume that Δ and Δ' do not hold. Then Γ' implies $\forall x'(R(x', \bar{z}) \rightarrow y \neq x')$, that is, $\neg R(y, \bar{z})$. By 4, $R(x, y\bar{z})$. But then Γ implies $\varphi(x, y, \bar{z})$.

Hence SUB_{cc} is sound for relational models $\langle \mathscr{A}, R \rangle$ satisfying 4.

Now suppose φ is not derivable from the Keisler axioms; by Theorem 3 there exists a canonical relational model $\langle \mathscr{A}, R\rangle$ for $\neg\varphi$ and the Keisler axioms[6]. By Theorem 4, $\langle \mathscr{A}, R\rangle$ satisfies 1–5 above; hence this model shows that **LCC**$+\neg\varphi$ is consistent. ⊣

§6. Non-existence of correspondents for completeness.

6.1. Restrictions imposed in the Sahlqvist theorem. In this section we show that not all formulas have a correspondent for completeness. First we prove this for the truth definition we are working with; later the result is generalized.

The following theorem also shows why occurrences of $\Diamond_x$ in the antecedent are forbidden in Theorem 4.

THEOREM 6. *$\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$ does not have a correspondent in the sense of completeness.*

PROOF. Let us call $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$ A. Although A does have a frame correspondent, namely $\forall x\forall\bar{y}\forall\bar{z}(R(x,\bar{y}) \to R(x,\bar{y}\bar{z}))$, (see van Benthem and Alechina [1993]) we show that it does not have a correspondent for completeness. Assume that $A^\dagger$ exists. The following axioms are consistent with A: $\Diamond_x x = x$ and $\forall y\neg\Diamond_y x = y$. Hence, together with A they have a model where $A^\dagger$ holds. From Proposition 1 follows that $A^\dagger$ implies $\forall x\forall\bar{y}\forall\bar{z}(R(x,\bar{y}) \to R(x,\bar{y}\bar{z}))$. Since the correspondents of the other two axioms (being their standard translations) hold in every model, we have a model where $\forall x\forall\bar{y}\forall\bar{z}(R(x,\bar{y}) \to R(x,\bar{y}\bar{z}))$, $\exists xR(x)$ and $\forall x\neg R(x,x)$ hold simultaneously. But this is impossible. ⊣

An immediate consequence of the theorem is that "$\Box$ over $\wedge$"-combination in the antecedent cannot be allowed, since the axiom considered above can be written equivalently as

$$\Box_x(\varphi \wedge \psi) \to \Box_x\varphi.$$

Also, we have

COROLLARY 1. *Extensionality, that is $\forall x(\varphi \equiv \psi) \to (\Box_x\varphi \equiv \Box_x\psi)$ does not have a correspondent in the sense of completeness.*

PROOF. One can check that extensionality is consistent with $\Diamond_x x = x$ and $\neg\Diamond_y x = y$, and that extensionality implies $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. Hence if extensionality would have a correspondent for completeness, it would imply the frame correspondent of $\Diamond_x\varphi \to \Diamond_x(\varphi \vee \psi)$. The rest of the argument is the same as above. ⊣

COROLLARY 2. *$\Diamond_x\varphi \to \Box_x\varphi$ does not have a correspondent in the sense of completeness.*

PROOF. The proof is analogous; the consistency can be shown by constructing a model where D is infinite, $R(x,\bar{y}) \Leftrightarrow \bigwedge_i x \neq y_i$, and for each predicate P^n, $V(P^n) = D^n$. ⊣

[6]Alternatively, take the model given by Keisler's completeness theorem and define R as a countable intersection.

COROLLARY 3. *$\exists x(x \neq y \wedge \varphi(x,y)) \to \Diamond_x(\varphi(x,y) \vee \psi)$ does not have a correspondent in the sense of completeness.*

PROOF. Analogous. ⊣

The clause of Theorem 4 forbidding occurrences of the same predicate letter with different free variables is also necessary. Suppose that there is a variable in the antecedent not occurring in the consequent, as in $\Box_y(\Box_x P(x,y) \to \Box_z P(x,z))$, is equivalent to $\Diamond_y(\Box_x P(x,y) \wedge \top(x)) \to \Box_z P(x,z)$, and $\forall y(\Box_x P(x,y) \to \Box_z P(x,z))$ is equivalent to $\exists y \Box_x P(x,y) \to \Box_z P(x,z)$.

This shows that the class of weak Sahlqvist formulas is strictly smaller than the class of all Sahlqvist formulas and that none of the conditions of Theorem 4 can be dropped.

6.2. Other truth definitions. Let us first of all compare the results of this paper with results previously obtained for modal logic. Recall that a modal logic L is called *first order complete* if there is a set Δ of first order sentences in the language $\{R, =\}$ such that

$\vdash_L \varphi$ if and only if for every Kripke model M: if $M \models \Delta$ then $M \models \varphi$;

Sahlqvist's theorem tells us that modal logics axiomatized by Sahlqvist formulas, such as K4, S4, S5, etc. are first order complete. When we transfer this concept of first order completeness to generalized quantifiers (cf. Definition 15), we see that there is an important quantifier logic axiomatized by Sahlqvist formulas which is not first order complete, namely the "almost all" quantifier axiomatized by *L1–L4* below plus $\Diamond_x \varphi \to \Diamond_x(\varphi \vee \psi)$ (which is equivalent to *L1 – L5* given the minimal logic). We do have first order completeness for logics axiomatized by weak Sahlqvist formulas, but since extensionality is not weak Sahlqvist, this result is not very interesting taken by itself. Indeed, we can obtain even stronger negative results, as follows. The reader may have observed that for generalized quantifier logics a stronger form of first order completeness can be defined: the truth definition we considered above is universal, but nothing prevents us from considering more complex truth definitions. Might we not obtain a first order correspondent to extensionality this way? E.g., the following truth definition studied in Jervell [1975], Mijajlovic [1985] and Krynicki [1990] trivially yields extensionality:

$$\Box_x \varphi(x) \text{ if and only if } \exists y \forall x(R(x,y) \to \varphi(x)),$$

where R is a new binary predicate and y does not occur free in φ. But in this case Keisler's axiom for "co-countably many"

$$\forall x \Box_y \varphi \wedge \Box_x \forall y \varphi \to \Box_y \forall x \varphi$$

corresponds to a schema, not to a first order condition as above.

This is not accidental. For instance, for the quantifier "almost all" it can be shown that any truth definition, however complex, involving an accessibility relation R, will make at least one axiom correspond to a schema.

In this subsection, we shall consider even more general truth definitions involving a relation $R(x, Y)$, where Y is a finite *subset* of the domain. Consideration of this relation is natural, because one might argue that dependence really is a relation between objects and finite sets of objects (compare algebraic or linear dependence). Such truth definitions, where quantifiers over finite sets are allowed, will be called *weak second-order*. The truth definition that we employed up till now can be expressed in the new language as follows:

$$\Box_x \varphi(x, y_1, \dots, y_n) \Leftrightarrow$$

$$\exists Y[\forall x(R(x, Y) \to \varphi(x, y_1, \dots, y_n)) \wedge \bigwedge_i y_i \in Y \wedge \forall z \in Y(\bigvee_i z = y_i)].$$

For any given weak second-order definition, we may now ask whether quantifier axioms have correspondents for completeness in the language $\{R, \in, =\}$.

THEOREM 7. *For any weak second-order truth definition, the conjunction of the Friedman axioms*

L1 $\Diamond_x x = x$
L2 $\Box_x x \neq y$
L3 $\Box_x \varphi \wedge \Box_x \psi \to \Box_x(\varphi \wedge \psi)$
L4 $\Box_x \Box_y \varphi \to \Box_y \Box_x \varphi$
L5 $\forall x(\varphi \to \psi) \to (\Box_x \varphi \to \Box_x \psi)$

does not have a correspondent for completeness.

PROOF. Choose a weak second-order truth definition and suppose that the conjunction of the Friedman axioms does have a correspondent for completeness. Then the following statement would be true:

(C) $\exists X \exists D \exists R(X = D^{<\omega} \wedge R \subseteq D \times X \wedge$ 'R satisfies the correspondent for completeness for the conjunction of the Friedman axioms').

The part in quotes is Δ_0 since all second-order quantifiers range over X. Furthermore the formula $X = D^{<\omega}$ is Σ_1, hence the entire statement is Σ_1.

From the Levy-Shoenfield Absoluteness Lemma (cf. Jech [1978], p. 120) it follows that (C) holds in the constructible universe L. We now show that this is a contradiction.

Observe first that by using our truth definition in L, we may add a generalized quantifier Q to the language $\{\in, =\}$, where $\in, =$ have the standard interpretation; Q will satisfy the conjunction of the Friedman axioms *for this language* since R satisfies its correspondent for completeness. In L, define a relation S by

$$S(x, \bar{y}) =_{df} \bigwedge_\varphi Qx\varphi(x, \bar{y}) \to \varphi(x, \bar{y})$$

and a new quantifier $\Box$ by $\Box_x \varphi(x, \bar{y}) =_{df} \forall x(S(x, \bar{y}) \to \varphi(x, \bar{y}))$. We show that we must have $\Box_x \varphi(x, \bar{y}) \equiv Qx\varphi(x, \bar{y})$. The direction from right to left is trivial. For the converse it suffices if Q is closed under countable intersections in L, for in that case

(a) $\Box_x \varphi(x, \bar{y}) \equiv \forall x((\bigwedge_\psi Qx\psi(x, \bar{y}) \to \psi(x, \bar{y})) \to \varphi(x, \bar{y}))$
(b) $\bigwedge_\psi Qx\psi(x, \bar{y}) \to Qx\psi(x, \bar{y}) \equiv Qx(\bigwedge_\psi Qx\psi(x, \bar{y}) \to \psi(x, \bar{y}))$

whence (by *L5*) $Qx\varphi(x, \bar{y})$.

To show that Q is closed under countable intersections in L, argue as follows: let Q be defined on D (i.e., $Q \subseteq \mathscr{P}(D)$ and $D \in Q$), and let $(A_n)_{n \in \omega}$ be in L, such that $A_n \subseteq D$ and for all n, $Qx(x \in A_n)$. We want to show $Qx(x \in \bigcap_n A_n)$.

Define a relation $T(x, y)$ by

$$T(x,y) =_{df} \forall n(x \in A_n^c \rightarrow \forall m(y \in A_m) \vee \exists m > n(y \in A_m^c)) \vee \forall n(x \in A_n \wedge y \in D).$$

Then $\forall x Q y T(x,y)$: for if $x \in \bigcap_n A_n$, then $y \in D$, and if for some n, $x \notin A_n$, then $y \in \bigcap_{m \leq n} A_m$. Hence $QxQyT(x,y)$ and by *L4* $QyQxT(x,y)$.

If $y \in A_m^c$, then $\{x : T(x,y)\} = \bigcap_n A_n \cup \bigcup_{n \leq m} A_n^c$; and if $y \in \bigcap_m A_m$, then $\{x : T(x,y)\} = D$. It follows that $\bigcap_m A_m \subseteq \{y : QxT(x,y)\}$. If we have equality, also $Qy(y \in \bigcap_m A_m)$. Otherwise, let m be such that $y \in A_m^c$ and $QxT(x,y)$. It follows that $Qx(x \in \bigcap_n A_n \cup \bigcup_{n \leq m} A_n^c)$, whence $Qx(x \in \bigcap_n A_n)$.

Hence $\Box_x \varphi(x, \bar{y}) \equiv Qx\varphi(x, \bar{y})$, and it follows that $\Box$ also satisfies the Friedman axioms. Applying the Sahlqvist algorithm to the Friedman axioms (note that we now have obtained a canonical relational model, and that the Friedman axioms do not have existential quantifiers in the antecedent, hence their correspondents hold in every canonical model) shows that S satisfies

(a) $\exists x S(x)$
(b) $\neg S(x,x)$
(c) $S(x, \bar{y}\bar{z}) \rightarrow S(x, \bar{y})$
(d) $S(x, y\bar{z}) \wedge S(y, \bar{z}) \rightarrow S(y, x\bar{z})$.

Furthermore, rewriting extensionality in terms of S shows that S satisfies, for every formula of the form $st(\psi)$

$$\exists x(S(x,\bar{y}) \wedge \neg st(\psi)(x,\bar{y})) \rightarrow \exists x(S(x,\bar{y}\bar{z}) \wedge \neg st(\psi(x,\bar{y})))$$

that is

$$(\sharp) \qquad \forall x(S(x,\bar{y}\bar{z}) \rightarrow st(\psi(x,\bar{y}))) \rightarrow \forall x(S(x,\bar{y}) \rightarrow st(\psi(x,\bar{y}))).$$

We show that L does not have a definable well-ordering, which is a contradiction.

Suppose that $<$ is a definable well-ordering on D. Define $P(x,\bar{z})$ as $S(x,\bar{z}) \wedge \forall y(S(y, x\bar{z}) \rightarrow x \leq y)$. Since $\exists \bar{z} \forall x \neg P(x, \bar{z})$ implies (with the Axiom of Dependent Choice, which holds in L) the existence of an infinite descending chain, we must have $\forall \bar{z} \exists x P(x, \bar{z})$. Then we can choose x_0 with $S(x_0)$ and $\forall y(S(y,x_0) \rightarrow x_0 \leq y)$ and x_1 with $S(x_1, x_0)$ and $\forall y(S(y, x_0 x_1) \rightarrow x_1 \leq y)$. Due to the property ($\sharp$) the following holds:

$$\forall y(S(y, x_1) \rightarrow x_1 \leq y).$$

$S(x_0) \wedge S(x_1, x_0)$ implies together with (d) $S(x_0, x_1)$, therefore $x_1 \leq x_0$. This yields $x_0 = x_1$, a contradiction with $S(x_1, x_0)$ by (b) (cf. Theorem 1.5.3 in van Lambalgen [1994]). ⊣

A slight rearrangement of the argument above yields

COROLLARY 4. *Let Q be a weak second-order definable filter quantifier satisfying $QxQy\varphi \rightarrow QyQx\varphi$. Then Q is a principal filter.*

§7. Undecidability of the correspondence problem. In modal logic, the problem whether a formula has a first order frame correspondent is undecidable. This was proved by Chagrova [1991]. It is easy to show that this result implies undecidability of the frame correspondence problem for generalized quantifiers (cf. van

Benthem and Alechina [1993]). The same result holds for the correspondence for completeness problem, even for logic which does not have ordinary quantifiers.

THEOREM 8. *For any weak second-order truth definition, it is undecidable whether a formula in the language with generalized quantifiers has a correspondent for completeness.*

PROOF. It is undecidable whether a formula without ordinary quantifiers is satisfiable in Friedman's logic (Steinhorn [1985b]). Let F be Friedman's logic; F can be written as $F_0 + EXT$, where EXT is extensionality, and F_0 is a conjunction of weak Sahlqvist formulas.

Let A be any formula with □'s but no first order quantifiers.

- If $F_0 + EXT + A$ is satisfiable, then it does not have a correspondent for completeness by the argument in L.
- If $F_0 + EXT + A$ is not satisfiable, then $\bot$ is a correspondent for completeness.

Hence if we could decide whether a schema has a correspondent for completeness, we could also decide satisfiability in Friedman's logic. ⊣

§8. Digraphs on variables.

8.1. Negation. We wish to provide a semantics for a fragment of $\mathscr{L}(IVar)$ corresponding to partially ordered complexes of generalized quantifiers. Partially ordered quantifiers Q (also known as Henkin quantifiers or branching quantifiers) can be defined formally as follows: Q is a relational system $\langle \mathscr{A}_Q, <_Q \rangle$, where $\mathscr{A}_Q$ is a finite set of first order quantifiers with distinct variables and $<_Q$ is an asymmetric partial order on $\mathscr{A}_Q$.

A *dependency* of a p.o. quantifier Q is an element of $<_Q$ of the form $\langle \forall a, \exists b \rangle$; the *essential order* of Q is the set of all dependencies of Q. A little reflection shows that $<_Q$ can be captured formally by means of suitable indexed variables, and indeed the semantics for $\mathscr{L}(IVar)$ will be such that all Henkin quantifiers can be represented. For example, the Henkin quantifier

$$\begin{pmatrix} \forall y \exists x \\ \forall u \exists v \end{pmatrix} \varphi(x, y, u, v)$$

(where φ is first order) is represented by $\forall y \exists x_y \forall u \exists x_u (y, x_y, u, x_u)$, in the sense that both formulas receive the semantical interpretation

$$\exists F \exists G \forall y \forall u \varphi(y, F(y), u, G(u)).$$

Clearly, both in the case of Henkin quantifiers and in that of $\mathscr{L}(IVar)$, we can construct much more complex formulas by means of $\neg$; the question is how to interpret these formulas. We illustrate the difficulty using Harel's [1979] semantics. For Harel, the following formulas are all wellformed:

(a) $\neg \forall y \exists x_y \forall u \exists x_u \varphi(y, x_y, u, x_u)$
(b) $\forall y \exists x_y \neg (\forall u \exists x_u) \varphi(y, x_y, u, x_u)$
(c) $\neg (\forall u \exists x_u) \forall y \exists x_y \varphi(y, x_y, u, x_u)$,

where the brackets indicate the scope of $\neg$. These formulas receive the following interpretations:

(a) $\forall F \forall G \exists y \exists u \neg \varphi(y, F(y), u, G(u))$

(b) $\exists F \forall G \forall y \exists u \neg\varphi(y, F(y), u, G(u))$
(c) $\forall F \exists G \exists y \exists u \neg\varphi(y, F(y), u, G(u))$.

We see that, firstly, $\neg$ introduces new dependencies (in (b), y depends on u; in (c), y depends on G), and, secondly, that (b) and (c) receive different interpretations, although $\forall y \exists x_y \forall u \exists x_u \varphi(y, x_y, u, x_u)$ and $\forall u \exists x_u \forall y \exists x_y \varphi(y, x_y, u, x_u)$ are equivalent. In other words, the order in which we write the independent prefixes now matters; and neither (b) nor (c) is a representation of

$$\begin{pmatrix} \forall y \exists x \\ \neg\forall u \exists v \end{pmatrix} \varphi(x, y, u, v).$$

Part of the trouble seems to be that the negation of a universal quantifier introduces a dependency, but that the nature of this dependency is underdetermined. Since we have no intuition whatsoever about the behaviour of dependencies in the presence of negation, we shall restrict ourselves to a fragment where these problems do not arise. We believe, however, that a semantics for a fragment including negations of prefixes should be based on the idea that partially ordered quantifiers are connected to games with imperfect information.

8.2. Semantics for interpretable formulas.

DEFINITION 19. A *prefix* is a sequence of quantifiers $Q_1 t_1 \cdots Q_n t_n$ such that for all i, all variables occurring in the index of $t_i \in IVar$ are bound by quantifiers $Q_1 t_1 \cdots Q_{i-1} t_{i-1}$. ⊣

DEFINITION 20. The set of *interpretable* formulas is defined as follows:
1. Quantifier-free formulas of $\mathcal{L}(IVar)$ are interpretable.
2. If P is a prefix and φ is an interpretable formula such that no variable bound in φ occurs in P, then $P\varphi$ is an interpretable formula. ⊣

We shall provide two semantics for interpretable formulas; the first (Definition 22) adequate for the case of full substitution rules (where we may take $\forall$ binding $x \in Var$ only), the second (Definition 23) appropriate for the case when there are no substitution rules, i.e., when indexed variables have to be restricted to a relation R.

DEFINITION 21. If P is a prefix, we define the *second-order part* $sop(P)$ of P to be $\exists \sharp_0(t_1) \cdots \exists \sharp_0(t_k)$, where each t_i is an indexed variable bound by $\exists$.

The *first order part* $fop(P)$ of P consists of the sequence of all universal quantifiers in P. ⊣

DEFINITION 22. Let $\mathcal{A}$ be a first order model. Satisfaction of interpretable formulas is given by a map $\sharp$ from $Form(\mathcal{L}(ITerm))$ into a second-order language:
1. $\sharp(\neg)A(t_1, \ldots, t_n) = (\neg)A(\sharp(t_1), \ldots, \sharp(t_n))$.
2. $\neg, \wedge, \vee$ as in Definition 6.
3. If for a prefix P, $P\varphi$ is interpretable, $\sharp P\varphi = sop(P) fop(P) \sharp\varphi$ (where indexed variables in $fop(P)$ are replaced by new ordinary variables).

We say that an interpretable formula φ is satisfied in $\mathcal{A}$ under an assignment f if $\sharp\varphi$ is satisfied in $\mathcal{A}$ under f in the second-order sense. The reader may check that, for linear prefixes, this definition is equivalent to Definition 6. ⊣

It would be no problem to extend the first order semantics to a larger class of formulas where also negations of prefixes are allowed, thus obtaining a system equivalent with second-order logic (Harel [1979]); but as remarked above we shall refrain from doing so, for want of a perspicuous interpretation.

Lastly, we provide the interpretable formulas with a relational semantics, appropriate for models $\langle \mathscr{A}, R \rangle$.

Definition 23. Relation semantics for the interpretable formulas is given by a map $\sharp$ from $Form(\mathscr{L}(ITerm))$ to a second-order language containing R, which can be described as follows.

Given an interpretable formula φ, replace successively each quantifier $\exists t$ by $\exists t(R(t, \text{ind}(t)) \wedge \cdots)$ and each quantifier $\forall t$ by $\forall x(R(x, \text{ind}(t)) \rightarrow \cdots)$ (where x is a new variable). Write the resulting expression in prenex form and define $\sharp\varphi$ to be the formula obtained by applying Definition 22. We say that φ is satisfied in $\langle \mathscr{A}, R \rangle$ under an assignment f if $\sharp\varphi$ is satisfied in $\langle \mathscr{A}, R \rangle$ under f the second-order sense. ⊣

The definition can be easily extended to many-sorted languages with indexed variables, corresponding to models with several R predicates.

Example 10. Take a standard model $\langle \mathscr{A}, Q \rangle$ for the quantifier 'co-countably many' and expand $\mathscr{A}$ to a model $\langle \mathscr{A}, Q, R \rangle$ by taking countable intersections. On this model we may now interpret the branching quantifier

$$\begin{pmatrix} \forall x Q y \\ \forall u Q v \end{pmatrix} \varphi(x, y, u, v).$$

Similarly, we may define branching compounds of many different generalized quantifiers. This should be compared to Barwise's [1979] definition.

8.3. Digraphs. Lastly, we investigate the graphical interpretation of branching compounds of generalized quantifiers. Henkin quantifiers can be represented as a partial order on variables, labelled by first order quantifiers. Interpretable formulas of $\mathscr{L}(Ivar)$ can also be represented as a labelled structure on variables, but in this case the structure is not always a partial order.

First we introduce some graph-theoretic terminology (see Harary, Norman and Cartwright [1965]). A *digraph* (from directed graph) is a pair $D = \langle V, A \rangle$, where V is the set of vertices and A is a set of ordered pairs $\langle u, v \rangle$, where $u \neq v \in V$; elements of A are called arcs. u is a *transmitter* in D if there is no v such that $\langle v, u \rangle \in A$; u is a *receiver* if there is no v such that $\langle u, v \rangle \in A$. A *path* is a sequence of arcs $\langle u_0, v_0 \rangle \cdots \langle u_n, v_n \rangle$ such that $v_i = u_i + 1$; this path is a *cycle* if in addition $u_0 = v_n$. A digraph is *acyclic* if it does not contain a cycle.

We shall now construct acyclic digraphs corresponding to indexed variables and prefixes. We need an auxiliary

Definition 24. If $t \in IVar$ is of the form $t = x_{y_1, \dots, y_n}$, we say that t *immediately depends* on $y_1, \dots, y_n$. ⊣

Given $t \in IVar$, we construct an acyclic digraph $D(t)$ as follows:

$$V = \{s \mid s \text{ occurs in the construction tree of } t\}$$
$$A = \{\langle u, v \rangle \mid u, v \in V, v \text{ immediately depends on } u\}.$$

$D(t)$ has the following properties: between any two vertices there is at most one arc, the transmitters are variables of the form $y \in Var$ or $x_\emptyset$, and there is a unique receiver (namely t).

If t occurs in a linear formula, $D(t)$ is transitive in the sense that the relation A is transitive (since for such t, the relation of immediate dependency is transitive).

Conversely, let $D = \langle V, A \rangle$ be an acyclic digraph with unique reciever, then we can find $t \in IVar$ such that D is isomorphic to $D(t)$:

- label the transmitters of D by ordinary variables or by variables with empty index
- if v is a vertex such that all u with $\langle u, v \rangle \in A$ have been labelled, label v by x indexed by the labels of immediate predecessors of v (where x is new).

If t is the label of the receiver, D is isomorphic to $D(t)$; the assumption that D is acyclic guarantees that each vertex will be labelled (cf. Harary et al. [1965], Theorem 10.1).

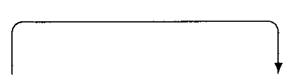
More generally, we may view prefixes as acyclic digraphs labelled by quantifiers. Take a prefix P and construct a digraph $D(P) = \langle V, A \rangle$ by

V is the set of variables which hereditarily occur in P
$A = \{\langle u, v \rangle \mid u, v \in V, v \text{ immediately depends on } u\}$.

If we label $D(P)$ in the obvious way by quantifiers, we obtain a graphical representation of P. In general, $D(P)$ will have several receivers; but if P is part of a linear formula, $D(P)$ is a transtive acyclic digraph with unique receiver. As above, there is also a converse: if D is an acyclic digraph, then D is isomorphic to $D(P)$ for some prefix P.

It follows that we can dispense with indexed variables in favour of a representation using acyclic digraphs. Let D be an acyclic digraph such that $V \subseteq Var$, and each vertex is labelled by a quantifier, then we can find an isomorphic D' on $IVar$ which together with the corresponding labelling defines a prefix. Conversely, any labelled acyclic digraph arising from a prefix is isomorphic to a labelled acyclic digraph on a subset of Var. Hence we may view the formalism of indexed variables as a linear representation of acyclic digraphs labelled by quantifiers.

That we really need digraphs and not just partial orders, is illustrated by the following example. In the case of generalized quantifiers, the prefixes $\forall y \exists x_y \exists x_{x_y}$ and $\forall y \exists x_y \exists x_{y,x_y}$ have different interpretations, namely as $\forall y \exists x \exists z (R(x, y) \wedge R(z, x) \wedge \cdots)$ and $\forall y \exists x \exists z (R(x, y) \wedge R(z, xy) \wedge \cdots)$ respectively. These correspond to the following digraphs:

 and 

However, the first digraph is not a partial order.

Since acyclic digraphs on ordinary variables and indexed variables are interchangeable, we could (perhaps more perspicuously) reformulate the proof systems for generalized quantifiers in terms of sequents labelled by sets of digraphs. To do this precisely would require an article in itself, so we leave to the reader the pleasure of playing with digraphs to verify the assertions made here[7].

For a start, a proof step in such a labelled calculus of sequents may consist in the transformation of a formula, a digraph, or both; in any case the transformation will depend on both input parameters. For instance, quantifiers can be introduced only on variables which are receivers in a digraph; after introducing the quantifiers, all arcs leading to the variable are deleted.

Substitution rules such as $SUBr$ or SUB_{ext} correspond to operations on digraphs. SUB_{ext} can be viewed as sanctioning the addition or deletion of certain arcs; and the action of SUB_r is best described as licensing taking a quotient of the digraph for the equivalence relation of intersubstitutibility. The reader may verify (using the correspondent computed in Example 8) that the substitution rule corresponding to the axiom $\Box_x\Box_y\varphi \rightarrow \Box_y\Box_x\varphi$ consists of reversing an arc in a digraph. Note that this correspondent is in Horn form.

However, SUB_{cc} (based on a correspondent not in Horn form) does *not* lead to an operation on digraphs. If we reformulate the proof of Keisler's axiom from SUB_{cc} (in Section 2.5) in terms of SUB'_{cc}, then the argument can be seen as involving a dichotomy: either $\exists x'_{\bar{z}}(y = x'_{\bar{z}})$ or $\forall x'_{\bar{z}}(y \neq x'_{\bar{z}})$. In terms of digraphs, this means that we make some assumptions on the sturcture of the digraph, *assumptions which are later discharged.* If we were to reformulate these systems in a natural deduction format, this would imply that there are also improper inference rules involving digraphs.

It is instructive to compare this with the proof theory of stationary logic developed by Barwise, Kaufmann and Makkai [1979,1981]. Stationary logic was designed to smooth out certain rough edges of Keisler's logic. Its main object of study is the quantifier $aa\ s$, interpreted as 'for almost all countable subsets s', where 'almost all' refers to the cub filter. Although s is a second-order variable, the techniques of the present paper can be applied to compute correspondents of the axioms determining aa; for example, the correspondent for completeness of the diagonal intersection axiom is $R(s, \bar{z}) \rightarrow (x \in s \rightarrow R(s, x\bar{z})$. The diagonal intersection axiom implies Keisler's axiom for 'co-countably many', but observe that the correspondent of the former now is a Horn formula.

The authors represent the dependence relation by a linear order on first and second-order variables, such that s depends only on variables which occur to its left; each sequent is labelled by such a linear order, and the non-trivial proof rules consist in manipulations with this linear order. We meet some old friends here: the (D) and (Add) rules of stationary logic, which allow one to delete or add variables under certain conditions, are the same as our SUB_{ext}. The rule corresponding to the diagonal intersection axiom can be read off from the correspondent given above; it is an operation on the linear order. It is also much 'nicer' in the sense that the derivation of the diagonal intersection axiom does not need CUT, unlike our derivation of Keisler's axiom. It seems likely that this difference is due to the fact

[7]The presentation in terms of digraphs might be useful to develop proof systems which approximate the (Σ^1_1) set of validities of Henkin quantifiers. A pleasant feature is that the non-linear representation of formulas allows us to introduce quantifiers in a non-deterministic manner. For the connection between *NP* and Henkin quantifiers, see Blass and Gurevich [1986].

that the correspondent of the diagonal intersection axiom is in Horn form, whence it translates into an operation on a linear order on the variables.

§9. **Conclusion.** The main thesis of this paper is that proof systems for generalized quantifiers can conveniently be constructed by giving attention to relationships of dependence between variables. Many properties of quantifiers determine dependencies, which sometimes can be made explicit. There are at least two ways to do this: one may endow variables with structure, or one may consider formulas to be pairs $\langle D, \varphi \rangle$, where D is a digraph on the variables occurring in φ. Correspondingly, proof rules for dependencies take one of two forms: substitution rules for indexed variables, or operations on (or more generally, arguments involving) digraphs. There appear to be formal connections between this type of substructural logic and logics obtained by deleting the propositonal structural rules; for instance, contraction together with suitable identification of formulas differing only in bound variables has the same effect as a substitution rule. However, the precise nature of this correspondence remains to be explored.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF AMSTERDAM
PLANTAGE MUIDERGRACHT 24
1018 TV AMSTERDAM, THE NETHERLANDS

E-mail: N.Alechina@cs.bham.ac.uk
michiell@fwi.uva.nl