

Working Paper

Equilibrium Programming Using Proximal-Like Algorithms

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Abstract

We consider problems where solutions – called equilibria – emerge as fixed points of an extremal mapping. Examples include convex programming, convex – concave saddle problems, many noncooperative games, and quasi – monotone variational inequalities. Using Bregman functions we develop proximal – like algorithms for finding equilibria. At each iteration we allow numerical errors or approximate solutions.

Key words: Proximal minimization, mathematical programming, Bregman functions.

EQUILIBRIUM PROGRAMMING USING PROXIMAL-LIKE ALGORITHMS

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1. INTRODUCTION Numerous problems in optimization and economics reduce to find a vector x^* satisfying the fixed point condition

$$x^* \in \operatorname{argmin}\{F(x^*, x) : x \in X\}. \quad (1.1)$$

Here X is a nonempty closed convex subset of some Euclidean space E , and the bivariate function $F: X \times X \rightarrow \mathbb{R}$ is convex in its second coordinate. E is endowed with the standard inner product $\langle \cdot, \cdot \rangle$, generating the customary norm $\|\cdot\|$.

Our purpose is to solve (1.1). Usually this is a well defined task since solutions - henceforth named *equilibria* - are indeed available under general conditions:

Proposition 1 (Existence of equilibrium). *Suppose X is nonempty compact convex, and $F(x, y)$ is jointly lower semicontinuous, separately continuous in x and convex in y . Then (1.1) admits at least one solution.*

Proof. The correspondence $X \ni x \rightarrow \operatorname{argmin}\{F(x, y) : y \in X\}$ has nonempty convex values and closed graph. Hence by Kakutani's theorem there exists a fixed point. ♥

For computational reasons we shall restrict attention to a certain class of equilibrium problems.

Definition Problem (1.1) is said to be of *saddle type* if for every equilibrium x^* and $x \in X$ we have

$$F(x, x^*) \leq F(x, x). \quad (1.2)$$

Problems fitting format (1.1) and satisfying (1.2) abound, as illustrated by important examples in Section 2. A prominent case included there, namely *monotone variational inequalities*, helps to put the subsequent development in perspective. Indeed, given a mapping $X \ni x \rightarrow m(x) \in E$, let $F(x, y) = \langle m(x), y - x \rangle$. Then x^* solves (1.1) $\Leftrightarrow \langle m(x^*), x - x^* \rangle \geq 0, \forall x \in X$. Moreover, (1.2) would follow from the monotonicity: $\langle m(x) - m(x^*), x - x^* \rangle \geq 0$. Granted this last property, it is well known that *proximal point algorithms* (Rockafellar 1976), (Güler 1991) give good convergence, but they are often hard to execute.

This motivates us to consider here new versions of proximal-like algorithms, especially adapted to the unifying framework (1.1). Section 3 states the said algorithms, all inspired by the iteration $x^{k+1} \in \operatorname{argmin}\{F(x^k, x) : x \in X\}$. In line with recent developments of Censor & Zenios (1992), Eckstein (1993), Chen & Teboulle (1993), Bertsekas & Tseng (1994) we shall accommodate Bregman functions and tolerate approximate evaluations. A main novelty is the procedure where regularization is done twice at every stage: first to predict the next iterate, thereafter to update the current point. Section 4 contains the convergence analysis.

2. EXAMPLES This section offers a list of problems all fitting format (1.1). We begin with

Convex minimization Let $F(x, y) = f(y)$ with $f: X \rightarrow \mathbb{R}$ convex. Then x^* solves (1.1) $\Leftrightarrow x^* \in \operatorname{argmin}\{f(x) : x \in X\}$. In this instance (1.2) is automatically satisfied. ♥

Convex-concave saddle problems Let $X = X_1 \times X_2$ be a product of two nonempty closed convex sets, $F(x, y) = L(y_1, x_2) - L(x_1, y_2)$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$, and L convex-concave. Then x^* solves (1.1) $\Leftrightarrow x^*$ is a saddle point of L . The saddle property (1.2) holds in this case as well. ♥

Noncooperative games with convex costs Generalizing the saddle problem, let individual $i \in I$, (I finite), incur convex cost $f_i(x_{-i}, x_i)$ in own decision $x_i \in X_i$, the latter set being nonempty closed convex. Here x_{-i} is short notation for actions taken by i 's rivals. Let $X := \prod X_i$ and $F(x, y) := \sum_i f_i(x_{-i}, y_i)$. Then x^* solves (1.1) $\Leftrightarrow x^*$ is

a Nash equilibrium. Property (1.2) is somewhat stringent in this case. In particular, it holds when $F(x,x) - F(x,y)$ is convex in x . For a discussion see Flåm & Ruszczynski (1994), Antipin & Flåm (1994). ♥

Variational inequalities Let $X \ni x \rightarrow G(x)$ be a correspondence with nonempty compact convex values. When $F(x,y) := \sup\{\langle g, y - x \rangle : g \in G(x)\}$, we get that x^* solves (1.1) $\Leftrightarrow \exists g^* \in G(x^*)$ such that $\langle g^*, x - x^* \rangle \geq 0, \forall x \in X$. Here condition (1.2) holds if G is *quasi-monotone at equilibrium* x^* in the sense that for all $x \in X$

$$\sup\{\langle g^*, x - x^* \rangle : g^* \in G(x^*)\} \geq 0 \quad \Rightarrow \quad \sup\{\langle g, x - x^* \rangle : g \in G(x)\} \geq 0. \heartsuit$$

Successive approximations Related to variational inequalities is the following optimization procedure. Let $f: X \rightarrow \mathbb{R}$ be convex and differentiable. Then, with $F(x,y) = f(x) + \langle f'(x), y - x \rangle$, we have that x^* solves (1.1) $\Leftrightarrow x^* \in \operatorname{argmin}\{f(x) : x \in X\}$. In this instance (1.2) is automatically satisfied.

Likewise, if $X \ni x \rightarrow G(x)$ is differentiable with $G'(x)$ positive semidefinite, and $F(x,y) = \langle G(x), y - x \rangle + \langle y - x, G'(x)(y - x) \rangle / 2$, then x^* solves (1.1) $\Leftrightarrow \langle G(x^*), x - x^* \rangle \geq 0, \forall x \in X$. ♥

3. ALGORITHMS This section proposes two procedures to solve (1.1). Both are amendments of

$$x^{k+1} \in \operatorname{argmin}\{F(x^k, x) : x \in X\}. \quad (3.1)$$

Our motivation stems from three deficiencies of (3.1). *Firstly*, it is unreasonable - at least in practice - to insist that argmin in (3.1) be computed exactly at every stage k . Rather one should tolerate some error $\epsilon_k \geq 0$. *Secondly*, the argmin operation - whether executed exactly or not - may cause instabilities. In particular, this happens often when $F(x,y)$ is affine in y . (See the above examples on variational inequalities). *Thirdly*, (3.1) reflects some myopia in minimizing at the current outcome x^k in lieu of at some predicted point, henceforth denoted x^{k+} .

These considerations lead us to replace (3.1) by more stable and flexible algorithms. For their statement we need to recall the notion of a *Bregman function*.

Definition Let S be an open convex subset of the ambient Euclidean space E . Then $\psi : \text{cl}S \rightarrow \mathbb{R}$ is baptized a **Bregman function** with zone S and "distance"

$$D(x,y) := \psi(x) - \psi(y) - \langle \psi'(y), x - y \rangle$$

if the following conditions hold:

- (i) ψ is continuously differentiable on S ;
- (ii) ψ is strictly convex continuous on $\text{cl}S$;
- (iii) for any number $r \in \mathbb{R}$ and points $x \in \text{cl}S$, $y \in S$ the two level sets
 $\{x \in \text{cl}S : D(x,y) \leq r\}$ and $\{y \in S : D(x,y) \leq r\}$

are both bounded;

- (iv) $S \ni y^k \rightarrow y \Rightarrow D(y, y^k) \rightarrow 0$;
- (v) if $\{x^k\}$ and $\{y^k\}$ are bounded sequences such that $y^k \rightarrow y \in \text{cl}S$ and $D(x^k, y^k) \rightarrow 0$, then $x^k \rightarrow y$.

Examples of such functions are given by Censor & Zenios (1992), Teboulle (1992), Eckstein (1993), Chen & Teboulle (1993). Generalizations are found in Kiwiel (1994a). (Of particular importance and convenience is the instance $\psi = \|\cdot\|^2/2$, yielding $D(x,y) = \|x-y\|^2/2$). Since X is bounded condition (iii) is not needed in the sequel. Now, with this notion in hand, employing a fixed Bregman function ψ we shall consider iterative procedures of the type

$$x^{k+1} \in \varepsilon_k\text{-argmin}\{\alpha_k F(x^{k+}, x) + D(x, x^k) : x \in X\}, \quad (3.2)$$

the initial point $x^0 \in X$ being arbitrary. An explanation of (3.2) is in order. The parameter $\varepsilon_k \geq 0$ there is an error tolerance. For asymptotic accuracy we invariably posit that

$$\sum_k \varepsilon_k^{1/2} < +\infty. \quad (3.3)$$

The other parameter $\alpha_k > 0$ in (3.2) is a matter of relative free choice. It should be bounded away from 0 and $+\infty$. More will be said about appropriate specifications later. The penalty term

$$D(x, x^k) = \psi(x) - \psi(x^k) - \langle \psi'(x^k), x - x^k \rangle$$

in (3.2), being the "distance" associated to a fixed *Bregman function* ψ with zone $S \supset X$, is intended to lend some inertia and stability to the adjustment process. Finally, the vector x^{k+} in (3.2) stands for a "prediction" or approximation of x^{k+1} to be defined in two alternative manners. One simply requires $x^{k+} = x^{k+1}$. The other makes for a special step to find x^{k+} , going as follows

$$x^{k+} \in \varepsilon_k\text{-argmin}\{\alpha_k F(x^k, x) + D(x, x^k) : x \in X\} \quad (3.4)$$

Algorithms of the sort (3.2-4), or akin to this procedure, have been studied recently by Antipin & Flåm (1994), Bertsekas & Tseng (1994), Kiwiel (1994b), Chen & Teboulle (1993), Eckstein (1993). However, none of these accomodate as much generality as done here. Typically these studies focus on convex minimization, or make the choice $\varepsilon_k = 0$, or employ $\psi = \|\cdot\|^2/2$. Our purpose is to lift these restrictions.

4. CONVERGENCE Throughout the rest we assume that the hypotheses of Proposition 1 and condition (1.2) are all in vigour. Also, we posit that the Bregman function ψ has a zone S containing X , with Lipschitz continuous gradient. Specifically, there exists some constant $L > 0$, such that for any error tolerance ε used in the sequel it holds

$$\|\psi'(x) - \psi'(y)\| \leq L\|x - y\|. \quad (4.1)$$

whenever $x \in X$ and $\text{dist}(y, X) \leq \varepsilon^{1/2}$. Three auxiliary results are needed.

Lemma 1 Suppose a function f is finite-valued convex near some nonempty closed convex subset X of the ambient Euclidean space. For fixed $\xi \in X$, and error tolerance $\varepsilon \geq 0$ let

$$x^+ \in \varepsilon\text{-argmin}\{f(x) + D(x, \xi) : x \in X\}.$$

Then, for some $\delta \in [0, \varepsilon]$ and all $x \in X$,

$$f(x) + D(x, \xi) \geq f(x^+) + D(x^+, \xi) + D(x, x^+) - \delta - (L+1)(\varepsilon - \delta)^{1/2} \|x - x^+\|.$$

Proof. The ε -optimality of x^+ implies that

$$0 \in \varepsilon\text{-}\partial[f + D(\cdot, \xi) + I_X](x^+)$$

where $\varepsilon\text{-}\partial$ denotes the ε -subdifferential operator, and I_X is the convex indicator of X (i.e., I_X equals 0 on X , and $+\infty$ elsewhere). By Hiriart-Urruty & Lemarechal (1991, Thm. XI 3.1.1) there exist "subgradients"

$$s_1 \in \varepsilon_1\text{-}\partial f(x^+), \quad s_2 \in \varepsilon_2\text{-}\partial D(\cdot, \xi)(x^+), \quad s_3 \in \varepsilon_3\text{-}\partial I_X(x^+)$$

with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ such that

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon \quad \text{and} \quad 0 = s_1 + s_2 + s_3. \quad (4.2)$$

Now, $s_1 \in \varepsilon_1 \partial f(x^+)$ implies

$$f(x) \geq f(x^+) + \langle s_1, x - x^+ \rangle - \varepsilon_1 \quad \text{for all } x \in X.$$

Adding the *three-point identity* (see Chen & Teboulle 1993)

$$D(x, \xi) = D(x^+, \xi) + D(x, x^+) + \langle \psi'(x^+) - \psi'(\xi), x - x^+ \rangle$$

to the above subgradient inequality, we obtain

$$f(x) + D(x, \xi) \geq f(x^+) + D(x^+, \xi) + D(x, x^+) + \langle s_1 + \psi'(x^+) - \psi'(\xi), x - x^+ \rangle - \varepsilon_1 \quad (4.3)$$

In turn, $s_2 \in \varepsilon_2 \partial D(\cdot, \xi)(x^+)$ implies $s_2 = S_2 - \psi'(\xi)$ for some $S_2 \in \varepsilon_2 \partial \psi'(x^+)$. By the Brönsted-Rockafellar theorem (see Hiriart-Urruty & Lemarechal 1993, Thm. XI, 4.2.1) there exists $y \in B(x^+, \varepsilon_2^{1/2})$ such that $\|\psi'(y) - S_2\| \leq \varepsilon_2^{1/2}$. Drawing upon these facts and (4.2) we have

$$\begin{aligned} s_1 + \psi'(x^+) - \psi'(\xi) &= s_1 + s_2 + s_3 + \psi'(x^+) - S_2 - s_3 \\ &= \psi'(x^+) - S_2 - s_3 = \psi'(x^+) - \psi'(y) + \psi'(y) - S_2 - s_3 \end{aligned}$$

so, using $\langle s_3, x - x^+ \rangle \leq \varepsilon_3$, it follows that $\langle s_1 + \psi'(x^+) - \psi'(\xi), x - x^+ \rangle$

$$\begin{aligned} &= \langle \psi'(x^+) - \psi'(y) + \psi'(y) - S_2 - s_3, x - x^+ \rangle \\ &\geq -(\|\psi'(x^+) - \psi'(y)\| + \|\psi'(y) - S_2\|)\|x - x^+\| - \langle s_3, x - x^+ \rangle \\ &\geq -(L\varepsilon_2^{1/2} + \varepsilon_2^{1/2})\|x - x^+\| - \varepsilon_3. \end{aligned}$$

Using this last inequality in (4.3) the desired conclusion follows immediately with $\delta = \varepsilon_1 + \varepsilon_3$ and $\varepsilon_2 = \varepsilon - \delta$. ♥

Lemma 2 Suppose a function f is finite-valued convex near some nonempty closed convex subset X of the ambient Euclidean space. Then

$$x^* \in \operatorname{argmin}\{f(x) + D(x, x^*) : x \in X\} \Leftrightarrow x^* \in \operatorname{argmin}\{f(x) : x \in X\}.$$

Proof. $\Rightarrow$ By Lemma 1, $f(x) + D(x, x^*) \geq f(x^*) + D(x^*, x^*) + D(x, x^*)$ for all $x \in X$, whence $f(x) \geq f(x^*)$ for all $x \in X$. Conversely, when $f(x) \geq f(x^*)$ for all $x \in X$, it holds that $f(x) + D(x, x^*) \geq f(x^*) + D(x^*, x^*)$ for all $x \in X$. ♥

Lemma 3 Suppose $\{a_k\}, \{b_k\}, \{c_k\}$ are sequences of nonnegative numbers such that $\sum_k b_k < +\infty$, and

$$a_{k+1} \leq a_k + b_k - c_k.$$

Then $\{a_k\}$ converges, and $\sum_k c_k < +\infty$.

Proof. From $a_K + \sum_{k < K} c_k \leq a_0 + \sum_{k < K} b_k$ it follows that $\{a_k\}$ is bounded and $\sum_k c_k < +\infty$. Let a be any cluster point of $\{a_k\}$. The inequality $a_k \leq a_K + \sum_{k \geq K} b_k$ valid for all $k > K$, implies that $\{a_k\}$ has no cluster point $> a$, whence $\{a_k\}$ converges. ♥

Theorem 1 (Convergence under "correct" predictions). For arbitrary initial $x^0 \in X$, the process (3.2) with $x^{k+} = x^{k+1}$ converges to equilibrium.

Proof. For any equilibrium x^* Lemma 1 yields $\alpha_k F(x^{k+1}, x^*) + D(x^*, x^k) \geq$

$$\alpha_k F(x^{k+1}, x^{k+1}) + D(x^*, x^{k+1}) + D(x^{k+1}, x^k) - \delta_k - (L+1)(\varepsilon_k - \delta_k)^{1/2} \|x^* - x^{k+1}\|$$

for some $\delta_k \in [0, \varepsilon_k]$. Invoking now the saddle property $F(x^{k+1}, x^*) \leq F(x^{k+1}, x^{k+1})$ we have

$$D(x^*, x^k) \geq D(x^*, x^{k+1}) + D(x^{k+1}, x^k) - \delta_k - (L+1)\varepsilon_k^{1/2} \|x^* - x^{k+1}\|$$

Using here (3.3), the boundedness of X , and Lemma 3 it obtains from the last inequality that $D(x^*, x^k)$ converges, and $\sum_k D(x^{k+1}, x^k) < +\infty$. In particular, $D(x^{k+1}, x^k) \rightarrow 0$. Let x^* be an accumulation point of $\{x^k\}$. Then, for some subsequence K , $\lim_{k \in K} x^k = \lim_{k \in K} x^{k+1} = x^*$, and $\lim_{k \in K} \alpha_k = \alpha > 0$. Passing to the limit along this subsequence in (3.2) we obtain

$$x^* \in \operatorname{argmin}\{\alpha F(x^*, x) + D(x, x^*) : x \in X\}$$

which by Lemma 2 is equivalent to (1.1). Thus $\{x^k\}$ clusters to an equilibrium x^* , and $\{D(x^*, x^k)\}$ converges to zero. It follows that the entire sequence $\{x^k\}$ converges to x^* . ♥

When $F(x,y)$ is subdifferentiable in y near X , $M(x) := \partial_y F(x,x) + \partial I_X(x)$, $\varepsilon_k = 0$, and $\psi = \|\cdot\|^2/2$, the procedure of Thm.1 is tantamount to the exact proximal point algorithm of Rockafellar (1976). To wit, the iteration in Thm. 1 then comes in the form $x^{k+1} = (I + \alpha_k M)^{-1}(x^k)$, recently generalized by Eckstein (1993). The requirement $x^{k+} = x^{k+1}$ in Thm.1, may make however, for laborious iterations (3.2). Essentially, the difficulty stems from the fact that (1.1) has two related features, namely: prediction in the first variable and optimization in the second. (3.4) serves to separate these two aspect from each other. For success in these matters we need some smoothness of F , and the parameters α_k must not be too large. Specifically, we assume there exists a constant $\Lambda > 0$ such that on X we have

$$\|F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y) - F(x+\Delta x, y) + F(x, y)\| \leq 2\Lambda \{D(x, x+\Delta x)D(y+\Delta y, y)\}^{1/2} \quad (4.4)$$

This seemingly strange condition simplifies, when $\psi = \|\cdot\|^2/2$, to

$$\|F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y) - F(x+\Delta x, y) + F(x, y)\| \leq \Lambda \|\Delta x\| \|\Delta y\|,$$

which holds when X is compact and F is continuously differentiable.

Theorem 2 (Convergence under regularized predictions). *Suppose $\{\alpha_k \Lambda\}$ is contained in a closed subinterval of $]0,1[$ with Λ satisfying (4.4). Then for arbitrary initial $x^0 \in X$, the process (3.2- 4) converges to equilibrium.*

Proof. Applying Lemma 1 to situation (3.4) we get $\alpha_k F(x^k, x^{k+1}) + D(x^{k+1}, x^k) \geq$

$$\alpha_k F(x^k, x^{k+}) + D(x^{k+}, x^k) + D(x^{k+1}, x^{k+}) - \delta_k - (L+1)(\varepsilon_k - \delta_k)^{1/2} \|x^{k+1} - x^{k+}\|.$$

The same Lemma 1 applied to (3.2) yields, when x^* is any equilibrium,

$$\alpha_k F(x^{k+}, x^*) + D(x^*, x^k) \geq$$

$$\alpha_k F(x^{k+}, x^{k+1}) + D(x^{k+1}, x^k) + D(x^*, x^{k+1}) - \delta_k - (L+1)(\varepsilon_k - \delta_k)^{1/2} \|x^* - x^{k+1}\|.$$

Adding these two inequalities we have

$$\alpha_k [F(x^k, x^{k+1}) - F(x^{k+}, x^{k+1}) - F(x^k, x^{k+}) + F(x^{k+}, x^*)] \geq$$

$$D(x^{k+}, x^k) + D(x^{k+1}, x^{k+}) - \delta_k - (L+1)(\varepsilon_k - \delta_k)^{1/2} \|x^{k+1} - x^{k+}\| +$$

$$D(x^*, x^{k+1}) - \delta_k - (L+1)(\epsilon_k - \delta_k)^{1/2} \|x^* - x^{k+1}\| - D(x^*, x^k)$$

Now invoke the saddle property $F(x^{k+1}, x^*) \leq F(x^{k+1}, x^{k+1})$ and the Lipschitz condition (4.4) to have $2\alpha_k \Lambda \{D(x^{k+1}, x^k) D(x^{k+1}, x^{k+1})\}^{1/2} \geq$

$$\alpha_k [F(x^k, x^{k+1}) - F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1}) + F(x^{k+1}, x^{k+1})] \geq$$

$$\alpha_k [F(x^k, x^{k+1}) - F(x^{k+1}, x^{k+1}) - F(x^k, x^{k+1}) + F(x^{k+1}, x^*)].$$

Combining the two last strings of inequalities we get

$$D(x^*, x^k) \geq D(x^*, x^{k+1}) + \{D(x^{k+1}, x^k)^{1/2} - \alpha_k \Lambda D(x^{k+1}, x^{k+1})^{1/2}\}^2 +$$

$$[1 - (\alpha_k \Lambda)^2] (D(x^{k+1}, x^{k+1}) - 2\delta_k - (L+1)(\epsilon_k - \delta_k)^{1/2} \{\|x^{k+1} - x^{k+1}\| + \|x^* - x^{k+1}\|\}).$$

This yields - by (3.3), the boundedness of X , and Lemma 3 - that $D(x^*, x^k)$ converges and

$$\sum_k \{D(x^{k+1}, x^k)^{1/2} - \alpha_k \Lambda D(x^{k+1}, x^{k+1})^{1/2}\}^2 + [1 - (\alpha_k \Lambda)^2] (D(x^{k+1}, x^{k+1}) - 2\delta_k - (L+1)(\epsilon_k - \delta_k)^{1/2} \{\|x^{k+1} - x^{k+1}\| + \|x^* - x^{k+1}\|\}) < +\infty.$$

It follows that $D(x^{k+1}, x^k) \rightarrow 0$ and $D(x^{k+1}, x^{k+1}) \rightarrow 0$. Let x^* be an accumulation point of $\{x^k\}$. Then, for some subsequence K , $\lim_{k \in K} x^k = \lim_{k \in K} x^{k+1} = \lim_{k \in K} x^{k+1} = x^*$, and $\lim_{k \in K} \alpha_k = \alpha > 0$. Passing to the limit along this subsequence in (3.2) we obtain

$$x^* \in \operatorname{argmin}\{\alpha F(x^*, x) + D(x, x^*) : x \in X\}$$

which by Lemma 2 is equivalent to (1.1). Thus $\{x^k\}$ clusters to an equilibrium x^* , and $\{D(x^*, x^k)\}$ converges to zero. It follows that the entire sequence $\{x^k\}$ converges to x^* . ♥

Clearly, in (3.4) one might use a sequence $\{\epsilon_{k+1}\}$ of errors different from $\{\epsilon_k\}$ but also satisfying (3.3).

When f is convex differentiable on X , $\epsilon_k = 0$, and $F(x, y) = \langle f'(x), y - x \rangle$, the steps of Thm. 2 assume the form: $\langle f'(x^{k+1}), x - x^{k+1} \rangle \geq 0$ for all $x \in X$, reminiscent of the extragradient method of Korpelevich (1976).

It appears interesting to incorporate variational convergence of functions $F^k \rightarrow F$, and sets $X^k \rightarrow X$, as done by Alart & Lemaire (1991). However, this falls outside the scope of this paper.

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