Empirical Wavelet Transform

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Outline

- Introduction - EMD
- 1D Empirical Wavelets
  - Definition
  - Experiments
- 2D Extensions
  - Tensor product case
  - Ridgelet case
  - Experiments

Time-frequency signal analysis

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- Hilbert-Huang transform (EMD + Hilbert transform)

Empirical Mode Decomposition (EMD): Principle

Goal: decompose a signal $f(t)$ into a finite sum of Intrinsic Mode Functions (IMF) $f_k(t)$:

$$f(t) = \sum_{k=0}^{N} f_k(t)$$

where an IMF is an AM-FM signal:

$$f_k(t) = F_k(t)\cos\left(\varphi_k(t)\right) \qquad \text{where } F_k(t), \varphi_k'(t) > 0 \ \forall t.$$

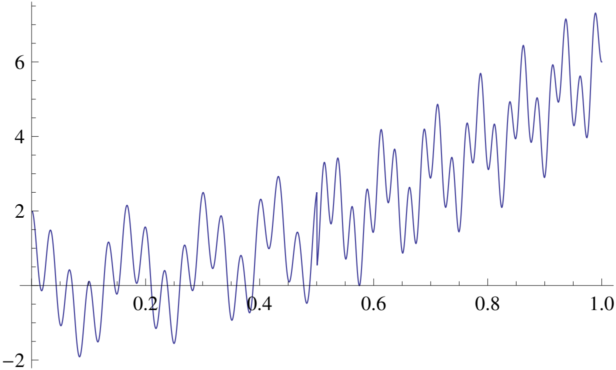
Main assumption: F_k and φ_k' vary much slower than φ_k.

Huang et al.[1] propose a pure algorithmic method to extract the different IMF.

[1] The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis, Proc. Royal Society London A., vol.454, pp.903–995,1998

Empirical Mode Decomposition (EMD): Algorithm

Initialization: $f^0 = f$
while all IMF are no extracted **do**
$r_0^k = f^k$
while r_n^k is not an IMF (Sifting process) **do**
Upper envelope $\bar{u}(t)$ (maxima + spline) of $r_n^k(t)$
Lower envelope $\underline{l}(t)$ (minima + spline) of $r_n^k(t)$
Mean envelope $m(t) = (\bar{u}(t) + \underline{l}(t))/2$
IMF candidate $r_{n+1}^k(t) = r_n^k(t) - m(t)$
end while
$f^{k+1} = f^k - r_{n+1}^k$
end while



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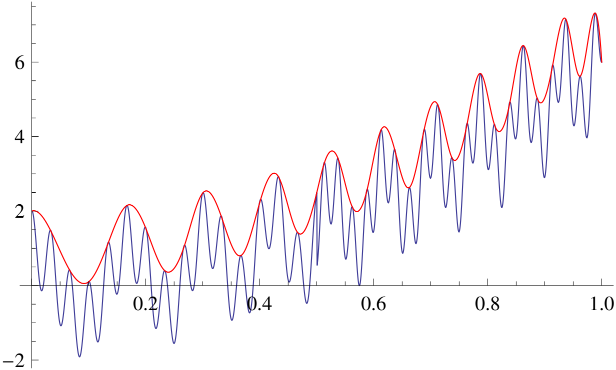

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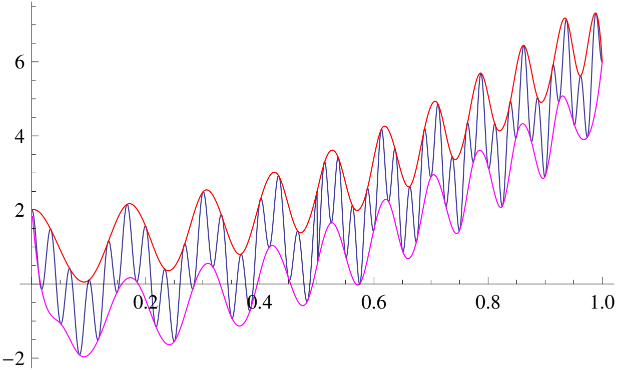

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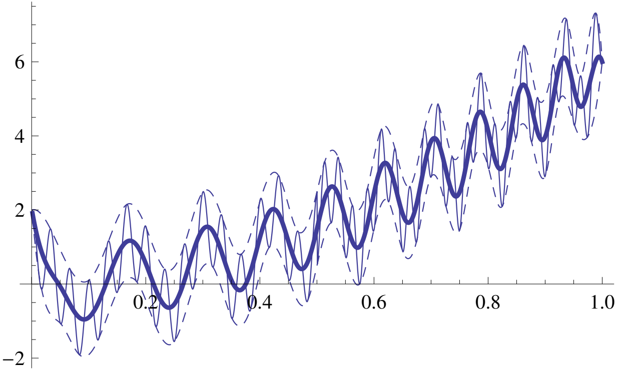
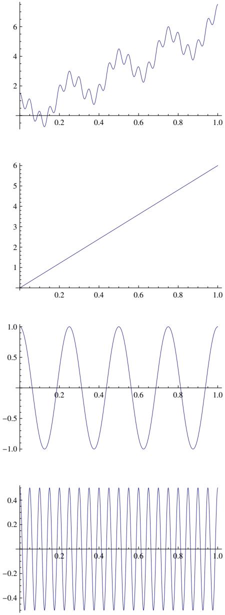
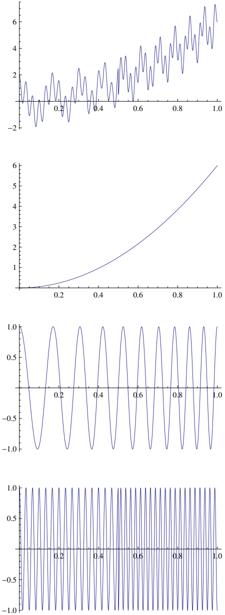


Example of EMD: input signals

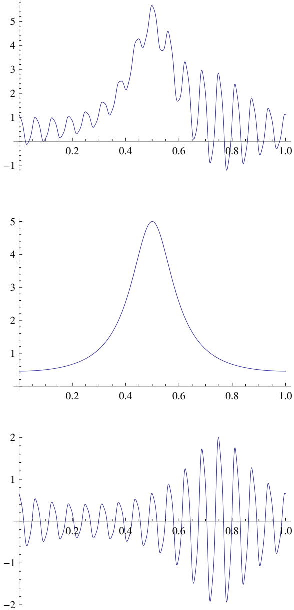
$$f_{Sig1}(t) = 6t + \cos(8\pi t) + 0.5\cos(40\pi t)$$



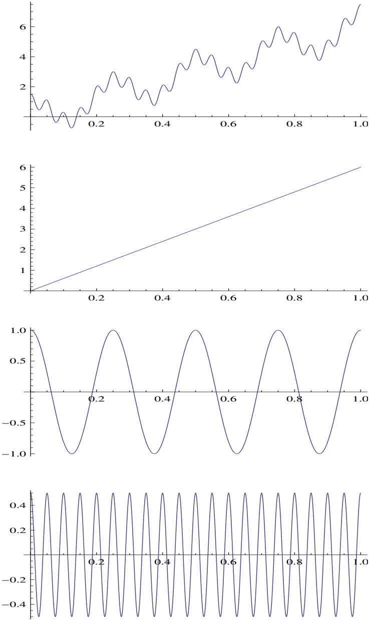
$$f_{Sig2}(t) = 6t^2 + \cos(10\pi t + 10\pi t^2) + \chi(t > 0.5)\cos(80\pi t - 15\pi) + \chi(t \leq 0.5)\cos(60\pi t)$$

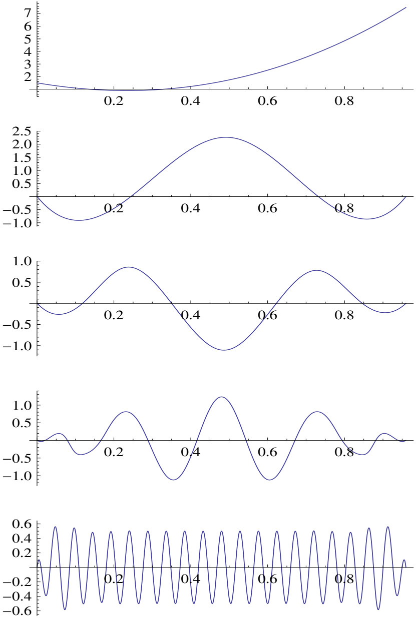


$$f_{Sig3}(t) = \frac{1}{1.2+\cos(2\pi t)} + \frac{1}{1.5+\sin(2\pi t)} + \cos(32\pi t + \cos(64\pi t))$$



Example of EMD: f_{Sig1}





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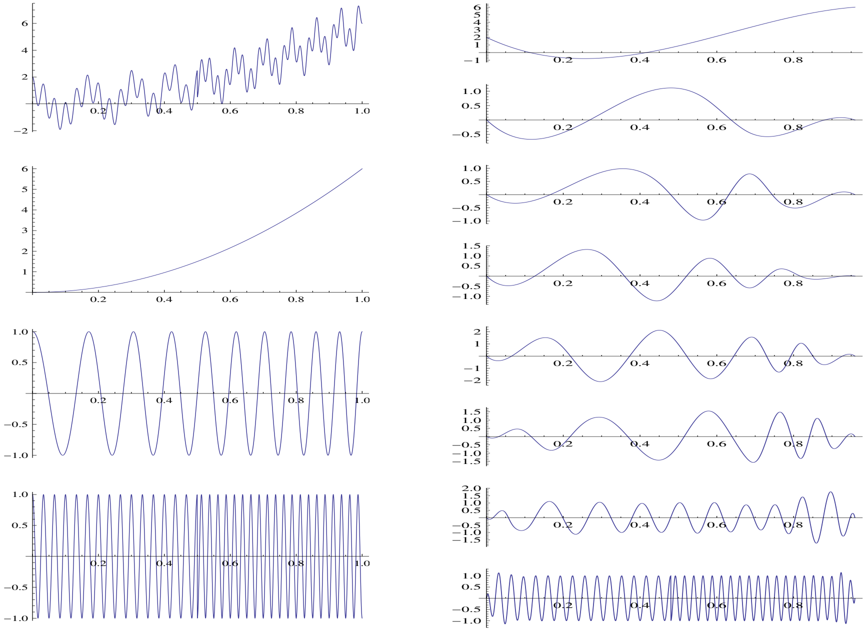

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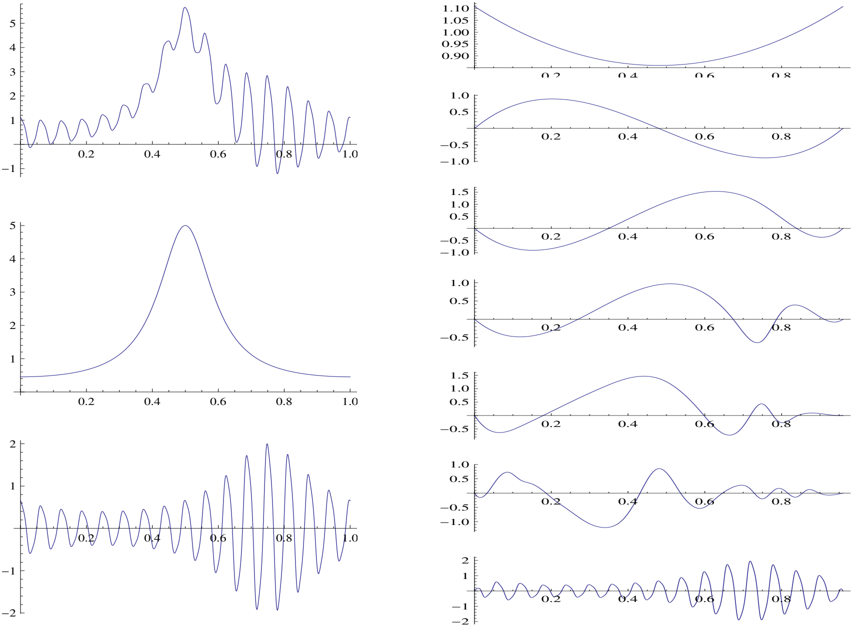
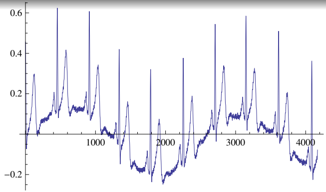
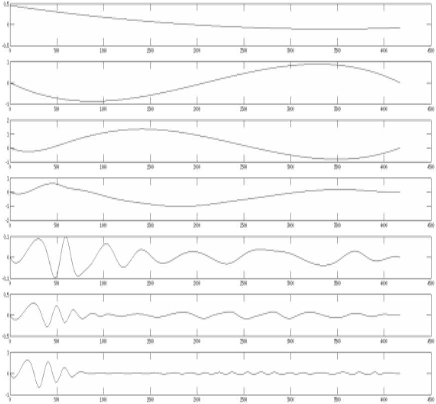
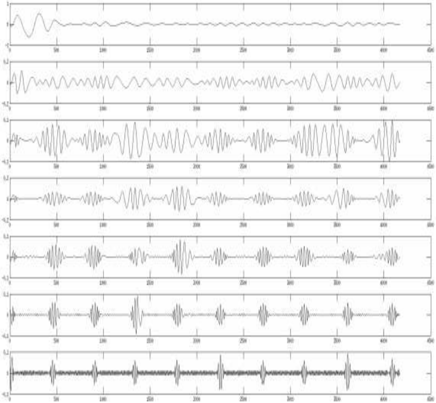


Example of EMD: f_{Sig4} - ECG







Hilbert-Huang Transform

Hilbert transform

$$\mathcal{H}_f(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{f(\tau)}{t-\tau} d\tau$$

Property: if $f_k(t) = F_k(t) \cos(\varphi_k(t))$ then

$$f_k^*(t) = f_k(t) + \imath \mathcal{H}_{f_k}(t) = F_k(t) e^{\imath \varphi_k(t)}$$

$\Rightarrow$ we can extract $F_k(t)$ and the instantaneous frequency $\frac{d\varphi_k}{dt}(t)$.

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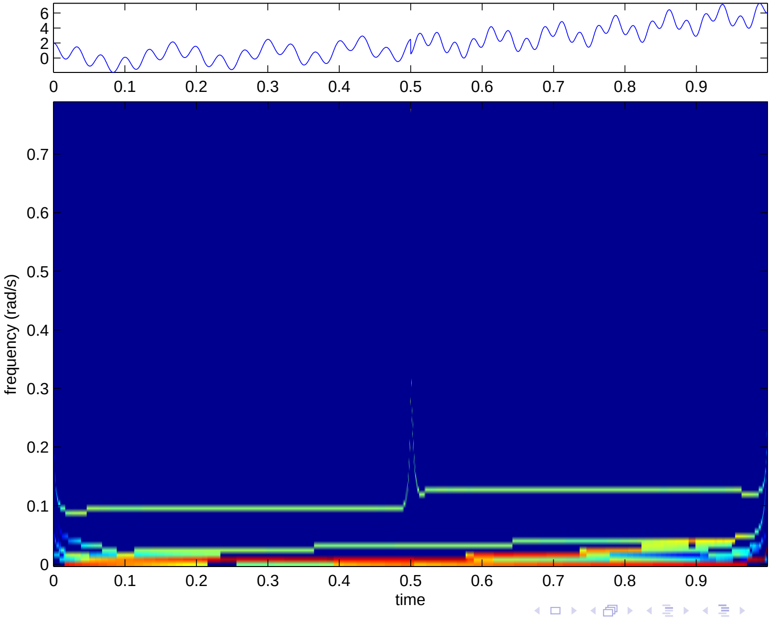
HHT

For each IMF k, we extract F_k and $\frac{d\varphi_k}{dt}(t)$ and accumulate the information in the time-frequency plane.

HHT of f_{sig2}


6
4
2
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
frequency (rad/s)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
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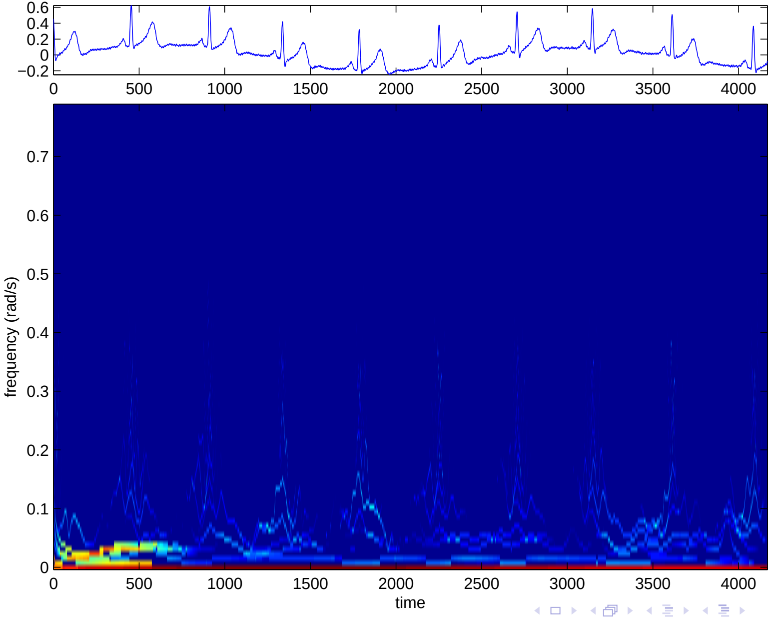
time

HHT of f_{sig4} - ECG



EMD: Issues and Properties

- Useful to analyze real signals.
- Implementation dependent.
- Problem: it's a nonlinear algorithm which has no mathematical theory $\Rightarrow$ difficult to predict and understand its output and behavior in the general case.
- Experimental property: seems to behave as an adaptive filter bank (Flandrin et al.[2])

[2] Empirical mode decomposition as a filter bank, IEEE Signal Processing Letters, vol.11, No.2, pp.112–114, 2004

Key ideas about wavelets

Wavelets ⇔ filtering

$$\begin{aligned}\mathcal{WT}_f(m,n) &= a_0^{-m/2}\int f(t)\psi(a_0^{-m}t - nb_0)dt \\ &= a_0^{-m/2}\int f(t)\psi\left(\frac{t - na_0^m b_0}{a_0^m}\right)dt \\ &= (f \star \psi_m)(na_0^m b_0)\end{aligned}$$

where $\psi_m(s) = \psi\left(\frac{-s}{a_0^{-m}}\right)$.

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$\Rightarrow$ Wavelets can be built both in the temporal or Fourier domains.

Empirical wavelet transform (EWT): Concept

Idea:

Combining the strength of wavelet's formalism with the adaptability of EMD.

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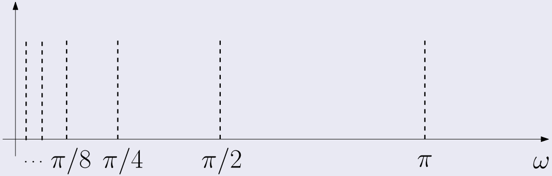
Wavelets are equivalent to filter banks $\rightarrow$ (dyadic) decomposition of the Fourier line



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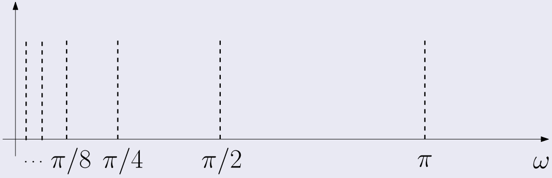

Does not necessarily correspond to "modes" positions.

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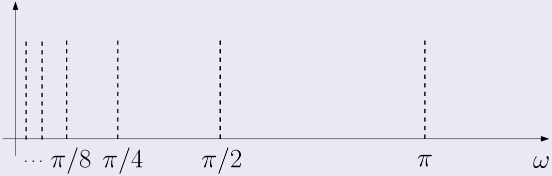

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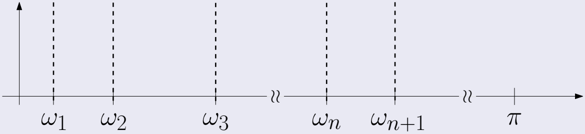
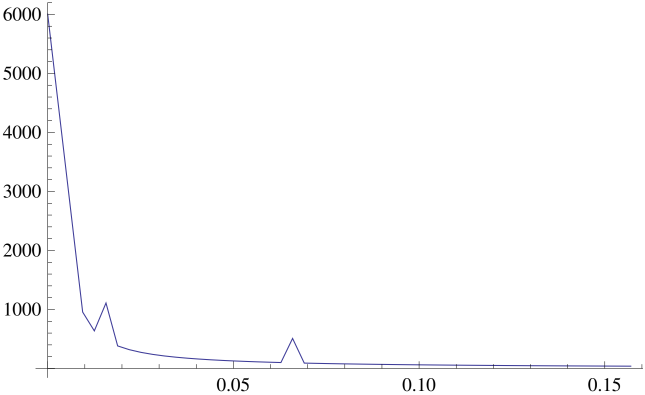
EWT $\rightarrow$ adaptive decomposition of the Fourier line



EWT: finding the modes

Fourier spectrum segmentation:

- Find the local maxima.
- Take support boundaries as the middle between successive maxima.



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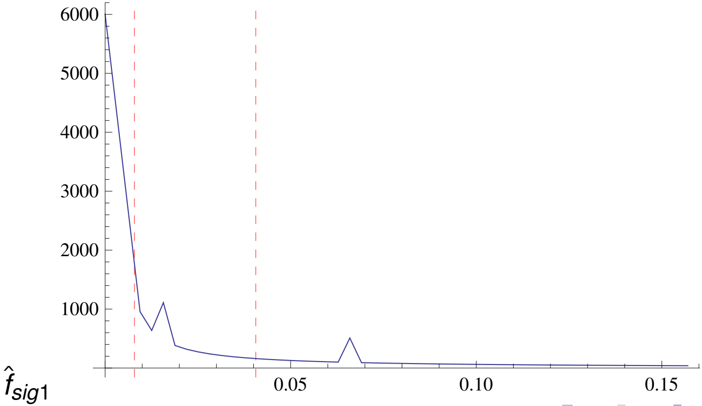

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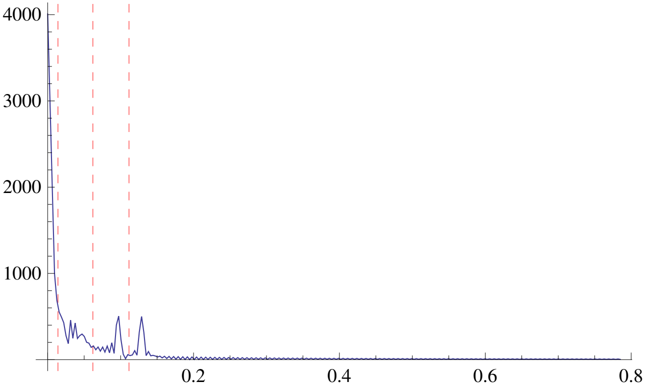
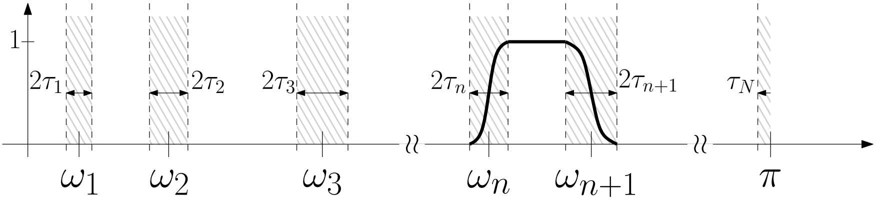


$\hat{f}_{sig2}$

EWT: filter bank construction (1/3)

Notations

- ω_n: support boundaries
- τ_n: half the length of the "transition phase"



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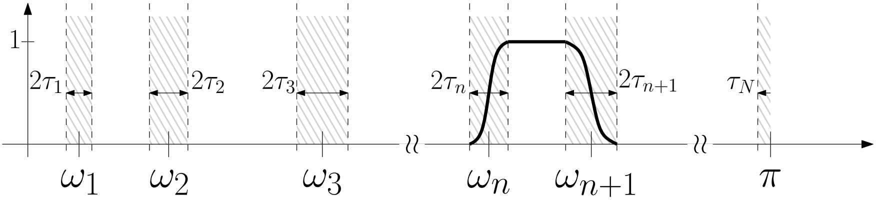


In practice we choose $\tau_n = \gamma\omega_n$

EWT: filter bank construction (2/3)

Scaling function spectrum

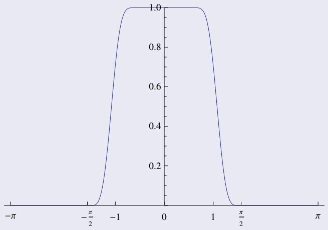
$$\hat{\phi}_n(\omega) = \begin{cases} 1 & \text{if } |\omega| \leq (1-\gamma)\omega_n \\ \cos\left[\frac{\pi}{2}\beta\left(\frac{1}{2\gamma\omega_n}(|\omega| - (1-\gamma)\omega_n)\right)\right] & \text{if } (1-\gamma)\omega_n \leq |\omega| \leq (1+\gamma)\omega_n \\ 0 & \text{otherwise} \end{cases}$$

Wavelet spectrum

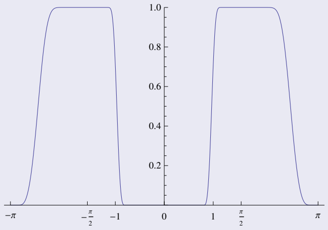
$$\hat{\psi}_n(\omega) = \begin{cases} 1 & \text{if } (1+\gamma)\omega_n \leq |\omega| \leq (1-\gamma)\omega_{n+1} \\ e^{-\imath\frac{\omega}{2}}\cos\left[\frac{\pi}{2}\beta\left(\frac{1}{2\gamma\omega_{n+1}}(|\omega| - (1-\gamma)\omega_{n+1})\right)\right] & \\ & \text{if } (1-\gamma)\omega_{n+1} \leq |\omega| \leq (1+\gamma)\omega_{n+1} \\ e^{-\imath\frac{\omega}{2}}\sin\left[\frac{\pi}{2}\beta\left(\frac{1}{2\gamma\omega_n}(|\omega| - (1-\gamma)\omega_n)\right)\right] & \\ & \text{if } (1-\gamma)\omega_n \leq |\omega| \leq (1+\gamma)\omega_n \\ 0 & \text{otherwise} \end{cases}$$

EWT: filter bank construction (3/3)

Scaling function spectrum for $\omega_n = 1$ and $\gamma = 0.5$



Wavelet spectrum for $\omega_n = 1$, $\omega_{n+1} = 2.5$ and $\gamma = 0.2$



EWT: property and example (1/2)

Proposition

If $\gamma < \min_n \left(\frac{\omega_{n+1}-\omega_n}{\omega_{n+1}+\omega_n} \right)$, then the set $\{\phi_1(t), \{\psi_n(t)\}_{n=1}^N\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

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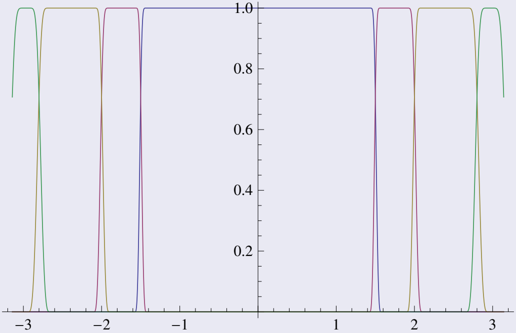
Filter Bank for $\omega_n \in \{0, 1.5, 2, 2.8, \pi\}$ with $\gamma = 0.05 < 0.057$



EWT: property and example (2/2)

Detail coefficients:

$$\mathcal{W}_f^{\mathcal{E}}(n,t) = \langle f, \psi_n \rangle = \int f(\tau)\overline{\psi_n(\tau - t)}d\tau$$
$$= \left(\hat{f}(\omega)\overline{\hat{\psi}_n(\omega)}\right)^{\vee},$$

Approximation coefficients (convention $\mathcal{W}_f^{\mathcal{E}}(0,t)$):

$$\mathcal{W}_f^{\mathcal{E}}(0,t) = \langle f, \phi_1 \rangle = \int f(\tau)\overline{\phi_1(\tau - t)}d\tau$$
$$= \left(\hat{f}(\omega)\overline{\hat{\phi}_1(\omega)}\right)^{\vee},$$

The reconstruction:

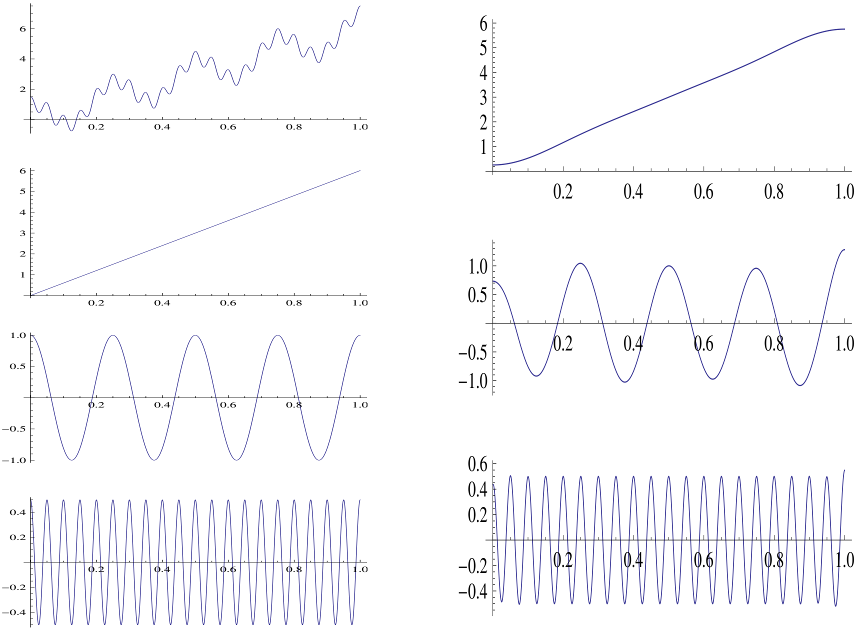
$$f(t) = \mathcal{W}_f^{\mathcal{E}}(0,t) \star \phi_1(t) + \sum_{n=1}^{N} \mathcal{W}_f^{\mathcal{E}}(n,t) \star \psi_n(t)$$
$$= \left(\widehat{\mathcal{W}_f^{\mathcal{E}}}(0,\omega)\hat{\phi}_1(\omega) + \sum_{n=1}^{N} \widehat{\mathcal{W}_f^{\mathcal{E}}}(n,\omega)\hat{\psi}_n(\omega)\right)^{\vee}.$$

EWT: algorithm

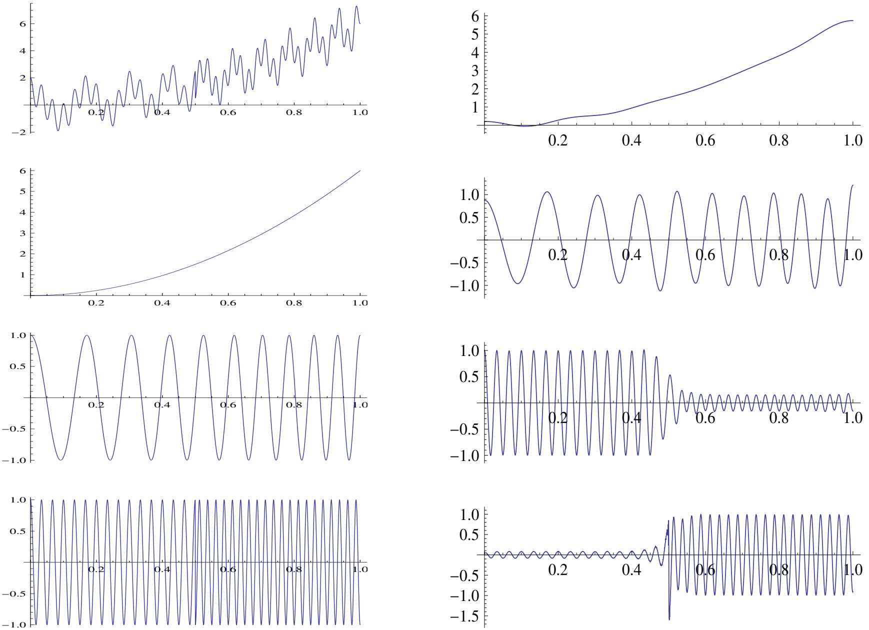
Input: f, N (number of scales)

1. Fourier transform of $f \to \hat{f}$.
2. Compute the local maxima of $\hat{f}$ on $[0, \pi]$ and find the set $\{\omega_n\}$.
3. Choose $\gamma < \min_n \left(\frac{\omega_{n+1} - \omega_n}{\omega_{n+1} + \omega_n} \right)$.
4. Build the filter bank.
5. Filter the signal to get each component.

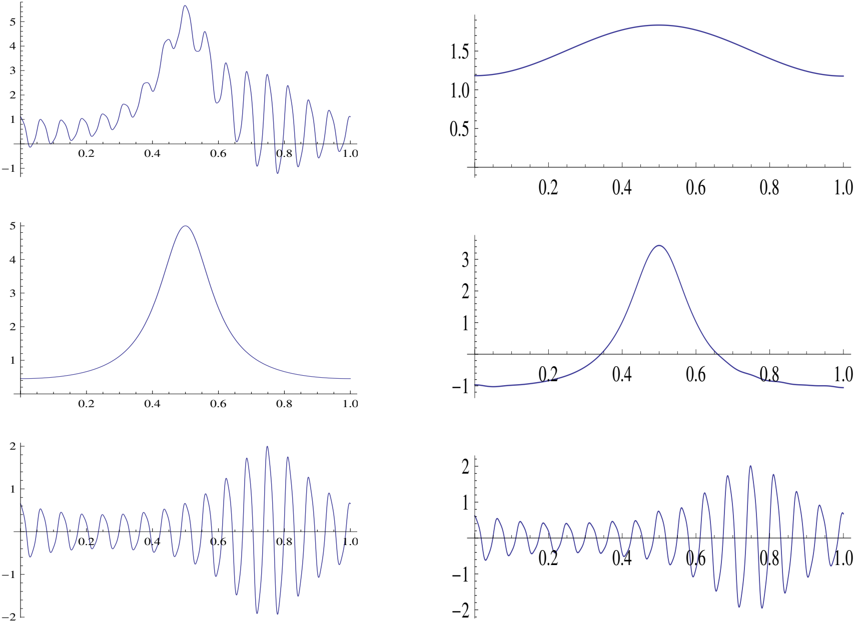
Experiment: f_{Sig1}

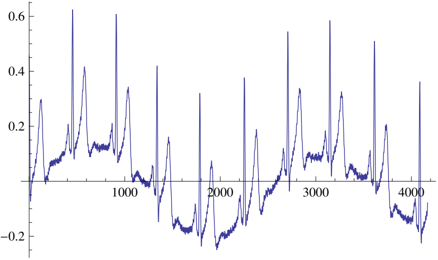


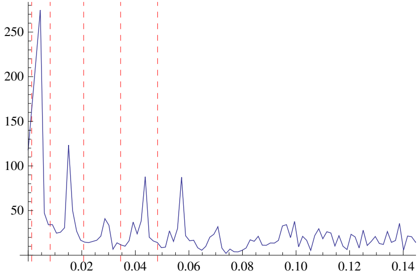
Experiment of EMD: f_{Sig2}

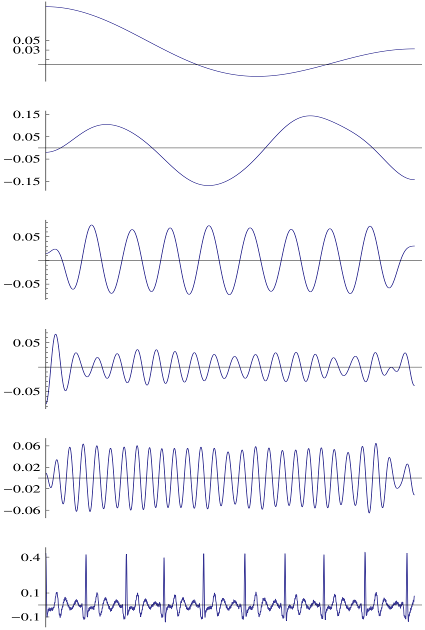


Experiment of EMD: f_{Sig3}

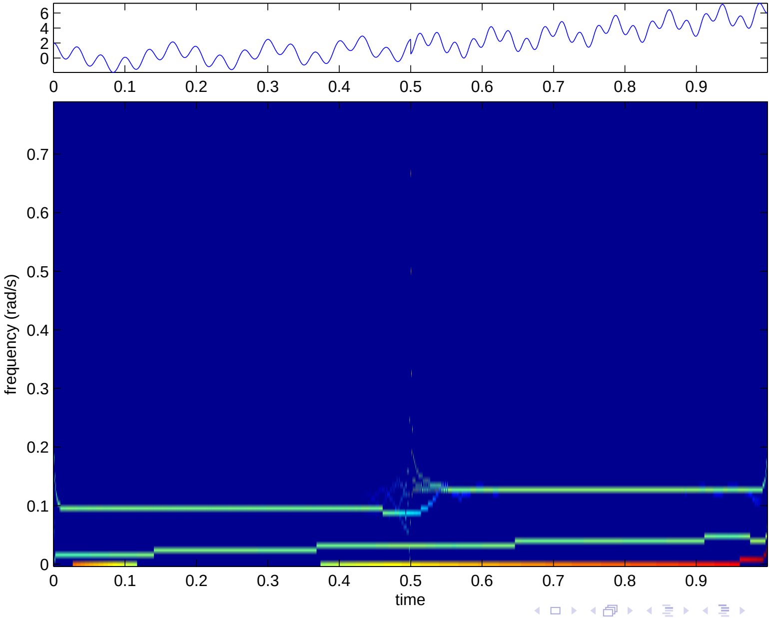



0.6
0.4
0.2
−0.2
1000
2000
3000
4000

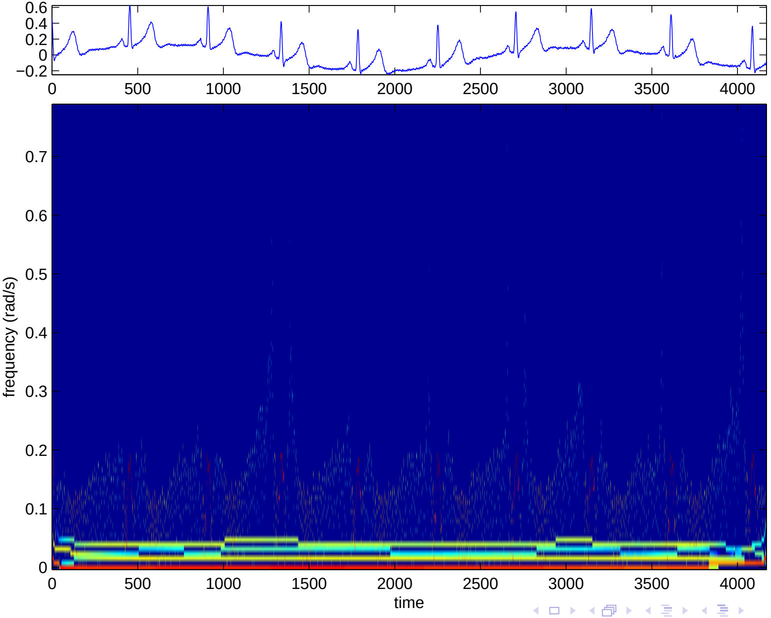

250
200
150
100
50
0.02
0.04
0.06
0.08
0.10
0.12
0.14


0.05
0.03
0.15
0.05
−0.05
−0.15
0.05
−0.05
0.05
−0.05
0.06
0.02
−0.02
−0.06
0.4
0.1
−0.1

Time-Frequency representation of f_{sig2}



Time-Frequency representation of f_{sig4}



2D - Extension

joint work with Giang Tran and Stan Osher

2D Extension - “Tensor product” approach

Like the “classic” wavelet transform $\rightarrow$ process rows then columns

but. . .

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but. . .

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$\Rightarrow$ Idea: “Mean Filter Banks”

2D Extension - Tensor product algorithm

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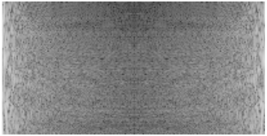
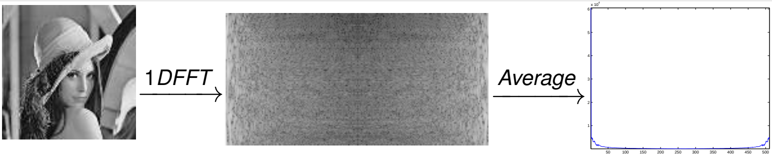
$\xrightarrow{1DFFT}$



2D Extension - Tensor product algorithm



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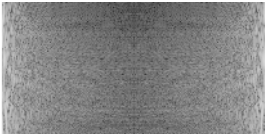
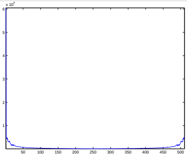
$\xrightarrow{1DFFT}$

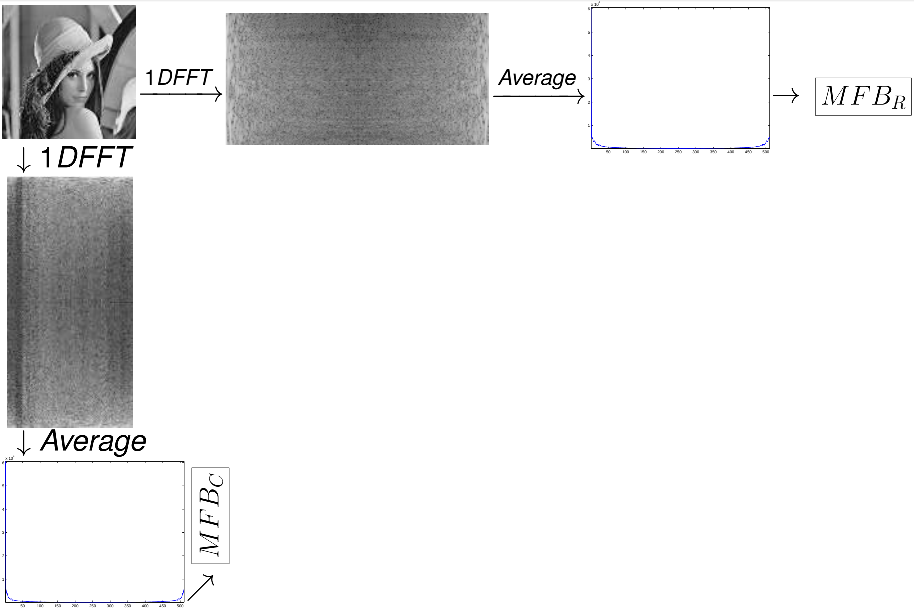


$\xrightarrow{Average}$



$\rightarrow$ $\boxed{MFB_R}$

2D Extension - Tensor product algorithm



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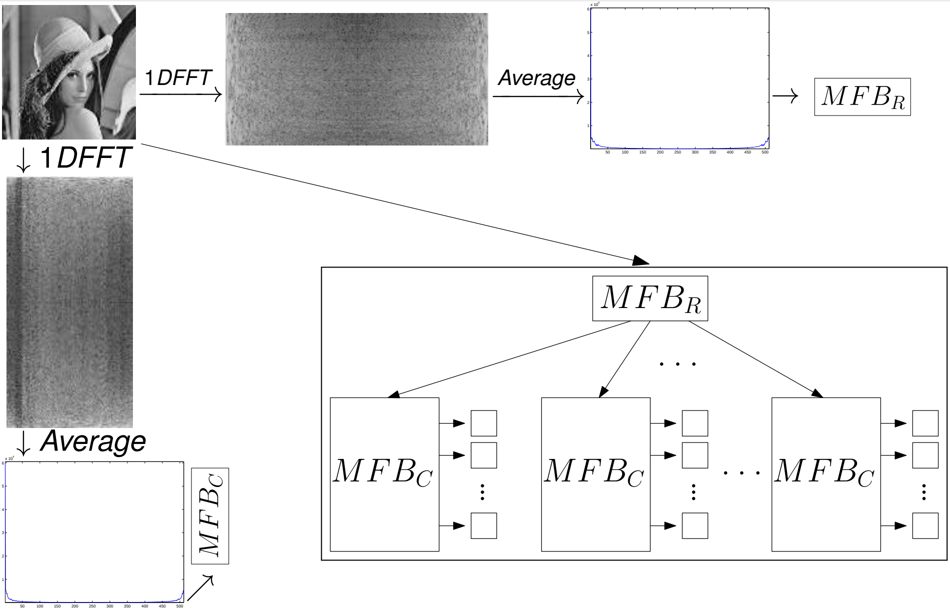
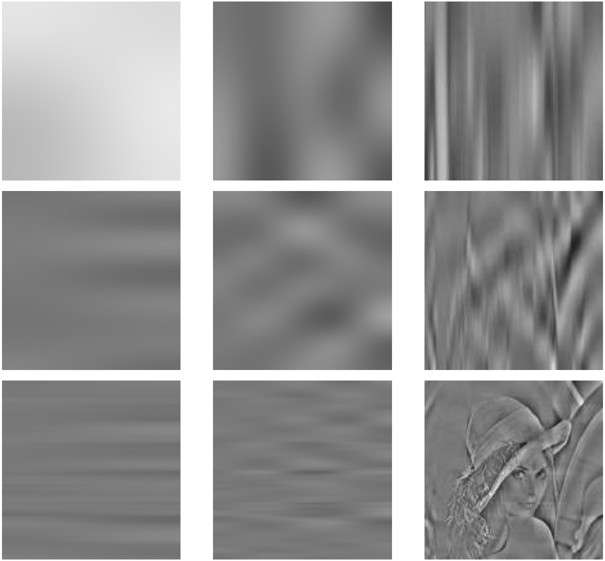
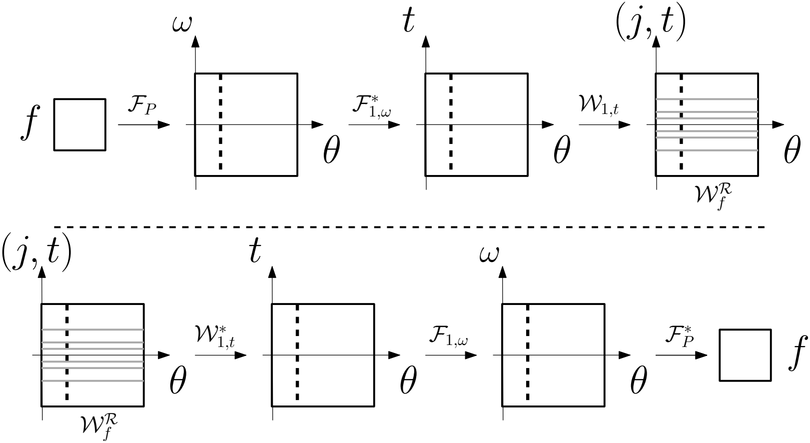


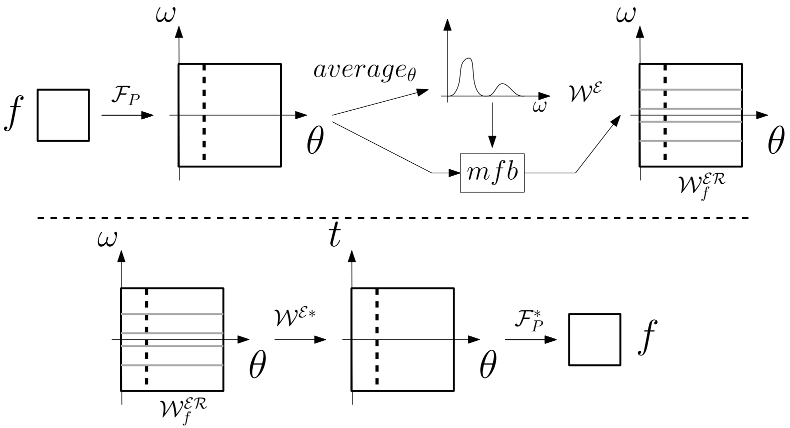
2D Extension - Example



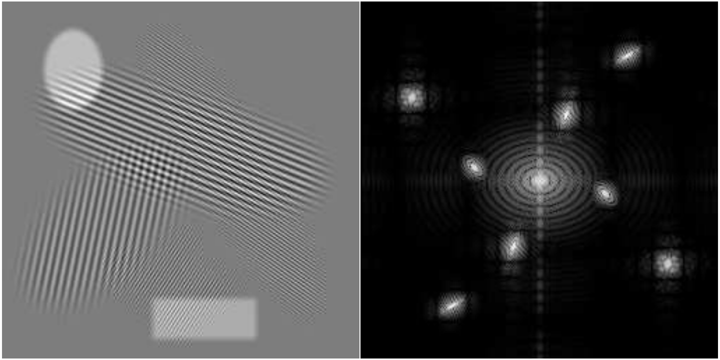
Classic Ridgelets



Empirical Ridgelets



2D Extension - Ridgelet: a first example



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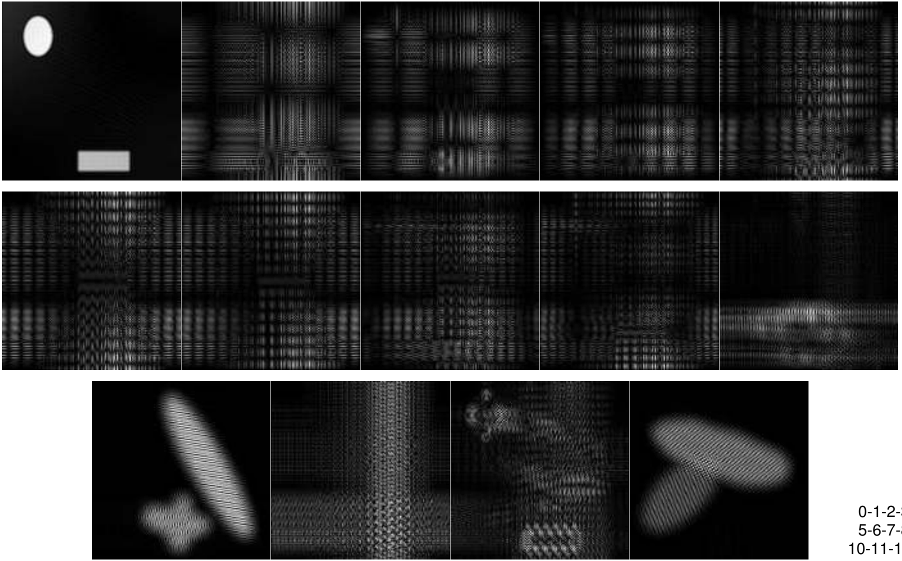

0-1-2-3-4
5-6-7-8-9
10-11-12-13

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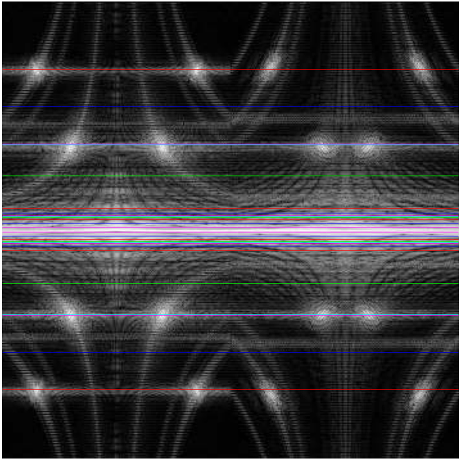

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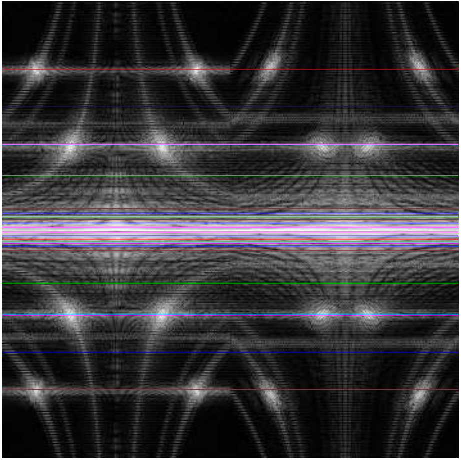

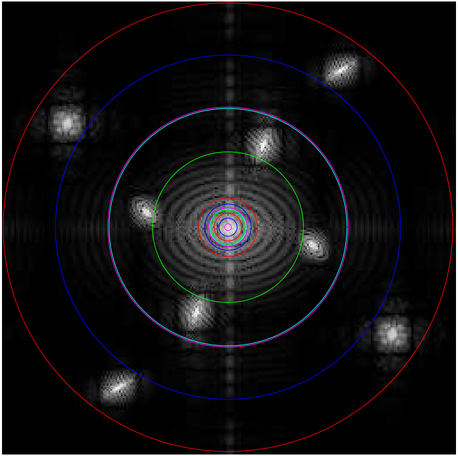

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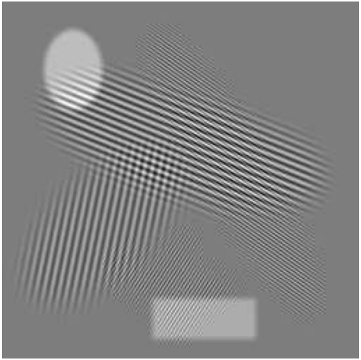

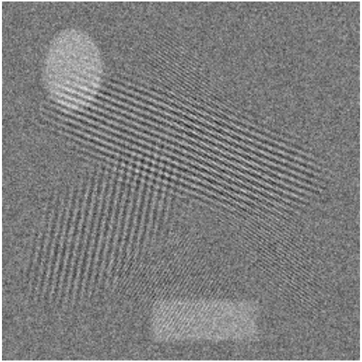

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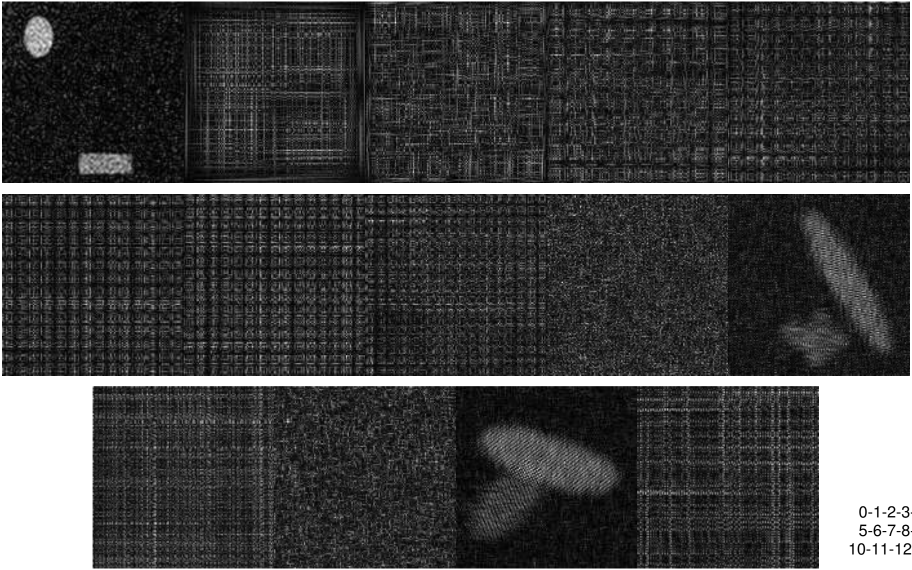

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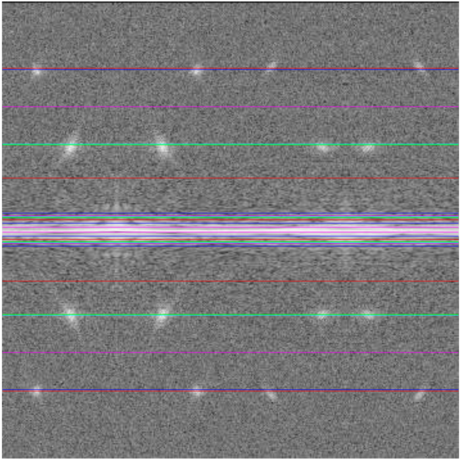

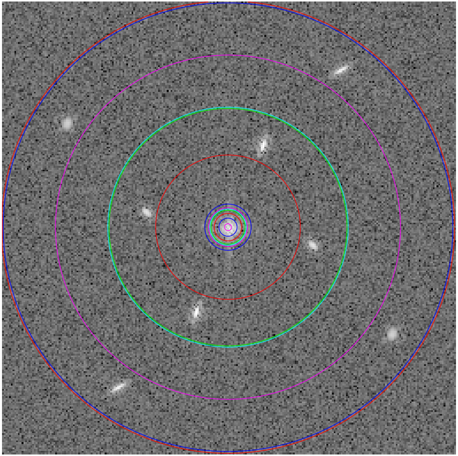

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- Fast algorithm.
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THANK YOU!

PS: Jack, I'm from UCLA and on the job market ;-)