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DIAGONALIZABLE NORMAL OPERATORS

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ABSTRACT. If the image $\varphi(A)$ of a normal operator A on a separable Hilbert space $\mathcal{H}$ is a diagonal operator for some nonzero representation φ of $B(\mathcal{H})$ (that annihilates the compact operators), then A must itself be a diagonal operator on $\mathcal{H}$ (with countable spectrum). This yields an "algebraic" characterization of the closure of the range of a derivation induced by a diagonal operator.

1. **Introduction.** If A is a bounded normal operator on a separable Hilbert space $\mathcal{H}$ can one find a faithful C^*-representation φ of the full algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ such that the eigenvectors of $\varphi(A)$ span the representation space $\mathcal{H}_\varphi$, that is, such that $\varphi(A)$ is a diagonal operator on $\mathcal{H}_\varphi$? The question was raised by B. E. Johnson in connection with some work on derivations [4]. Our purpose here is to supply the answer: only if A is already a diagonal operator on $\mathcal{H}$. We show also that $\varphi(A)$ is a diagonal operator for a nonzero representation φ of $\mathcal{B}(\mathcal{H})$ that annihilates the compact operators if and only if A is a diagonal operator with countable spectrum.

The nondiagonalizability result as just stated hardly seems surprising. But it is interesting to note that the proof seems to require a deep result only recently discovered. Moreover, Johnson's question is reasonable because an affirmative answer would provide a satisfying result about derivations, or better, because Berberian [1] has shown that one *can* find a faithful representation φ of $\mathcal{B}(\mathcal{H})$ such that each point of the spectrum of $\varphi(A)$ is an eigenvalue.

2. **Diagonalizability.** For $A \in \mathcal{B}(\mathcal{H})$ we shall denote by δ_A the inner derivation $X \to AX - XA$ on $\mathcal{B}(\mathcal{H})$ and by $\mathcal{K}$ the ideal of compact operators on $\mathcal{H}$.

THEOREM 1. *Let A be a normal operator on a separable Hilbert space $\mathcal{H}$. The following conditions are equivalent*:

(*1*) *There exists a nonzero representation φ of $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}_\varphi$ (not necessarily faithful) such that $\varphi(A)$ is a diagonal operator on $\mathcal{H}_\varphi$.*

(*2*) *A is a diagonal operator on $\mathcal{H}$.*

(*3*) *Each positive operator in the norm closure $\mathcal{R}(\delta_A)^-$ of the range of δ_A is compact.*

(*4*) *Each projection in $\mathcal{R}(\delta_A)^-$ has finite rank.*

PROOF. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are trivial. Suppose (1)

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holds so that $\varphi(A)e_i = \lambda_i e_i$ for some orthonormal basis $\{e_i\}$ of $\mathcal{H}_\varphi$, and let $Z = \lim \delta_A(X_n)$ be a positive operator in $\mathfrak{R}(\delta_A)^-$. Then

$$\begin{aligned}(\varphi(Z)e_i, e_i) &= \lim[(\varphi(A)\varphi(X_n)e_i, e_i) - (\varphi(X_n)\varphi(A)e_i, e_i)] \\ &= \lim[\lambda_i(\varphi(X_n)e_i, e_i) - \lambda_i(\varphi(X_n)e_i, e_i)] = 0.\end{aligned}$$

Hence $\sqrt{\varphi(Z)}\,e_i = 0$ for each i so that $\varphi(Z) = 0$. Thus $Z \in \ker(\varphi) \subset \mathcal{K}$ because $\mathcal{H}$ is separable. Thus (1) $\Rightarrow$ (3).

We complete the proof by showing (4) $\Rightarrow$ (2), or what is the same, that if A is not a diagonal operator then $\mathfrak{R}(\delta_A)^-$ contains an infinite rank projection. By replacing A with its restriction to the orthocomplement of the span of its eigenvectors, there is no loss of generality in assuming that A itself has no eigenvectors.

Let B be the direct sum of countably many copies of A acting in the usual way on the space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots$. Then by a theorem of I. D. Berg [2] there is a unitary transformation U mapping $\mathcal{H}$ onto $\tilde{\mathcal{H}}$ and a compact operator K on $\mathcal{H}$ such that $U^{-1}BU = A + K$.

Now A and B have no eigenvalues and consequently $\mathfrak{R}(\delta_A)^-$ and $\mathfrak{R}(\delta_B)^-$ respectively contain all the compact operators on $\mathcal{H}$ and $\tilde{\mathcal{H}}$ respectively [6]. Hence $\mathfrak{R}(\delta_A)^- = U^{-1}\mathfrak{R}(\delta_B)^- U$. Also, for the same reason, if P_0 is any nonzero projection of finite rank, we can choose a sequence $X_n \in \mathfrak{B}(\mathcal{H})$ such that $\|\delta_A(X_n) - P_0\| \leqslant n^{-1}$. Let $\tilde{X}_n$ be the direct sum of countably many copies of X_n and let $\tilde{P}$ be the direct sum of as many copies of P_0. Then $\|\delta_B(\tilde{X}_n) - \tilde{P}\| \leqslant n^{-1}$ so that $\tilde{P} \in \mathfrak{R}(\delta_B)^-$, and consequently, $P = U^{-1}\tilde{P}U$ is a projection of infinite rank in $\mathfrak{R}(\delta_A)^-$.

3. **Essential diagonalizability.** A diagonal operator A on a separable space $\mathcal{H}$ has only countably many eigenvalues of course, but the spectrum itself can be any prescribed compact subset of the plane. However, if A is also diagonalizable by a representation of the Calkin algebra, there is a severe restriction on the spectrum. The first assertion of the next theorem was pointed out to me by C. Foiaş.

THEOREM 2. *Let A be a normal operator on a separable Hilbert space $\mathcal{H}$.*

(1) If $\varphi(A)$ is a diagonal operator for some nonzero representation φ of $\mathfrak{B}(\mathcal{H})$ with $\varphi(\mathcal{K}) = 0$, then the spectrum of A is countable.

(2) Conversely, if A has countable spectrum, then $\varphi(A)$ is a diagonal operator for any nonzero representation φ of $\mathfrak{B}(\mathcal{H})$.

PROOF. (1) Suppose that the spectrum $\sigma(A)$ of A is not countable. Then there is a continuous measure μ with support contained in $\sigma(A)$ [5, p. 176]. (For example, take $\mu = \nu \circ f^{-1}$ where ν is Haar measure on the compact abelian group $G = \{0, 1\}^{\mathbf{N}}$ and f is a homeomorphism from G into $\sigma(A)$.) Let B_0 be the operator defined by multiplication by the independent variable in $L^2(\mu)$. Then A and $B = B_0 \oplus A$ have the same essential spectrum, so that by Berg's theorem [2] there is a unitary operator U from $L^2(\mu) \oplus \mathcal{H}$ onto $\mathcal{H}$ and a compact operator K with $UBU^{-1} = A + K$. But then $\varphi(UBU^{-1}) = \varphi(A + K) = \varphi(A)$ is diagonal so that (Theorem 1) UBU^{-1}, and therefore B itself, is diagonal. This is a contradiction since B_0 has no eigenvalues.

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(2) Suppose that A is a diagonal operator on $\mathcal{H}$ with countable spectrum and let φ be a nonzero representation of $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}_\varphi$. Let $\mathfrak{M} = \varphi(1)\mathcal{H}_\varphi$. If $X \in \mathcal{B}(\mathcal{H})$ then $\varphi(X) = \varphi_0(X) \oplus 0$ on $\mathcal{H}_\varphi = \mathfrak{M} \oplus \mathfrak{M}^\perp$ so that φ_0 is a representation of $\mathcal{B}(\mathcal{H})$ on $\mathfrak{M}$ and $\varphi_0(1)$ is the identity operator on $\mathfrak{M}$. In particular, the operator $\varphi_0(A)$ has spectrum contained in $\sigma(A)$ and is therefore countable. It suffices to show, therefore, that a normal operator B on a Hilbert space $\mathfrak{M}$ having countable spectrum is diagonal. This fact is well known: if $\mathfrak{M}_0$ is the span of the eigenvectors of B then $\mathfrak{M}_1 = \mathfrak{M} \ominus \mathfrak{M}_0$ reduces to 0; otherwise $\sigma(B|\mathfrak{M}_1)$, being countable, must have an isolated point and this is necessarily an eigenvalue of B.

REMARK 1. L. G. Brown has observed that Theorem 2 is valid as stated with the weaker hypothesis that the operator A is essentially normal, i.e., that $\pi(A) = A + \mathcal{K}$ is a normal element of the quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}$: if $\sigma(A)$ is uncountable choose a normal operator B_0 with no eigenvalues such that $\sigma(B_0) \subset \sigma(A)$ and let $B = A \oplus B_0 \oplus B_0 \oplus \cdots$. Then $\mathcal{R}(\delta_B)^-$ contains a projection of infinite rank so that $\varphi(B)$ is not a diagonal operator for any nonzero representation φ of $\mathcal{B}(\mathcal{H})$. Hence if $\varphi(\mathcal{K}) = 0$ then $\varphi(A)$ is also not a diagonal operator because $\pi(A)$ and $\pi(B)$ are unitarily equivalent [3].

2. After this paper was completed the author discovered a preprint of John G. Aiken, *An application of direct integral theory to a question of Calkin*, [Notices Amer. Math. Soc. **21** (1974), A493]. Aiken constructs a diagonal operator A such that $T_u(A)$ is not a diagonal operator for any of the "generalized limit" representations T_u of $\mathcal{B}(\mathcal{H})/\mathcal{K}$ introduced by J. W. Calkin, thereby answering a question explicitly(!) raised in the latter's famous paper [Ann. of Math. (2) **42** (1941), 839-873. MR 3, 208].

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