COHERENT MEASURES OF RISK

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July 22, 1998

Abstract. In this paper we study both market risks and non-market risks, without complete markets assumption, and discuss methods of measurement of these risks. We present and justify a set of four desirable properties for measures of risk, and call the measures satisfying these properties "coherent". We examine the measures of risk provided and the related actions required by SPAN, by the SEC/NASD rules and by quantile based methods. We demonstrate the universality of scenario-based methods for providing coherent measures. We offer suggestions concerning the SEC method. We also suggest a method to repair the failure of subadditivity of quantile-based methods.

Key words and phrases. aggregation of risks, butterfly, capital requirement, coherent risk measure, concentration of risks, currency risk, decentralization, extremal events risk, insurance risk, margin requirement, market risk, mean excess function, measure of risk, model risk, net worth, quantile, risk-based capital, scenario, shortfall, subadditivity, tail value at risk, value at risk.

The authors acknowledge financial support from Société Générale for this work. The views expressed are those of the authors. They thank for useful discussions on a related paper, participants of the following meetings: Boston University Mathematical Finance Day, March 31, 1996, University of Waterloo Insurance and Finance Conference, May 24, 1996, StudienZentrum Gerzensee Symposium in Financial Markets, July 15-26, 1996, Latsis Symposium, ETH Zürich, September 24-25, 1996, Universität Karlsruhe Geld, Finanzwirtschaft, Banken, Versicherungen Tagung, December 11-13, 1996, Aarhus University Workshop, February 25-March 1, 1997. They also thank D. Madan, discussant at the French Finance Association International Meeting, Geneva, June 1996, F. Diebold, discussant at the Federal Reserve Bank of Atlanta 1997 Financial Markets Conference, R. Bliss, Ph. Boyle, V. Brousseau, P. Embrechts, A. Hoffman, W. Neuefeind, Ch. Petitmengin, P. Poncet, J. Renegar, E. Shiu as well as a referee of an earlier version of this paper.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

We provide in this paper a *definition* of risks (market risks as well as non-market risks) and present and justify a unified framework for the analysis, construction and implementation of *measures* of risk. We do not assume completeness of markets. These measures of risk can be used as (extra) capital requirements, to regulate the risk assumed by market participants, traders, insurance underwriters, as well as to allocate existing capital.

For these purposes, we:

(1) Define "acceptable" future random net worths (see Section 2.1) and provide a set of axioms about the set of acceptable future net worths (Section 2.2);

(2) Define the measure of risk of an unacceptable position *once a reference, "prudent," investment instrument has been specified*, as the minimum extra capital (see Section 2.3) which, invested in the reference instrument, makes the future value of the modified position become acceptable;

(3) State axioms on measures of risk and relate them to the axioms on acceptance sets. We argue that these axioms should hold for any risk measure which is to be used to effectively regulate or manage risks. We call risk measures which satisfy the four axioms *coherent*;

(4) Present, in Section 3, a (simplified) description of three existing methods for measuring market risk: the "variance-quantile" method of value-at-risk (VaR), the margin system SPAN (Standard Portfolio Analysis of Risk) developed by the Chicago Mercantile Exchange, and the margin rules of the Securities and Exchanges Commission (SEC), which are used by the National Association of Securities Dealers (NASD);

(5) Analyze the existing methods in terms of the axioms and show that the last two methods are essentially the same (i.e., that when slightly modified they are mathematical duals of each other);

(6) Make a specific recommendation for the improvement of the NASD-SEC margin system (Section 3.2);

(7) Examine in particular the consequences of using value at risk for risk management (Section 3.3);

(8) Provide a general representation for all coherent risk measures in terms of "generalized scenarios" (see Section 4.1), by applying a consequence of the separation theorem for convex sets already in the mathematics literature;

(9) Give conditions for extending into a coherent risk measure a measurement already agreed upon for a restricted class of risks (see Section 4.2);

(10) Use the representation results to suggest a specific coherent measure (see Section 5.1) called tail conditional expectation, as well as to give an example of construction of a coherent measure out of measures on separate classes of risks, for example credit risk and market risk (see Section 5.2).

(11) Our axioms are not restrictive enough to specify a unique risk measure. They instead characterize a large class of risk measures. The choice of precisely which measure to use (from this class) should presumably be made on the basis of additional economic considerations. Tail conditional expectation is, under some assumptions, the least expensive among these which are coherent and accepted by regulators since being more conservative than the value at risk measurement.

A non technical presentation of part of this work is given in [ADEH].

2. Definition of risk and of coherent risk measures

This section accomplishes the program set in (1), (2) and (3) above, in the presence of different regulations and different currencies.

2.1 Risk as the random variable: future net worth.

Although several papers (including an earlier version of this one) define risk in terms of *changes* in values between two dates, we argue that because risk is related to the variability of the *future value* of a position, due to market changes or more generally to uncertain events, it is better to instead consider future values only. Notice indeed that there is no need for the initial costs of the components of the position to be determined from universally defined market prices (think of over-the-counter transactions). The principle of "bygones are bygones" leads to this "future wealth" approach.

The basic objects of our study shall therefore be the random variables on the set of states of nature at a future date, interpreted as possible future values of positions or portfolios currently held. A first, crude but crucial, measurement of the risk of a position will be whether its future value belongs or does not belong to the subset of *acceptable risks*, as decided by a supervisor like:

(a) a *regulator* who takes into account the unfavorable states when allowing a risky position which may draw on the resources of the government, for example as a guarantor of last resort;
(b) an *exchange's clearing firm* which has to make good on the promises to all parties, of transactions being securely completed;
(c) an investment *manager* who knows that his firm has basically given to its traders an exit option where the strike "price" consists in being fired in the event of big trading losses on one's position.

In each case above, there is a trade-off between severity of the risk measurement, and level of activities in the supervised domain. The axioms and characterizations we shall provide do not single out a specific risk measure, and additional economic considerations have to play a role in the final choice of a measure.

For an *unacceptable risk* (i.e. a position with an unacceptable future net worth) one remedy may be to alter the position. Another remedy is to look for some commonly accepted instruments which, added to the current position, make its future value become acceptable to the regulator/supervisor. The current cost of getting enough of this or these instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.

For simplicity, we consider only one period of uncertainty $(0, T)$ between two dates 0 and T. The various currencies are numbered by $i\,, 1 \leq i \leq I$ and, for each of them, one "reference" instrument is given, which carries one unit of date 0 currency i into r_i units of date T currency i . Default free zero coupon bonds with maturity at date T may be chosen as particularly simple reference instruments in their own currency. Other possible reference instruments are mentionned in Section 2.3, right before the statement of Axiom T.

The period $(0, T)$ can be the period between hedging and rehedging, a fixed interval like two weeks, the period required to liquidate a position, or the length of coverage provided by an insurance contract.

We take the point of view of an investor subject to regulations and/or supervision in country 1. He considers a portfolio of securities in various currencies.

Date 0 exchange rates are supposed to be one, while e_i denotes the random number of units of currency 1 which one unit of currency i buys at date T.

An investor's initial portfolio consists of positions A_i , $1 \leq i \leq I$, (possibly within some institutional constraints, like the absence of short sales and a "congruence" for each currency between assets and liabilities). The position A_i provides $A_i(T)$ units of currency i at date T. We call *risk* the investor's *future net worth* $\sum_{1 \leq i \leq I} e_i \cdot A_i(T)$.

Remark. The assumption of the position being held during the whole period can be relaxed substantially. In particular, positions may vary due to the agent's actions or those of counterparties. In general, we can consider the risk of following a strategy (which specifies the portfolio held at each date as a function of the market events and counterparties' actions) over an arbitrary period of time. Our current results in the simplified setting represent a first step.

2.2 Axioms on Acceptance sets, i.e. sets of acceptable future net worths.

We suppose that the set of all possible states of the world at the end of the period is known, but the probabilities of the various states occurring may be unknown or not subject to common agreement. When we deal with market risk, the state of the world might be described by a list of the prices of all securities and all exchange rates, and we assume that the set of all possible such lists is known. Of course, this assumes that markets at date T are liquid; if they are not, more complicated models are required, where we can distinguish the risks of a position and of a future net worth, since, with illiquid markets, the mapping from the former to the latter may not be linear.

Notation.

(a) We shall call Ω the set of states of nature, and assume it is finite. Considering Ω as the set of outcomes of an experiment, we compute the final net worth of a position for each element of Ω. It is a random variable denoted by X. Its negative part, $\max(-X, 0)$, is denoted by X^- and the supremum of X^- is denoted by $\|X^-\|$. The random variable identically equal to 1 is denoted by $\mathbf{1}$. The indicator function of state ω is denoted by $\mathbf{1}_{\{\omega\}}$.

(b) Let $\mathcal{G}$ be the set of all risks, that is the set of all real valued functions on Ω. Since Ω is assumed to be finite, $\mathcal{G}$ can be identified with R^n, where $n = card(\Omega)$. The cone of non-negative elements in $\mathcal{G}$ shall be denoted by L_+, its negative by L_-.

(c) We call $\mathcal{A}_{i,j}, j \in J_i$, a set of final net worths, expressed in currency i, which, in country i, are accepted by regulator/supervisor j.

(d) We shall denote $\mathcal{A}_i$ the intersection $\bigcap_{j \in J_i} \mathcal{A}_{i,j}$ and use the generic notation $\mathcal{A}$ in the listing of axioms below.

We shall now state axioms for acceptance sets. Some have an immediate interpretation while the interpretation of the third one will be more easy in terms of risk measure (see Axiom S in Section 2.3.) The rationale for Axioms 2.1 and 2.2 is that a final net worth which is always nonnegative does not require extra capital, while a net worth which is always (strictly) negative certainly does.

Axiom 2.1. *The acceptance set $\mathcal{A}$ contains L_+.*

Axiom 2.2. *The acceptance set $\mathcal{A}$ does not intersect the set L_{--} where*

$$L_{--} = \{X \mid \text{ for each } \omega \in \Omega \ , \ X(\omega) < 0\}.$$

It will also be interesting to consider a stronger axiom:

Axiom 2.2'. *The acceptance set $\mathcal{A}$ satisfies $\mathcal{A} \cap L_- = \{0\}$.*

The next axiom reflects risk aversion on the part of the regulator, exchange director or trading room supervisor.

Axiom 2.3. *The acceptance set $\mathcal{A}$ is convex.*

A less natural requirement on the set of acceptable final net worths is:

Axiom 2.4. *The acceptance set $\mathcal{A}$ is a positively homogeneous cone.*

2.3 Correspondence between acceptance sets and measures of risk.

Sets of acceptable future net worths are the primitive objects to be considered in order to describe acceptance or rejection of a risk. We present here how, *given* some "reference instrument", there is a natural way to define a measure of risk by describing how close or how far from acceptance a position is.

Definition 2.1. *A measure of risk is a mapping from $\mathcal{G}$ into $\mathbb{R}$.*

In Section 3 we shall speak of a *model-dependent* measure of risk when an explicit probability on Ω is used to construct it (see e.g. Sections 3.1 and 3.3), and of a *model-free* measure otherwise (see e.g. Section 3.2). Model-free measures can still be used in the case where only risks of positions are considered.

When positive, the number $\rho(X)$ assigned by the measure ρ to the risk X will be interpreted (see Definition 2.2 below) as the minimum extra cash the agent has to *add* to the risky position X, *and* to invest "prudently", that is in the reference instrument, to be allowed to proceed with his plans. If it is negative, the cash amount $-\rho(X)$ can be withdrawn from the position, or received as restitution as in the case of organized markets for financial futures.

Remark 1. The reader may be surprised that we define a measure of risk on the *whole* of $\mathcal{G}$. Why, in particular, should we consider a risk, a final net worth, like the constant $-\mathbf{1}$? No one would or could willingly enter into a deal which for sure entails a negative of final net worth equal to 1 ! Let us provide three answers:

(a) we want to extend the accounting procedures dealing with future certain bad events (like loss in inventories, degradation [wear and tear] of physical plant), into measurement procedures for future uncertain bad events;
(b) actual measurements used in practice seem to be indeed defined only for risks where *both* states with positive and states with negative final net worth exist. Section 4.2 shows that, under well-defined conditions, they can be extended without ambiguity to measurements for *all* functions in $\mathcal{G}$;
(c) multiperiod models may naturally introduce at some intermediate date the prospect of such final net worths.

Remark 2. It has been pointed out to us that describing risk "by a single number" involves a great loss of information. However, the actual decison about taking a risk or allowing one to take it is fundamentally binary, of the "yes or no" type, and we claimed at the beginning of Section 2.1 that this is the actual origin of risk measurement.

Remark 3. The expression "cash" deserves some discussion in the case of a publicly traded company. It refers to an increase in equity. The amount $\rho(X)$ may, for example, be used to lower the amount of debt in the balance sheet of the company.

We define a correspondence between acceptance sets and measures of risk.

Definition 2.2. *Risk measure associated to an acceptance set: given the total rate of return r on a reference instrument, the risk measure associated to the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{G}$ to $I\!R$ denoted by $\rho_{\mathcal{A},r}$ and defined by*

$$\rho_{\mathcal{A},r}(X) = \inf\{m \mid m \cdot r + X \in \mathcal{A}\}.$$

Remark. Acceptance sets allow us to address a question of importance to an international regulator and to the risk manager of a multinational firm, namely the invariance of acceptability of a position with respect to a change of currencies. If, indeed, we have for each currency $i, 1 \leq i \leq I$, $e_i \cdot \mathcal{A}_i = \mathcal{A}_1$ then, for each position providing an acceptable future net worth X in currency i, the same position provides a future net worth $e_i/e_j \cdot X$ in currency j, which is also acceptable. The situation is more complex for unacceptable positions. If a position requires an extra initial cash of $\rho_{\mathcal{A}_i,r_i}(X)$ units to be invested in the i-th reference instrument, it is not necessarily true that this amount is equal to the number $\rho_{\mathcal{A}_j,r_j}(X)$ of initial units deemed sufficient by the regulation(s) in country j, if invested in the j-th reference instrument, even though we supposed the initial exchange rate to be 1.

Definition 2.3. *Acceptance set associated to a risk measure: the acceptance set associated to a risk measure ρ is the set denoted by $\mathcal{A}_\rho$ and defined by*

$$\mathcal{A}_\rho = \{X \in \mathcal{G} \mid \rho(X) \leq 0\}.$$

We consider now several possible properties for a risk measure ρ defined on $\mathcal{G}$. They will be related, in Section 2.4, to the axioms stated above concerning acceptance sets. For clarity we label the new axioms with letters.

The first requirement ensures that the risk measure is stated in the same units as the final net worth, except for the use of the reference instrument. This particular asset is modeled as having the initial price 1 and a strictly positive price r (or *total return*) in any state of nature at date T. It is the regulator's (supervisor's) responsibility to accept for r possible random values as well as values smaller than 1.

Axiom T means that adding (resp. subtracting) the sure initial amount α to the initial position and investing it in the reference instrument, simply decreases (resp. increases) the risk measure by α.

Axiom T. *Translation invariance: for all $X \in \mathcal{G}$ and all real numbers α, we have $\rho(X + \alpha \cdot r) = \rho(X) - \alpha$.*

Remark 1. Axiom T ensures that, for each X, $\rho(X + \rho(X) \cdot r) = 0$. This equality has a natural interpretation in terms of the acceptance set associated to ρ (see Definition 2.3 above.)

Remark 2. By insisting on references to cash and to time, Axiom T clearly indicates that our approach goes much farther than the interpretation given by Wang of an earlier version of this paper: [Wan], page 3, indeed claims that "the main function of a risk measure is to properly rank risks."

Axiom S. *Subadditivity: for all X_1 and $X_2 \in \mathcal{G}$, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.*

We contend that this property, which could be stated in the following brisk form "a merger does not create extra risk," is a natural requirement:

(a) if an exchange's risk measure were to fail to satisfy this property, then, for example, an individual wishing to take the risk $X_1 + X_2$ may open two accounts, one for the risk X_1 and the other for the risk X_2, incurring the smaller margin requirement of $\rho(X_1) + \rho(X_2)$, a matter of concern for the *exchange*;

(b) if a firm were forced to meet a requirement of extra capital which did not satisfy this property, the firm might be motivated to break up into two separately incorporated affiliates, a matter of concern for the *regulator*;

(c) bankruptcy risk inclines *society* to require less capital from a group without "firewalls" between various business units than it does require when one "unit" is protected from liability attached to failure of another "unit";

(d) suppose that two desks in a firm compute in a decentralized way, the measures $\rho(X_1)$ and $\rho(X_2)$ of the risks they have taken. If the function ρ is subadditive, the *supervisor* of the two desks can count on the fact that $\rho(X_1) + \rho(X_2)$ is a feasible guarantee relative to the global risk $X_1 + X_2$. If indeed he has an amount m of cash available for their joint business, he knows that imposing limits m_1 and m_2 with $m = m_1 + m_2$, allows him to decentralise his cash constraint into two cash constraints, one per desk. Similarly, the firm can allocate its capital among managers.

Axiom PH. *Positive homogeneity: for all $\lambda \geq 0$ and all $X \in \mathcal{G}$, $\rho(\lambda X) = \lambda\rho(X)$.*

Remark 1. If position size directly influences risk (for example, if positions are large enough that the time required to liquidate them depend on their sizes) then we should consider the consequences of lack of liquidity when computing the future net worth of a *position*. With this in mind, Axioms S and PH about mappings from *random variables* into the reals, remain reasonable.

Remark 2. Axiom S implies that $\rho(nX) \leq n\rho(X)$ for $n = 1, 2, \ldots$. In Axiom PH we have imposed the reverse inequality (and require equality for all positive λ) to model what a government or an exchange might impose in a situation where no netting or diversification occurs, in particular because the government does not prevent many firms to all take the same position.

Remark 3. Axioms T and PH imply that for each α, $\rho(\alpha \cdot (-r)) = \alpha$.

Axiom M. *Monotonicity: for all X and $Y \in \mathcal{G}$ with $X \leq Y$, we have $\rho(Y) \leq \rho(X)$.*

Remark. Axiom M rules out the risk measure defined by $\rho(X) = -\mathbf{E}_{\mathbb{P}}[X] + \alpha \cdot \sigma_{\mathbb{P}}(X)$, where $\alpha > 0$ and where $\sigma_{\mathbb{P}}$ denotes the standard deviation operator, computed under the probability $\mathbb{P}$. Axiom S rules out the "semi-variance" type risk measure defined by $\rho(X) = -\mathbf{E}_{\mathbb{P}}[X] + \sigma_{\mathbb{P}}((X - \mathbf{E}_{\mathbb{P}}[X])^-)$.

Axiom R. *Relevance: for all $X \in \mathcal{G}$ with $X \leq 0$ and $X \neq 0$, we have $\rho(X) > 0$.*

Remark. This axiom is clearly necessary, but not sufficient, to prevent concentration of risks to remain undetected (see Section 4.3.)

We notice that for $\lambda > 0$, Axioms S, PH, M and R remain satisfied by the measure $\lambda \cdot \rho$, if satisfied by the measure ρ. It is not the case for Axiom T.

The following choice of required properties will define coherent risk measures.

Definition 2.4. *Coherence: a risk measure satisfying the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity, is called coherent.*

2.4 Correspondence between the axioms on Acceptance Sets and the axioms on Measures of risks.

The reader has certainly noticed that we claimed the acceptance set to be the fundamental object, and discussed the axioms mostly in terms of the associated risk measure. The following propositions show that this was reasonable.

Proposition 2.1. *If the set $\mathcal{B}$ satisfies Axioms 2.1, 2.2, 2.3 and 2.4, the risk measure $\rho_{\mathcal{B},r}$ is coherent. Moreover $\mathcal{A}_{\rho_{\mathcal{B},r}} = \bar{\mathcal{B}}$, the closure of $\mathcal{B}$.*

Proof of Proposition 2.1. (1) Axioms 2.2 and 2.3 ensure that for each $X, \rho_{\mathcal{B},r}(X)$ is a finite number.

(2) The equality $\inf\{p \mid X + (\alpha + p) \cdot r \in \mathcal{B}\} = \inf\{q \mid X + q \cdot r \in \mathcal{B}\} - \alpha$ proves that $\rho_{\mathcal{B},r}(X + r \cdot \alpha) = \rho(X) - \alpha$, and Axiom T is satisfied.

(3) The subadditivity of $\rho_{\mathcal{B}}$ follows from the fact that if $X + m \cdot r$ and $Y + n \cdot r$ both belong to $\mathcal{B}$, so does $X + Y + (m + n) \cdot r$ as Axioms 2.3 and 2.4 show.

(4) If $m > \rho_{\mathcal{B},r}(X)$ then for each $\lambda > 0$ we have $\lambda \cdot X + \lambda \cdot m \cdot r \in \mathcal{B}$, by Definition 2.3 and Axiom 2.4, and this proves that $\rho_{\mathcal{B},r}(\lambda \cdot X) \leq \lambda \cdot m$. If $m < \rho_{\mathcal{B},r}(X)$, then for each $\lambda > 0$ we have $\lambda \cdot X + \lambda \cdot m \cdot r \notin \mathcal{B}$, and this proves that $\rho_{\mathcal{B},r}(\lambda \cdot X) \geq \lambda \cdot m$. We conclude that $\rho_{\mathcal{B},r}(\lambda \cdot X) = \lambda \cdot \rho_{\mathcal{B},r}(X)$.

(5) Monotonicity of $\rho_{\mathcal{B},r}$ follows from the fact that if $X \leq Y$ and $X + m \cdot r \in \mathcal{B}$ then $Y + m \cdot r \in \mathcal{B}$ by use of Axioms 2.3 and 2.1, and of Definition 2.3.

(6) For each $X \in \mathcal{B}$, $\rho_{\mathcal{B},r}(X) \leq 0$ hence $X \in \mathcal{A}_{\rho_{\mathcal{B},r}}$. Proposition 2.2 and points (1) through (5) above ensure that $\mathcal{A}_{\rho_{\mathcal{B},r}}$ is closed, which proves that $\mathcal{A}_{\rho_{\mathcal{B},r}} = \bar{\mathcal{B}}$.

Proposition 2.2. *If a risk measure ρ is coherent, then the acceptance set $\mathcal{A}_\rho$ is closed and satisfies Axioms 2.1, 2.2, 2.3 and 2.4. Moreover $\rho = \rho_{\mathcal{A}_\rho,r}$.*

Proof of Proposition 2.2. (1) Subadditivity and positive homogeneity ensure that ρ is a convex function on $\mathcal{G}$, hence continuous, and that the set $\mathcal{A}_\rho = \{X \mid \rho(X) \leq 0\}$ is a closed, convex and homogeneous cone.

(2) Positive homogeneity implies that $\rho(0) = 0$. Together with monotonicity this ensures that the set $\mathcal{A}_\rho$ contains the positive orthant L_+ .

(3) Let X be in L_{--} with $\rho(X) < 0$. Axiom M ensures that $\rho(0) < 0$, a contradiction. If $\rho(X) = 0$, then we find $\alpha > 0$ such that $X + \alpha \cdot r \in L_{--}$, which provides, by use of Axiom T, the relation $-\alpha \geq 0$, a contradiction. Hence $\rho(X) > 0$, that is $X \notin \mathcal{A}_\rho$, which establishes Axiom 2.2.

(4) For each X, let δ be any number with $\rho_{\mathcal{A}_\rho,r}(X) < \delta$. Then $X + \delta \cdot r \in \mathcal{A}_\rho$, hence $\rho(X + \delta \cdot r) \leq 0$, hence $\rho(X) \leq \delta$, which proves that $\rho(X) \leq \rho_{\mathcal{A}_\rho,r}(X)$, that is $\rho \leq \rho_{\mathcal{A}_\rho,r}$.

(5) For each X, let δ be any number with $\delta > \rho(X)$, then $\rho(X + \delta \cdot r) < 0$ and $X + \delta \cdot r \in \mathcal{A}_\rho$, hence $\rho_{\mathcal{A}_\rho,r}(X + \delta \cdot r) \leq 0$. This proves that $\rho_{\mathcal{A}_\rho,r}(X) \leq \delta$ and that $\rho_{\mathcal{A}_\rho,r}(X) \leq \rho(X)$, hence $\rho_{\mathcal{A}_\rho,r} \leq \rho$.

Proposition 2.3. *If a set $\mathcal{B}$ satisfies Axioms 2.1, 2.2', 2.3 and 2.4, then the coherent risk measure $\rho_{\mathcal{B},r}$ satisfies the relevance axiom. If a coherent risk measure ρ satisfies the relevance axiom, then the acceptance set $\mathcal{A}_{\rho_{\mathcal{B},r}}$ satisfies Axiom 2.2'.*

Proof of Proposition 2.3. (1) For an X like in the statement of Axiom R we know that $X \in L_-$ and $X \neq 0$, hence, by Axiom 2.2', $X \notin \mathcal{B}$, which means $\rho_{\mathcal{B},r}(X) > 0$.

(2) For $X \in L_-$ and $X \neq 0$ Axiom R provides $\rho(X) > 0$ and $X \notin \mathcal{B}$.

3. Three currently used methods of measuring market risk

In this section, we give a (simplified) description of three currently used methods of measuring market risk:

a - SPAN [Sp] developed by the Chicago Mercantile Exchange,
b - the Securities Exchange Commission rules used by the National Association of Securities Dealers (see [NASD] and [Fed]), similar to rules used by the Pacific Exchange and the Chicago Board of Options Exchange
c - the quantile-based Value at Risk (or VaR) method [B], [D], [DP], [DPG], [Risk], [RM].

We examine the relationship of these three methods with the abstract approach provided in Section 2. We also suggest slightly more general forms for some of the methods. It will be shown that the distinction made above between model-free and model-dependent measures of risk *actually* shows up.

3.1 An organized exchange's rules: The SPAN computations.

To illustrate the SPAN margin system [Sp] (see also [Ma], pages 7-8), we consider how the initial margin is calculated for a simple portfolio consisting of units of a futures contract and of several puts and calls with a common expiration date on this futures contract. The SPAN margin for such a portfolio is computed as follows: First, fourteen "scenarios" are considered. Each scenario is specified by an up or down move of volatility combined with no move, or an up move, or a down move of the futures price by 1/3, 2/3 or 3/3 of a specified "range." Next, two additional scenarios relate to "extreme" up or down moves of the futures price. The measure of risk is the *maximum* loss incurred, using the full loss for the first fourteen scenarios and only 35% of the loss for the last two "extreme" scenarios. A specified model, typically the Black model, is used to generate the corresponding prices for the options under each scenario.

The calculation can be viewed as producing the maximum of the expected loss under each of sixteen probability measures. For the first fourteen scenarios the probability measures are point masses at each of the fourteen points in the space Ω of securities prices. The cases of extreme moves correspond to taking the convex combination $(0.35, 0.65)$ of the losses at the "extreme move" point under study and at the "no move at all" point (i.e., prices remain the same). We shall call these probability measures "generalized scenarios".

The account of the investor holding a portfolio is required to have sufficient current net worth to support the maximum expected loss. If it does not, then extra cash is required as margin call, in an amount equal to the "measure of risk" involved. This is completely in line with our interpretation of Definition 2.3.

The following definition generalizes the SPAN computation and presents it in our framework:

Definition 3.1. *The risk measure defined by a non-empty set $\mathcal{P}$ of probability measures or "generalized scenarios" on the space Ω and the total return r on a reference instrument, is the function $\rho_{\mathcal{P}}$ on $\mathcal{G}$ defined by*

$$\rho_{\mathcal{P}}(X) = \sup\{\mathbf{E}_{\mathbb{P}}[-X/r] \mid \mathbb{P} \in \mathcal{P}\}.$$

The scenario-based measures from Definition 3.1 are coherent risk measures:

Proposition 3.1. *Given the total return r on a reference instrument and the non-empty set $\mathcal{P}$ of probability measures, or "generalized scenarios", on the set Ω of states of the world, the risk measure $\rho_{\mathcal{P}}$ of Definition 3.1 is a coherent risk measure. It satisfies the relevance axiom if and only if the union of the supports of the probabilities $\mathbb{P} \in \mathcal{P}$ is equal to the set Ω.*

Proof of Proposition 3.1. Axioms PH and M ensure that a coherent risk measure satisfies Axiom R if and only if the negative of each indicator function $\mathbf{1}_{\{\omega\}}$ has a (strictly) positive risk measure. This is equivalent to the fact that any state belongs to at least one of the supports of the probabilities found in the set $\mathcal{P}$.

Section 4.1 shows that *each* coherent risk measure is obtained by way of scenarios.

3.2 Some model-free measures of risks: The SEC rules on final net worth.

The second example of a risk measure used in practice is found in the rules of the Securities and Exchange Commission and the National Association of Securities Dealers. Their common approach is to consider portfolios as formal *lists* of securities and impose "margin" requirements on *them*, in contrast to the SPAN approach which takes the random *variables* - gains and losses of the portfolios of securities - as basic objects to measure. In the terminology of [B] we have here something similar to a "standardized measurement method".

Certain spread positions like a long call and a short call of higher exercise price, both calls having same maturity date, are described in [NASD], page 8133, SEC rule 15c3-1a,(11), as requiring no margin (no "deduction"). No justification is given for this specification. We shall use the paper [RS] as the basis for explaining, for a simple example, the computation of margin according to these common rules.

Let A be a portfolio consisting of two long calls with strike 10, two short calls with strike 20, three short calls with strike 30, four long calls with strike 40 and one short call with strike 50. For simplicity assume all calls European and exercise dates equal to the end of the holding period. A simple graph shows that the final value of this position is never below -10, which should entail a margin deposit of *at most* 10.

Under the SEC method, the position A is represented or "decomposed" as a portfolio of long call spreads. No margin is required for a spread if the strike of the long side is less than the strike of the short side. A margin of $K - H$ is required for the spread consisting of a long call with strike K and a short call with strike H, when $H \leq K$. The margin resulting from a representation or " decomposition" is the sum of the margins attached to each call spread. The investor is presumably able to choose the best possible representation. A simple linear programming computation will show that 30 is the resulting minimum, that is much more than the negative of the worst possible future value of the position!

Remark 1. This 30 seems to be the result of an attempt to bound the largest payout which the investor might have to make at the end of the period. In this method, the current value of his account must be at least as large as the current value of the calls plus 30.

Remark 2. A careful reading of the SEC rules reveals that one must:

a - first mark the account (reference instruments *plus* calls) to market,
b - deduct the market value of the calls (long or short),

c - then deduct the various "margins" required for the spreads in the chosen decomposition (we shall call the total as the "margin,")
d - and then check that this is at least 0.

In the framework of Definition 2.3, this bears some analogy to

a - marking to market *both* the positions in the "risky" instruments as well as in the reference one,
b - subtract the market value of the risky part,
c - make sure that the difference is positive.

We now formalize the special role played by the call spreads, which we call "standard risks," and the natural margin requirements on them in the SEC rules approach to risk measurement, following the lines of [RS], Section 4 (see also [CR], pages 107-109). Given some underlying security, we denote by C_K the European call with exercise price K and exercise date equal to the end of the holding period, and by $S_{H,K}$ the spread portfolio consisting of "one long C_H, one short C_K" , which we also denote by $C_H - C_K$. These spreads shall be "standard risks" for which a simple rule of margin requirement is given. They are then used to "support" general portfolios of calls and provide conservative capital requirements.

We describe the extra capital requirement for a portfolio A consisting of a_H calls $C_H, H \in \mathcal{H}$, $\mathcal{H}$ a finite set of strikes. For simplicity we assume that $\sum_H a_H = 0$, i.e., we have no net long or short position. The exchange allows one to compute the margin for such a portfolio A by solving the linear programming problem:

$$\inf_{n_{H,K}} \sum_{H,K,H\neq K} n_{H,K}(H-K)^+ \tag{3.1}$$

under the conditions that

$$\text{for all } H,K,H\neq K \text{ we have } n_{H,K}\geq 0 \text{ and } A = \sum_{H,K,H\neq K} n_{H,K}S_{H,K}.$$

This program provides the holder of portfolio A with the cheapest decomposition ensuring that *each* spread showing in it has a non-negative net worth at date T.

Going one step farther than Rudd and Schroeder (pages 1374-1376) we write explicitly the dual program:

$$\sup_{\nu_K} \sum_K \nu_K a_K \tag{3.2}$$

where the *sup* is taken over all (ν_K) satisfying: $\nu_H - \nu_K \leq (H-K)^+$.

For the interpretation of this *dual* problem, we rewrite the preceding program with the negative π_K of the dual variables, getting:

$$\inf_{\pi_K} \sum_K \pi_K a_K \tag{3.3}$$

under the conditions that

$\pi_H - \pi_K \geq -(H-K)^+$, or $\pi_H - \pi_K \geq 0$ if $H < K$ and $\pi_H - \pi_K \geq K - H$ if $H > K$

the last inequalities being rewritten as

$$\pi_K - \pi_H \leq H - K \text{ if } H > K. \tag{3.4}$$

Notice that if we interpret π_H as the cash flows associated with the call C_H at expiration date T, the objective function in (3.3) is the cash flow of the portfolio A at expiration. The duality theorem of linear programming ensures that the worst payout to the holder of portfolio A, under all scenarios satisfying the constraints specified in problem (3.3), cannot be larger than the lowest margin accepted by the exchange. The exchange is therefore sure than the investor commitments will be fulfilled.

It is remarkable that the primal problem (3.1) did not seem to refer to a model of distribution for future prices of the call. Yet the duality results in an implicit set of states of nature consisting of call prices, with a surprise! Our example of portfolio A in the beginning of this Section has shown indeed that the exchange is, in some way, *too* secure, as we now explain.

That the cash flows of the calls must satisfy the constraints (3.4) specified for problem (3.3) (and indeed many other constraints such as convexity as a function of strike, see [Me], Theorem 8.4) is well known. For the specific portfolio A studied in Section 3.2, the set of strikes is $\mathcal{H} = \{10, 20, 30, 40, 50\}$, and an optimal primal solution is given by $n^*_{10,20} = 2, n^*_{40,50} = 1, n^*_{40,30} = 3$, all others $n^*_{H,K} = 0$, for a minimal margin of 30. The cash flows are given by $\pi^*_{10} = \pi^*_{20} = \pi^*_{30} = 10$ and $\pi^*_{40} = \pi^*_{50} = 0$, which provides the value -30 for the minimal cash flow of the portfolio at expiration. However, this minimal cash flow corresponds to cash flows for the individual options which cannot arise for any stock price. Indeed, if the stock price at expiration is S, the cash flow of C_H is $(S-H)^+$, which is obviously convex in H. Thus since $\pi^*_{20} + \pi^*_{40} < 2\pi^*_{30}$, these $\pi's$ cannot arise as cash flows for any terminal stock price. Briefly, there are *too many* scenarios considered, because some of them are *impossible* scenarios.

The convexity of the call price as function of the strike can be derived from the fact that a long "butterfly" portfolio as $B_{20} = C_{10} - 2C_{20} + C_{30}$ must have a positive price. Therefore, we submit this butterfly to the decomposition method and write it as a sum of spreads $S_{10,20} + S_{30,20}$, which requires a margin of 10. If we instead take the approach of Section 2, looking at random variables, more precisely at the random net worth at the end of the holding period, we realize that the butterfly never has negative net worth, or, equivalently, that the net loss it can suffer is never larger than its initial net worth. The butterfly portfolio should therefore be margin free, which would imply a margin of only 10 for the original portfolio $A = 2B_{20} + 2B_{30} - B_{40}$. In our opinion it is not coherent, in this setting, to have only the spreads $S_{H,K}$ (for $H \leq K$) as margin free portfolios. The method uses *too few* standard risks.

In Section 4.2 we present a framework for extensions of risk measurements of "standard risks" and give conditions under which our construction actually produces coherent measures of risk. The results of Section 4.1 on scenario representation of coherent measures will allow to interpret the extension in terms of scenarios attached to the original measurement.

3.3 Some model-dependent rules based on quantiles.

The last example of measures of risk used in practice is the "Value at Risk" (or VaR) measure. It is usually defined in terms of net wins or P/L and therefore ignores the difference between money at one date and money at a different date, which, for small time periods and a single currency, may be acceptable. It uses quantiles, which requires us to pay attention to discontinuities and *intervals* of quantile numbers.

Definition 3.2. *Quantiles: given $\alpha \in]0,1[$ the number q is an $\alpha -$ quantile of the random variable X under the probability distribution $\mathbb{P}$ if one of the three equivalent properties below is satisfied:*

a - $\mathbb{P}\left[X \leq q\right] \geq \alpha \geq \mathbb{P}\left[X < q\right]$,
b - $\mathbb{P}\left[X \leq q\right] \geq \alpha$ *and* $\mathbb{P}\left[X \geq q\right] \geq 1-\alpha$,
c - $F_X(q) \geq \alpha$ *and* $F_X(q-) \leq \alpha$ *with* $F_X(q_-) = \lim_{x \to q, x<q} F(x)$, *where* F_X *is the cumulative distribution function of* X.

Remark. The set of such α-quantiles is a closed interval. Since Ω is finite, there is a finite left-(resp. right-) end point q_α^- (resp. q_α^+) which satisfies $q_\alpha^- = \inf\{x \mid \mathbb{P}\left[X \leq x\right] \geq \alpha\}$ [equivalently $\sup\{x \mid \mathbb{P}\left[X \leq x\right] < \alpha\}$] (resp. $q_\alpha^+ = \inf\{x \mid \mathbb{P}\left[X \leq x\right] > \alpha\}$). With the exception of at most countably many α the equality $q_\alpha^- = q_\alpha^+$ holds. The quantile q_α^- is the number $F^{\leftarrow}(\alpha) = \inf\{x \mid \mathbb{P}\{X \leq x\} \geq \alpha\}$ defined in [EKM] Definition 3.3.5 (see also [DP]).

We formally define VaR in the following way:

Definition 3.3. *Value at risk measurement: given $\alpha \in]0,1[$, and a reference instrument r, the value-at-risk VaR_α at level α of the final net worth X with distribution $\mathbb{P}$, is the negative of the quantile q_α^+ of X/r, that is*

$$VaR_\alpha(X) = -\inf\{x \mid \mathbb{P}\left[X \leq x \cdot r\right] > \alpha\}.$$

Remark 1. Notice that what we are using for defining VaR_α is really the amount of additional capital that a VaR_α type calculation entails.

Remark 2. We have here what is called an “internal” model in [B], and it is not clear whether the (estimated) physical probability or a “well-chosen” subjective probability should be used.

We will now show that, while satisfying properties T, PH and M, VaR_α *fails* to satisfy the subadditivity property.

Consider as an example, the following two digital options on a stock, with the same exercise date T, the end of the holding period. The first option denoted by A (initial price u) pays 1000 if the value of the stock at time T is more than a given U, and nothing otherwise, while the second option denoted by B (initial price l) pays 1000 if the value of the stock at T is less than L (with $L < U$), and nothing otherwise.

Choosing L and U such that $\mathbb{P}\{S_T < L\} = \mathbb{P}\{S_T > U\} = 0.008$ we look for the 1% values at risk of the future net worths of positions taken by two traders writing respectively 2 options A and 2 options B. They are $-2 \cdot u$ and $-2 \cdot l$ respectively (r supposed to be one). By contrast, the positive number $1000 - l - u$ is the 1% value at risk of the future net worth of the position taken by a trader writing $A + B$. This implies that the set of acceptable net worths (in the sense of Definition 2.4 applied to the value at risk measure) is not convex. Notice that this is an even worse feature than the non-subadditivity of the measurement. We give below one more example of non-subadditivity.

Remark 1. We note that if quantiles are computed under a distribution for which all prices are jointly normally distributed, then the quantiles do satisfy subadditivity as long as probabilities of excedence are smaller than 0.5. Indeed, $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$ for each pair (X, Y) of random variables. Since for a normal random variable X we have

$$VaR_\alpha(X) = -(\mathbf{E}_{\mathbb{P}}[X] + \Phi^{-1}(\alpha) \cdot \sigma_{\mathbb{P}}(X)\,),$$

with Φ the cumulative standard normal distribution and since $\Phi^{-1}(0.5) = 0$, the proof of subadditivity follows.

Remark 2. Several works on quantile-based measures (see [D], [Risk], [RM]) consider mainly the *computational* and *statistical* problems they raise, without first considering the implications of this method of measuring risks.

Remark 3. Since the beginning of this century, casualty actuaries have been involved in computation and use of quantiles. The choice of initial capital controls indeed the probabilities of ruin at date T. Loosely speaking, "ruin" is defined in (retrospective) terms by the negativity, at date T, of the *surplus*, defined to be :

$$Y = \text{capital at date 0 + premium received - claims paid (from date 0 to date T)}.$$

Imposing an upper bound $1 - \alpha$ on the probability of Y being negative determines the initial capital via a quantile calculation (for precise information, see the survey article by Hans Bühlmann, *Tendencies of Development in Risk Theory*, in [Cen], pages 499-522).

Under some circumstances, related to Remark 1 above, (see [DPP], pages 157, 168), this "capital at risk" is a measure which possesses the subadditivity property. For some models the surplus represents the net worth of the insurance firm at date T. In general, the difficulty of assigning a market value to insurance liabilities forces us to distinguish surplus and net worth.

Remark 4. We do not know of organized exchanges using value at risk as the basis of risk measurement for margin requirements.

For a second example of non-subadditivity, briefly allow an infinite set Ω and consider two *independent* identically distributed random variables X_1 and X_2 having the same density 0.90 on the interval $[0, 1]$, the same density 0.05 on the interval $[-2, 0]$. Assume that each of them represents a future random net worth with positive expected value, that is a possibly interesting risk. Yet, in terms of quantiles, the 10% values at risk of X_1 and X_2 being equal to 0, whereas an easy calculation showing that the 10% value at risk of $X_1 + X_2$ is certainly larger than 0, we conclude that the individual controls of these risks do not allow directly a control of their sum, if we were to use the 10% value at risk.

Value at risk measurement also *fails* to recognise *concentration* of risks. A remarkably simple example concerning credit risk is due to Claudio Albanese (see [Alba]). Assume that the base rate of interest is zero, and that the spreads on all corporate bonds is 2%, while these bonds default, independently from company to company, with a (physical) probability of 1%. If an amount of $1,000,000$ borrowed at the base rate is invested in the bonds of a single company, the 5% value at risk of the resulting position is negative, namely $-20,000$, and there is "no risk".

If, in order to diversify, the whole amount is invested equally into bonds of one hundred different companies, the following happens in terms of value at risk.

Since the probability of at least two companies defaulting is greater than 0.18 it follows that the portfolio of bonds leads to a negative future net worth with a probability greater than 0.05 : *diversification* of the original portfolion has *increased* the measure of risk, while the "piling-up" of risky bonds issued by the same company had remained undetected. We should not rely on such "measure".

Value at risk also *fails* to encourage a reasonable *allocation* of risks among agents as can be seen from the simple following example. Let Ω consists of three states $\omega_1, \omega_2, \omega_3$ with respective probabilities $0.94, 0.03, 0.03$. Let two agents have the same future net worth X with $X(\omega_1) \geq 0, X(\omega_2) = X(\omega_3) = -100$. If one uses the 5% value at risk measure, one would not find sufficient an extra capital (for each agent) of 80. But this same capital would be found more than sufficient, for each agent, if, by a risk exchange, the two agree on the modified respective future net worths Y and Z, where $Y(\omega_1) = Z(\omega_1) = X(\omega_1), Y(\omega_2) = Z(\omega_3) = -120, Y(\omega_3) = Z(\omega_2) = -80$. This is not reasonable since the allocation $(X + 80, X + 80)$ Pareto dominates the allocation $(Y + 80, Z + 80)$ if the agents are risk averse.

In conclusion, the basic reasons to reject the value at risk measure of risks are the following:

(a) value at risk does not behave nicely with respect to addition of risks, even independent ones, creating severe aggregation problems.

(b) the use of value at risk does not encourage and, indeed, sometimes prohibits diversification, because value at risk does not take into account the *economic consequences* of the events the probabilities of which it controls.

4. Representation Theorems for Coherent Risk Measures

This section provides two representations of coherent risk measures. The first corresponds exactly to the SPAN example of Section 3.1 and the second is the proper generalisation of the NASD/SEC examples of Section 3.2. These representation results are used in Section 5.2 to provide an example of algorithm to measure risks in trades involving two different sources of randomness, once coherent measures of risks for trades dealing with only one of these sources have been agreed upon.

4.1 Representation of coherent risk measures by scenarios.

In this section we show that Definition 3.1 provides the most general coherent risk measure: *any* coherent risk measure arises as the supremum of the expected negative of final net worth for some collection of "generalized scenarios" or probability measures on states of the world. We continue to suppose that Ω is a finite set, otherwise we would also get finitely additive measures as scenarios.

The σ-algebra, 2^Ω, is the class of all subsets of Ω. Initially there is no particular probability measure on Ω.

Proposition 4.1. *Given the total return r on a reference investment, a risk measure ρ is coherent if and only if there exists a family $\mathcal{P}$ of probability measures on the set of states of nature, such that*

$$\rho(X) = \sup\{\mathbf{E}_{\mathbb{P}}[-X/r] \mid \mathbb{P} \in \mathcal{P}\}.$$

Remark 1. We note that ρ can also be seen as an insurance premium principle. In that case, denoting by $\mathbb{R}$ the physical measure, we find that the condition $\mathbb{R} \in \mathcal{P}$ (or in the convex hull of this set), is of great importance. This condition is translated as follows: for all $X \leq 0$ we have $\mathbf{E}_{\mathbb{R}}[-X/r] \leq \rho(X)$.

Remark 2. The more scenarios considered, the more conservative (i.e. the larger) is the risk measure obtained.

Remark 3. We remind the reader about Proposition 3.1. It will prove that Axiom R is satisfied by ρ if and only if the union of the supports of the probabilities in $\mathcal{P}$ is the whole set Ω of states of nature.

Proof of Proposition 4.1. (1) We thank a referee for pointing out that the mathematical content of Proposition 4.1, which we had proved on our own, is already in the book [Hu]. We therefore simply identify the terms in Proposition 2, Chapter 10 of [Hu] with these of our terminology of risks and risk measure.

(2) The sets Ω and $\mathcal{M}$ of [Hu] are our set Ω and the set of probabilities on Ω. Given a risk measure ρ we associate to it the functional E^* by $E^*(X) = \rho(-r \cdot X)$. Axiom M for ρ is equivalent to Property (2.7) of [Hu] for E^*, Axioms PH and T together are equivalent to Property (2.8) for E^*, and Axiom S is Property (2.9).

(3) The "if" part of our Proposition 4.1 is obvious. The "only if" part results from the "representability" of E^*, since Proposition (2.1) of [Hu] states that

$$\rho(X) = E^*(-X/r) = \sup\{\mathbf{E}_{\mathbb{P}}\,[-X/r] \mid \mathbb{P} \in \mathcal{P}_\rho\}$$

where $\mathcal{P}_\rho$ is defined as the set

$$\{\mathbb{P} \in \mathcal{M} \mid \text{ for all } X \in \mathcal{G}\colon \mathbf{E}_{\mathbb{P}}\,[X] \leq E^*(X) = \rho(-r \cdot X)\}$$

$$= \{\mathbb{P} \in \mathcal{M} \mid \text{ for all } Y \in \mathcal{G} \colon \mathbf{E}_{\mathbb{P}}[-Y/r] \le \rho(Y)\} .$$

Remark 1. Model risk can be taken into account by including into the set $\mathcal{P}$ a family of distributions for the future prices, possibly arising from other models.

Remark 2. Professor Bühlmann kindly provided us with references to works by Hattendorf, [Hat], Kanner, [Kan], and Wittstein, [Wit], which he had mentionned in his Göttingen presentation ([Bü]). These authors consider, in the case of insurance risks, possible losses *only*, neglecting the case of gains. For example, risk for a company providing annuities is linked to the random *excess* number of survivors over the expected number given by the lifetable. Several of these references, for example [Hat], §3, page 5, contain an example of a risk measure used in life insurance, namely the "mittlere Risico" constructed out of one scenario, related to the life table used by a company. It is defined as the mathematical expectation of the *positive part* of the loss, as "die Summe aller möglichen Verluste, jeden multipliciert in die Wahrscheinlichkeit seines Eintretens". This procedure defines a risk measure satisfying Axioms S, PH, M.

Remark 3. It is important to distinguish between a point mass scenario and a simulation trial: the first is chosen by the investor or the supervisor, while the second is chosen randomly according to a *distribution* they have prescribed beforehand.

Conclusion. The result in Proposition 4.1 completely explains the occurrence of the first type of actual risk measurement, the one based on scenarios, as described in Section 3.1. Any coherent risk measure appears therefore as given by a "worst case method", in a framework of generalized scenarios. At this point it should be emphasized that scenarios be announced to all traders within the firm (by the manager) or to all firms (by the regulator). In the first case, we notice that decentralization of risk management within the firm is only available *after* these announcements. Yet, in quantile-based methods, even after the announcements of individual limits, there remains a problem preventing decentralized risk management: two operators ignorant of each other's actions may well each comply with their individual quantile limits and yet no automatic procedure provides for an interesting upper bound for the measure of the joint risk due to their actions. As for the regulation case we allow ourselves to interpret a sentence from [Stu]: "regulators like Value at Risk, because they can regulate it" as pointing to the formidable task of building and announcing a reasonable set of scenarios.

4.2 Construction of coherent risk measures by extension of certain risk measurements.

We now formalize the attempts described in Section 3.2 to measure risks. Their basis is to impose margin requirements on certain basic portfolios considered as "standard risks", to use combinations of those risks to "support" other risks and then bound from above required capital, using the margins required for standard risks.

Definition 4.1. *Supports of a risk: given a set $\mathcal{Y}$ of functions on Ω, we consider a family, indexed by $\mathcal{Y}$, of nonnegative numbers $\mu = (\mu_Y)_{Y \in \mathcal{Y}}$, all of them but a finite number being zero, and we say that the couple (μ, γ), where γ is a real number,* *"supports" X, for $X \in \mathcal{G}$, provided*

$$X \geq \sum_{Y \in \mathcal{Y}} \mu_Y Y + \gamma \cdot r.$$

The set of all such (μ, γ) which support X will be denoted by $S_{\mathcal{Y}}(X)$.

The idea is now to use these "supports", made of "standard risks", to bound above possible extensions of a function Ψ defined on a subset of $\mathcal{G}$. A consistency condition is required to avoid supports leading to infinitely negative values.

Condition 4.1. *Given a set $\mathcal{Y}$ of functions on Ω, and a function $\Psi: \mathcal{Y} \longrightarrow I\!R$, we say that Ψ fulfills Condition 4.1 if for each support (μ, γ) of 0, we have the inequality $\sum_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma \geq 0$.*

Proposition 4.2. *Given a set $\mathcal{Y}$ of functions on Ω and a function $\Psi: \mathcal{Y} \longrightarrow I\!R$, the equality*

$$\rho_\Psi(X) = \inf_{(\mu,\gamma) \in S_{\mathcal{Y}}(X)} \sum_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma$$

defines a coherent risk measure ρ_Ψ, if and only if Ψ fulfills Condition 4.1. If so, ρ_Ψ is the largest coherent measure ρ such that $\rho \leq \Psi$ on $\mathcal{Y}$.

Proof of Proposition 4.2. (1) The necessity of Condition 4.1 is obvious.

(2) Since $(0, 0)$ is a support of the element $X = 0$ of $\mathcal{G}$ and since Condition 4.1 ensures that any support of 0 provides a nonnegative number we find that $\rho_\Psi(0) = 0$. Notice that if Condition 4.1 is violated, then we would get $\rho_\Psi(0) = -\infty$.

(3) Axiom S required from a coherent risk measure follows here from the relation $S_{\mathcal{Y}}(X_1 + X_2) \supset S_{\mathcal{Y}}(X_1) + S_{\mathcal{Y}}(X_2)$, and Axiom PH is satisfied since, given $\lambda > 0$, (μ, γ) supports X if and only if $(\lambda \cdot \mu, \lambda \cdot \gamma)$ supports $\lambda \cdot X$.

(4) For a support (μ, γ) of a risk X let us call the number $\sum_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma$ the "cost" of the support. By noticing for each risk X and each real α, that the support (μ, γ) for $X + \alpha \cdot r$ provides the support $(\mu, \gamma - \alpha)$ for X, at a cost lower by the amount α than the cost of the support of $X + \alpha \cdot r$ we find that $\rho_\Psi(X) = \rho_\Psi(X + \alpha \cdot r) + \alpha$. Axiom T is therefore satisfied by ρ_Ψ.

(5) Since for $X \leq Z$ we have $S_{\mathcal{Y}}(Z) \supset S_{\mathcal{Y}}(X)$, Axiom M is satisfied by ρ_Ψ.

(6) For any coherent measure ρ with $\rho \leq \Psi$ on $\mathcal{Y}$ we must have, for any support (μ, γ) of X, the inequality $\rho(X) \leq \sum_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma$ and therefore $\rho(X) \leq \rho_\Psi(X)$.

Remark. As opposed to the case of scenarios based measures of risks, the fewer initial standard risks are considered, the more conservative is the coherent risk measure obtained. This is similar to what happens with the SEC rules since Section 3.2 showed us that too many scenarios, and dually, too few standard risks, were considered.

Condition 4.1 allows one to consider the function ρ_Ψ in particular on the set $\mathcal{Y}$, the set of prespecified risks. There, it is clearly bounded above by the original function Ψ. An extra consistency condition will prove helpful to figure out whether ρ_Ψ is actually equal to Ψ on $\mathcal{Y}$.

Condition 4.2. *Given a set $\mathcal{Y}$ of functions on Ω and a function $\Psi: \mathcal{Y} \longrightarrow I\!R$, we say that Condition 4.2 is satisfied by Ψ if for each element $Z \in \mathcal{Y}$ and each support (μ, γ) of Z we have $\Psi(Z) \leq \sum_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma$.*

Remark. It is an easy exercise to prove that Condition 4.2 implies Condition 4.1.

Proposition 4.3. *Given a set $\mathcal{Y}$ of functions on Ω and a function $\Psi: \mathcal{Y} \longrightarrow I\!R_+$ satisfying Condition 4.2, the coherent risk measure ρ_Ψ is the largest possible extension of the function Ψ to a coherent risk measure.*

Proof of Proposition 4.3. (1) Condition 4.2 just ensures that the value at $Z \in \mathcal{Y}$ of the original function Ψ is bounded above by the sum obtained with any support of Z, hence also by their infimum $\rho_\Psi(Z)$, which proves that $\rho_\Psi = \Psi$ on $\mathcal{Y}$.

(2) Let ρ be any coherent risk measure, which is also an extension of Ψ. Since $\rho \leq \Psi$ on $\mathcal{Y}$, Proposition 4.3 ensures that $\rho \leq \rho_\Psi$.

Propositions 4.2 and 4.3 above applied to $(\mathcal{Y}, \Psi) = (\mathcal{G}, \rho)$, provide a statement similar to Proposition 4.1 about representation of coherent risk measures.

Proposition 4.4. *A risk measure ρ is coherent if and only if it is of the form ρ_Ψ for some Ψ fulfilling Condition 4.1.*

Remark. It can be shown that for a coherent risk measure ρ built as a ρ_Ψ, the following set of probabilities

$$\mathcal{P}_\Psi = \{\mathbb{P} \mid \text{ for all } X \in \mathcal{G} : \mathbf{E}_\mathbb{P}[-X/r] \leq \Psi(X)\}$$

is non-empty and verifies the property

$$\rho(X) = \sup\{\mathbf{E}_\mathbb{P}[-X/r] \mid \mathbb{P} \in \mathcal{P}_\Psi\}.$$

4.3 Relation between scenario probabilities and pricing measures.

The representation result in Proposition 4.1 allows us to approach the problem of risk concentration for *coherent* risk measures.

If the position consisting of the short Arrow-Debreu security corresponding to state of nature ω, has a non-positive measure of risk, that is bankruptcy in the state ω is "allowed", the market price of this security should also be non-positive. To formalize this observation we suppose an arbitrage free market, and denote by $\mathcal{Q}_r$ the closed convex set of pricing probability measures on Ω, using the instrument r as *numeraire*. Given the coherent risk measure $\rho_{\mathcal{B},r}$ associated to r and to an acceptance set B, simply denoted by ρ_r (see Proposition 2.2), it will be natural to assume the following condition:

Condition 4.3. *The closed convex set $\mathcal{P}_{\rho_r}$ of probability measures defining the coherent risk measure ρ_r has a non empty intersection with the closed convex set $\mathcal{Q}_r$ of probability pricing measures.*

When Condition 4.3 is satisfied, there is some $\mathbb{Q} \in \mathcal{Q}_r$ such that for any future net worth Y, $\mathbf{E}_\mathbb{Q}[-Y/r] \leq \rho_r(Y)$, hence if Y has a strictly negative price under $\mathbb{Q}$ it cannot be accepted. We interpret this fact in the following manner: if a firm can, by trading, add a position Y to its portfolio *and* receive cash at the same time, *without* having any extra capital requirement, then there is a bound to the quantity of Y which the firm can add this way without trigging a request for extra capital.

If Condition 4.3 is not satisfied, then there exists a future net worth Y such that

$$\sup\{\mathbf{E}_\mathbb{Q}[Y/r] \mid \mathbb{Q} \in \mathcal{Q}_r\} < \inf\{\mathbf{E}_\mathbb{S}[Y/r] \mid \mathbb{S} \in \mathcal{P}_{\rho_r}\}.$$

Hence for each pricing measure $\mathbb{Q}$ we have $\mathbf{E}_\mathbb{Q}[-Y/r] > \rho_r(Y)$ and therefore the future net worth $Z = Y + \rho_r(Y) \cdot r$ satisifies both conditions $\rho_r(Z) = 0$ and $\mathbf{E}_\mathbb{Q}[Z/r] < 0$. We have therefore an acceptable position with strictly negative price, a situation which may well lead to an undetected accumulation of risk.

5.1 A proposal: the "worst conditional expectation" measure of risk.

Casualty actuaries have been working for long computing pure premium for policies with deductible, using the conditional average of claim size, *given* that the claim exceeds the deductible, see [HK]. In the same manner, reinsurance treaties have involved the conditional distribution of a claim for a policy (or of the total claim for a portfolio of policies), *given* that it is above the ceding insurer's retention level. In order to tackle the question of "how bad is bad", which is *not* addressed by the value at risk measurement, some actuaries (see [Albr], [E]) have first identified the deductible (or retention level) with the quantile used in the field of financial risk measurement. We prove below that one of the suggested methods gets us close to *coherent* risk measures.

Considering the "lower partial moment" or expectation of the "shortfall", the presentation in [Albr] would translate, with our paper's notations, into measuring a risk X by the number $\mathbf{E}_{\mathbb{P}}\left[\min\left(0, -VaR_\alpha(X) - X\right)\right].$

The presentations in [BEK], [E], use instead the *conditional* expectation of the shortfall given that it is positive. The quoted texts (see also [EKM], Definition 3.4.6 as well as the methods indicated there to *estimate* the whole conditional distribution) present the terminology "mean excess function". We suggest the term tail conditional expectation since we do not consider the excess but the whole of the variable X :

Definition 5.1. *Tail conditional expectation (or "TailVaR"): given a base probability measure $\mathbb{P}$ on Ω, a total return r on a reference instrument and a level α, the tail conditional expectation is the measure of risk defined by*

$$TCE_\alpha(X) = -\mathbf{E}_{\mathbb{P}}\left[X/r \mid X/r \leq -VaR_\alpha(X)\right].$$

Definition 5.2. *Worst conditional expectation: given a base probability measure $\mathbb{P}$ on Ω, a total return r on a reference instrument and a level α, the worst conditional expectation is the coherent measure of risk defined by*

$$WCE_\alpha(X) = -\inf\{\mathbf{E}_{\mathbb{P}}\left[X/r \mid A\right] \mid \mathbb{P}\left[A\right] > \alpha\}.$$

Remark. TCE_α has been suggested as a possible ingredient of reinsurance treaties (see [A]).

Proposition 5.1. *We have the inequality $TCE_\alpha \leq WCE_\alpha$.*

Proof of Proposition 5.1. (1) Let us denote X/r by Y. If $F_Y(q^+_\alpha(Y)) > \alpha$ the set $A = \{\omega \mid Y(\omega) \leq q^+_{\alpha(Y)}\}$ is one used in the definition of WCE_α , hence the claim is true.

(2) If $F_Y(q^+_\alpha(Y)) = \alpha$ it follows from the definition of q^+_α and the monotonicity of F_Y that for each $\varepsilon > 0\,, F_Y(\varepsilon + q^+_\alpha(Y)) > \alpha$. Hence, setting $A_\varepsilon = \{\omega \mid Y(\omega) \leq \varepsilon + q^+_\alpha(Y)\}$ we get

$$WCE_\alpha(X) \geq -\mathbf{E}_{\mathbb{P}}\left[Y \mid A_\varepsilon\right] = -\frac{\mathbf{E}_{\mathbb{P}}\left[Y \cdot \mathbf{1}_{A_\varepsilon}\right]}{\mathbb{P}\left[A_\varepsilon\right]}\,.$$

Since F_Y is right-continuous, $\lim_{\varepsilon\to 0}\mathbb{P}\left[A_\varepsilon\right] = F_Y(q^+_\alpha(Y))$ and $A_\varepsilon \downarrow A_0$ so the right hand side has the limit $-\mathbf{E}_{\mathbb{P}}\left[Y \mid A_0\right] = TCE_\alpha(X)$.

The paper [Alba] makes numerical studies of portfolios built out of collection of risky bonds. It looks for a coherent measure which dominates the Value at Risk measurement and yet gets close to it on a specific bond portfolio.

We interpret and generalize this search as the problem of a firm constrained by the supervisors along the lines of the quantile risk measurement. Nevertheless, the firm wishes at the same time to operate on a coherent basis, at the lowest possible cost. Proposition 5.4 will provide circumstances where the firm's problem has a clear-cut solution.

Proposition 5.2. *For each risk X one has the equality*

$$VaR_\alpha(X) = \inf\{\rho(X) \mid \rho \textit{ coherent and } \rho \geq VaR_\alpha\}$$

The proof will use the following

Lemma 5.1. *If ρ is the coherent risk measure defined by a set $\mathcal{P}$ of probability measures, then $\rho \geq VaR_\alpha$ if and only if for each B with $\mathbb{P}[B] > \alpha$ and each $\varepsilon > 0$ there is a $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{Q}[B] > 1 - \varepsilon$.*

Proof of Lemma 5.1. (1) Necessity: take $X = -r \cdot \mathbf{1}_B$ where $\mathbb{P}[B] > \alpha$. Clearly $VaR_\alpha(-r \cdot \mathbf{1}_B) = 1$ and hence $\rho(-r \cdot \mathbf{1}_B) \geq 1$. This implies that for each $\varepsilon > 0$ there exists $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{Q}[B] \geq 1 - \varepsilon$.

(2) Sufficiency: let $-k = VaR_\alpha(X)$, then $\mathbb{P}[X \leq k \cdot r] \geq \alpha$ and for each $\delta > 0$ we have $\mathbb{P}[X \leq (k + \delta) \cdot r] > \alpha$.

Let $\mathbb{Q} \in \mathcal{P}$ be chosen such that $\mathbb{Q}[X \leq (k+\delta) \cdot r] \geq 1-\delta$. We obtain $\mathbf{E}_\mathbb{Q}[-X/r] \geq (-k-\delta) \cdot (1-\delta) - \delta \cdot \|X/r\|$. Since $\delta > 0$ was arbitrary we find that $\rho(X) \geq -k$.

Proof of Proposition 5.2. (1) Given any risk X let again $-k = VaR_\alpha(X)$. Then $\mathbb{P}[X \leq k \cdot r] \geq \alpha$ and for each $\delta > 0$, $\mathbb{P}[X \leq (k+\delta) \cdot r] > \alpha$. We will construct a coherent risk measure ρ such that $\rho \geq VaR_\alpha$ and $\rho(X) \leq VaR_\alpha(X) + \delta$.

(2) For any set B with $\mathbb{P}[B] > \alpha$, we must have $\mathbb{P}[B \cap \{X \geq k \cdot r\}] > 0$ and we can define h_B as $\mathbf{1}_{B \cap \{X \geq k \cdot r\}} / \mathbb{P}[B \cap \{X \geq k \cdot r\}]$ and $\mathbb{Q}_B = h_B \cdot \mathbb{P}$. Lemma 5.1 shows that the measure ρ built with all the $\mathbb{Q}_B$ dominates VaR_α, but for X we obtain $\rho(X) = \sup_{\mathbb{Q}_B} \mathbf{E}_{\mathbb{Q}_B}[-X/r] \leq -k = VaR_\alpha(X)$.

Definition 3.1 and Proposition 3.1 allow one to address a question by Ch. Petit-mengin, Société Générale, about the coherence of the TCE_α measure.

Proposition 5.3. *Assume that the base probability $\mathbb{P}$ on Ω is uniform. If X is a risk such that no two values of the discounted risk $Y = X/r$ in different states are ever equal, then $TCE_\alpha(X) = WCE_\alpha(X)$.*

Proof of Proposition 5.3. (1) Given $\alpha \in]0,1[$ let us denote $-VaR_\alpha(X)$ by q, the set $\{X \leq q \cdot r\}$ by B and the various values of $Y = X/r$ by $y_1 < y_2 < \cdots < y_n$.

(2) Let k be the integer with $0 \leq k < n$ such that $\alpha \in [\frac{k}{n}, \frac{k+1}{n})$. We will prove that $-VaR_\alpha(X) = q_\alpha^+(Y) = q = y_{k+1}$.

(3) For each $u > q$ we have

$$\frac{\#\{i \mid y_i \leq u\}}{n} > \alpha$$

hence the integer $\#\{i \mid y_i \leq u\}$ being strictly greater than $\alpha \cdot n$ is at least $k+1$.

(4) By taking $u = y_{k+1}$ we actually minimize the integer $\#\{i \mid y_i \leq u\}$ and therefore prove the point stated in (2).

(5) The set $Y(B)$ is the set $\{y_1, \dots, y_{k+1}\}$ and

$$TCE_\alpha(X) = -\mathbf{E}\,[X/r \mid X \leq q \cdot r] = -\frac{y_1 + \cdots + y_{k+1}}{k+1}\,.$$

(6) Any set C containing at least $k+1$ states of nature and different from B will provide values for $-Y$ averaging to strictly less than $TCE_\alpha(X)$, which therefore equals $WCE_\alpha(X)$.

Proposition 5.4. *Assume that the base probability $\mathbb{P}$ on Ω is uniform. If a coherent risk measure ρ only depends on the distribution of the discounted risk and is greater than the risk measure VaR_α, then it is greater than the WCE_α (coherent) risk measure.*

Proof of Proposition 5.4. (1) Given a risk X, we denote $-VaR_\alpha(X)$ simply by q and X/r by Y. The set $A = \{\omega \mid Y(\omega) \leq q\}$ has cardinality $p > n \cdot \alpha$ and A is written after possible renumbering as $A = \{\omega_1, \omega_2, \dots, \omega_p\}$ with $Y(\omega_i) \leq Y(\omega_{i+1})$ for $1 \leq i \leq p-1$.

(2) Define $\bar{Y}(\omega_i)$ for $i \leq p$ as $y^* = (Y(\omega_1) + \cdots + Y(\omega_p))/p = \mathbf{E}\,[Y \mid Y \leq q]$ and as $Y(\omega_i)$ otherwise.

(3) For a permutation σ of the first p integers, we define Y^σ by $Y^\sigma(\omega_i) = Y(\omega_{\sigma(i)})$ for $1 \leq i \leq p$, and $Y^\sigma(\omega_j) = Y(\omega_j)$ for $p+1 \leq j \leq n$. We then find that $\bar{Y}$ is also the average of the $p!$ random variables Y^σ.

(4) The assumption that for each risk Z, $\rho(Z)$ only depends on the distribution of Z/r implies that all the $\rho(r \cdot Y^\sigma)$ are equal to $\rho(X)$. The convexity of the function ρ then implies the inequality $\rho(X) \geq \rho(r \cdot \bar{Y})$.

(5) The last assumption made on ρ implies that $\rho(r \cdot \bar{Y}) \geq VaR_\alpha(r \cdot \bar{Y})$.

(6) We have $VaR_\alpha(r \cdot \bar{Y}) = -y^* = \mathbf{E}\,[-Y \mid Y \leq q]$ since for $i \leq p, \bar{Y}(\omega_i) \leq Y(\omega_p)$. Hence $\rho(X) \geq \mathbf{E}\,[-X/r \mid X \leq q \cdot r]$.

(7) For a dense set of random variables X on the finite state space Ω we have, by Proposition 5.3, the equality $\mathbf{E}\,[-X/r \mid X \leq q \cdot r] = WCE_\alpha(X)$ hence the inequality $\rho(X) \geq WCE_\alpha(X)$ holds for a dense set of elements X of $\mathcal{G}$.

(8) Both risk measures ρ and WCE_α are coherent, hence continuous functions on $\mathcal{G}$. The inequality $\rho \geq WCE_\alpha$ is therefore true on the whole of $\mathcal{G}$.

5.2 Construction of a measure out of measures on separate classes of risks.

It is important to realize that Proposition 4.3 can be applied to a set $\mathcal{Y}$ of risks having no structure. It can be the union of a family $(\mathcal{Y}_j)_{j \in J}$ of sets of risks, where, for each j a function (Ψ_j) is given on $\mathcal{Y}_j$, in such a way that $\Psi_j = \Psi_{j'}$ on $\mathcal{Y}_j \cap \mathcal{Y}_{j'}$. The function Ψ is then defined by its restrictions to each of the $\mathcal{Y}_j$.

The different sets $\mathcal{Y}_j$ may be exchange based risks on the one hand and over the counter risks on the other hand, or market risks and credit risks in a framework where a *joint* internal model would be looked for. Similarly, multi line aggregated combined risk optimisation tools (see [Sh], 1998) would call for combined measure of risks. The functions Ψ_j may come from preliminary rules given by exchanges and/or by regulators (see [B], 1996). Assuming that Condition 4.2 is being satisfied, which will depend on inequalities satisfied by the Ψ_j, Proposition 4.3 allows one to *mechanically* compute a coherent risk measure extending the family of the Ψ_j and dominating *any* other possible coherent risk measure *chosen* by exchanges and/or by regulators to extend the family of the Ψ_j. It therefore provides a conservative coherent tool for risk management.

In the special case of $\Omega = \Omega_1 \times \Omega_2$ with given coherent risk measures ρ_i, $i = 1, 2$, on $\mathcal{G}_i$, we define $\mathcal{Y}_i$ as the set of all functions on Ω which are of the form $f_i \circ pr_i$ where f_i is any function on Ω_i, and where pr_i is the projection of Ω on its i-th factor. We also define Ψ_i on $\mathcal{Y}_i$ by the equality $\Psi_i(f_i \circ pr_i) = \rho_i(f_i)$. Since $\mathcal{Y}_1 \cap \mathcal{Y}_2$ consists of the constants, the functions Ψ_1 and Ψ_2 are equal on it and they define a function Ψ on $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$ which satisfies Condition 4.2.

Let $\mathcal{P}_i$ be the set of scenarios defining ρ_i and let $\mathcal{P}$ be the set of probabilities on Ω with marginals in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. We claim that the risk measure ρ_Ψ on the set $\mathcal{G}$ of functions on Ω, that is the largest coherent risk measure extending both Ψ_1 and Ψ_2, is equal to the risk measure $\rho_{\mathcal{P}}$, generated, as in Definition 3.1, by the scenarios in $\mathcal{P}$.

Proposition 5.5. *The two coherent risk measures $\rho_{\mathcal{P}}$ and ρ_Ψ are equal.*

Proof of Proposition 5.5. The restriction of $\rho_{\mathcal{P}}$ to $\mathcal{Y}_i$ equals Ψ_i since for each function f_i on Ω_i we have

$$\begin{aligned}\rho_{\mathcal{P}}(f_i \circ pr_i) &= \sup\{\mathbf{E}_{\mathbb{P}}[-f_i \circ pr_i/r] \mid \mathbb{P} \circ pr_1^{-1} \in \mathcal{P}_1 \ , \mathbb{P} \circ pr_2^{-1} \in \mathcal{P}_2\} \\ &= \sup\{\mathbf{E}_{\mathbb{P}\circ pr_i^{-1}}[-f_i/r] \mid \mathbb{P} \circ pr_i^{-1} \in \mathcal{P}_i\} \\ &= \rho_i(f_i) = \Psi_i(f_i \circ pr_i),\end{aligned}$$

which proves that $\rho_{\mathcal{P}} \leq \rho_\Psi$.

To prove the reverse inequality we use point (3) in the proof of Proposition 4.1 and show that if a probability $\mathbb{Q}$ on Ω is such that for each function X on Ω, $\mathbf{E}_{\mathbb{Q}}[-X/r] \leq \rho_\Psi(X)$, then $\mathbb{Q}$ has its marginals $\mathbb{Q}_1$ and $\mathbb{Q}_2$ in $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Choose indeed $X = f_i \circ pr_i$. We find that $\mathbf{E}_{\mathbb{Q}}[-f_i \circ pr_i/r] = \mathbf{E}_{\mathbb{Q}_i}[-f_i/r]$ which proves that for each $f_i \in \mathcal{G}_i$ one has $\mathbf{E}_{\mathbb{Q}_i}[-f_i/r] \leq \rho_\Psi(f_i \circ pr_i)$, and therefore $\mathbb{Q}_i \in \mathcal{P}_i$.

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