An improved kernelization algorithm for Trivially Perfect Editing

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Abstract

In the Trivially Perfect Editing problem one is given an undirected graph $G = (V, E)$ and an integer k and seeks to add or delete at most k edges in G to obtain a trivially perfect graph. In a recent work, Dumas *et al.* [16] proved that this problem admits a kernel with $O(k^3)$ vertices. This result heavily relies on the fact that the size of trivially perfect modules can be bounded by $O(k^2)$ as shown by Drange and Pilipczuk [14]. To obtain their cubic vertex-kernel, Dumas *et al.* [16] then showed that a more intricate structure, so-called *comb*, can be reduced to $O(k^2)$ vertices. In this work we show that the bound can be improved to $O(k)$ for both aforementioned structures and thus obtain a kernel with $O(k^2)$ vertices. Our approach relies on the straightforward yet powerful observation that any large enough structure contains unaffected vertices whose neighborhood remains unchanged by an editing of size k, implying strong structural properties.

1 Introduction

In the Trivially Perfect Editing problem one is given an undirected graph $G = (V, E)$ and an integer k and seeks to *edit* (add or delete) at most k edges in G so that the resulting graph is trivially perfect (*i.e.* does not contain any cycle on four vertices nor path on four vertices as an induced subgraph). More formally we consider the following problem:

> Trivially Perfect Editing
>
> **Input**: A graph $G = (V, E)$, a *parameter* $k \in \mathbb{N}$
> **Question**: Does there exist a set $F \subseteq (V \times V)$ of size at most k such that the graph $H = (V, E \triangle F)$ is trivially perfect ?

Here $E \triangle F = (E \cup F) \setminus (E \cap F)$ denotes the symmetric difference between sets E and F. We define similarly the deletion (resp. completion) variant of the problem by only allowing to delete (resp. add) edges. Graph modification covers a broad range of well-studied problems that find applications in various areas. For instance, Trivially Perfect Editing has been used to define the community structure of complex networks by Nastos and Gao [30] and is closely related to the well-studied graph parameter tree-depth [20, 32]. Theoretically, some of the earliest NP-Complete problems are graph modification problems [19, 25]. Regarding edge (graph) modification problems, one of the most notable one is the Minimum Fill-in problem which aims at adding edges to a given graph to obtain a chordal graph (*i.e.* a graph that does not contain any induced cycle of length at least 4). In a seminal result, Kaplan *et al.* [24] proved that Mininum Fill-in admits a parameterized algorithm as well as a kernel containing $O(k^3)$ vertices. This result was later improved to $O(k^2)$ vertices by Natanzon *et al.* [31]. Parameterized complexity and kernelization algorithms provide a powerful theoretical framework to cope with decision problems.

Parameterized complexity A parameterized problem Π is a language of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. An instance of a parameterized problem is a pair (I, k) with $I \subseteq \Sigma^*$ and $k \in \mathbb{N}$, called the *parameter*. A parameterized problem is said to be *fixed-parameter tractable* if it can be decided in time $f(k) \cdot |I|^{O(1)}$. An equivalent definition of fixed-parameter tractability is the notion of *kernelization*. Given an instance (I, k) of a parameterized problem Π, a *kernelization algorithm* for Π (kernel for short) is a polynomial-time algorithm that outputs an equivalent instance (I', k') of Π such that $|I'| \leqslant h(k)$ for some function h depending on the parameter only and $k' \leqslant k$. It is well-known that a parameterized problem is fixed-parameter tractable if and only if it admits a kernelization algorithm (see *e.g.* [18]). Problem Π is said to admit a *polynomial kernel* whenever h is a polynomial.

Related work Since the work of Kaplan *et al.* [24] many polynomial kernels for edge modification problems have been devised (see *e.g.* [1, 2, 3, 11, 12, 14, 15, 23, 26]). There is also evidence that under some reasonable theoretical complexity assumptions, some graph modification problems do not admit polynomial kernels [8, 22, 27, 29]. We refer the reader to a recent comprehensive survey on kernelization for edge modification problems by Crespelle *et al.* [9]. The Trivially Perfect Editing problem has been well-studied in the literature [1, 4, 5, 14, 16, 21, 28, 30]. Recall that trivially perfect graphs are a subclass of chordal graphs that additionally do not contain any path on four vertices as an induced subgraph. These graphs are also known as *quasi-threshold* graphs. We note here that while the NP-Completeness of completion and deletion toward trivially perfect graphs has been known for some time [6, 33], the NP-Completeness of Trivially Perfect Editing remained opened until the work of Nastos and Gao [30]. Thanks to a result of Cai [7] stating that graph modification toward *any* graph class characterized by a finite set of forbidden induced subgraphs is fixed-parameter tractable, Trivially Perfect Editing is fixed-parameter tractable. Regarding kernelization algorithms, Drange and Pilipczuk [14] provided a kernel containing $O(k^7)$ vertices, a result that was recently improved to $O(k^3)$ vertices by Dumas *et al.* [16]. These results also work for the deletion and completion variants. For the latter problem, a recent result by Bathie *et al.* [1] improves the bound to $O(k^2)$ vertices.

As part of the proof for the size of their cubic vertex-kernel, Dumas *et al.* [16] subsequently showed the following result. The structures used in Theorem 1 shall be defined later.

Theorem 1 ([16]). *Let (G, k) be an instance*[1] *of* Trivially Perfect Editing *such that the sizes of its* trivially perfect modules *and* combs *are bounded by $p(k)$ and $c(k)$, respectively. If (G, k) is a* Yes*-instance then G has $O(k \cdot (p(k) + c(k))$ vertices.*

The cubic vertex-kernel of Dumas *et al.* [16] relied on a result of Drange and Pilipczuk [14] that proved that $p \in O(k^2)$ and then used new reduction rules implying that $c \in O(k^2)$.

Our contribution We provide reduction rules and structural properties on trivially perfect graphs that will imply an $O(k)$ bound for both functions p and c of Theorem 1. These new reduction rules allow us to prove the existence a quadratic vertex-kernel for Trivially Perfect

[1] As we shall see Section 3.1 the instance also needs to be further reduced under standard reduction rules.

Editing. To bound the size of trivially perfect modules by $O(k)$, we first reduce the ones that contain a large matching of non-edges with the use of a simple reduction rule. To bound the ones that do not contain such structures, we will rely on so-called *combs*, introduced by Dumas *et al.* [16]. Combs correspond to parts of the graph that induce trivially perfect graphs (but not necessarily modules) with strong properties on their neighborhoods. They are composed of two main parts, called the *shaft* and the *teeth*, that will be independently reduced to a size linear in k. The reduction rule dealing with shafts will ultimately allow us to bound the size of trivially perfect modules with no large matching of non-edges. Our approach relies on the straightforward yet powerful observation that any large enough structure contains unaffected vertices whose neighborhood remains unchanged by an editing of size k. Finally, we note that our kernel works for both the deletion and completion variants of the problem.

Outline Section 2 presents some preliminary notions and structural properties on (trivially perfect) graphs. Section 3 describes known as well as our additional reduction rules to obtain the claimed kernelization algorithm while Section 4 explain why our kernel is safe for the deletion variant of the problem. We conclude with some perspectives in Section 5.

2 Preliminaries

We consider simple, undirected graphs $G = (V, E)$ where V denotes the vertex set of G and $E \subseteq (V \times V)$ its edge set. We will sometimes use $V(G)$ and $E(G)$ to clarify the context. The open (respectively closed) neighborhood of a vertex $u \in V$ is denoted by $N_G(u) = \{v \in V \mid \{u, v\} \in E\}$ (respectively $N_G[u] = N_G(u) \cup \{u\}$). Given a subset of vertices $S \subseteq V$ the neighborhood of S is defined as $N_G(S) = \{v \in V(G) \setminus S \mid \{u, v\} \in E\}$. Similarly, given a vertex $u \in V$ and $S \subseteq V$ we let $N_S(u) = N_G(u) \cap S$. In all aforementioned cases we forget the subscript mentioning graph G whenever the context is clear. Given a subset of vertices $S \subseteq V$ we denote by $G[S]$ the subgraph induced by S, that is $G[S] = (S, E \cap (S \times S))$. In a slight abuse of notation, we use $G \setminus S$ to denote the induced subgraph $G[V \setminus S]$. A *connected component* is a maximal subset of vertices $S \subseteq V$ such that $G[S]$ is connected. A *module* of G is a set $M \subseteq V$ such that for all $u, v \in M$ it holds that $N(u) \setminus M = N(v) \setminus M$. Two vertices u and v are *true twins* whenever $N[u] = N[v]$, and a *critical clique* is a maximal set of true twins. A graph is *trivially perfect* if and only if it does not contain any C_4 (a cycle on 4 vertices) nor P_4 (a path on 4 vertices) as an induced subgraph. In the remaining of this section we describe characterizations and structural properties of trivially perfect graphs. The first one relies on the well-known fact that any connected trivially perfect graph contains a universal vertex (see *e.g.* [35]).

Definition 1 (Universal clique decomposition, [13])**.** *A* universal clique decomposition *(UCD) of a connected graph $G = (V, E)$ is a pair $\mathcal{T} = \big(T = (V_T, E_T), \mathcal{B} = \{B_t\}_{t \in V_T}\big)$ where T is a rooted tree and $\mathcal{B}$ is a partition of the vertex set V into disjoint nonempty subsets such that:*

- *if $\{v, w\} \in E$ and $v \in B_t$, $w \in B_s$ then s and t are on a path from a leaf to the root, with possibly $s = t$,*
- *for every node $t \in V_T$, the set B_t of vertices is the universal clique of the induced subgraph $G[\bigcup_{s \in V(T_t)} B_s]$, where T_t denotes the subtree of T rooted at t.*

A simple way of understanding Definition 1 is to observe that such a decomposition can be obtained by removing the set U of universal vertices of G and then recursively repeating this process on every trivially perfect connected component of $G \setminus U$. Drange *et al.* [13] showed that any connected graph admitting an UCD is trivially perfect, thus proving equivalence between both definitions. Using the notion of UCD, Dumas *et al.* [16] proved the following characterization for trivially perfect graphs that will be heavily used in our reduction rules. A collection of subsets $\mathcal{F} \subseteq 2^U$ over some universe U is a *nested family* if $A \subseteq B$ or $B \subseteq A$ holds for any $A, B \in \mathcal{F}$.

Lemma 1 ([16])**.** *Let $G = (V, E)$ be a graph, $S \subseteq V$ a maximal clique of G and $\{K_1, ..., K_r\}$ the set of connected components of $G \backslash S$. The graph G is trivially perfect if and only if the following conditions are verified:*

(i) $G[S \cup K_i]$ *is trivially perfect,* $1 \leqslant i \leqslant r$

(ii) $\bigcup_{1 \leqslant i \leqslant r} \{N_G(K_i)\}$ *is a nested family*

(iii) $(K_i \times N_G(K_i)) \subseteq E$, $1 \leqslant i \leqslant r$*. In other words,* K_i *is a module of* G*.*

In the remaining of this paper, a *k-editing of G into a trivially perfect graph* is a set $F \subseteq (V \times V)$ such that $|F| \leqslant k$ and the graph $H = (V, E \triangle F)$ is trivially perfect. Here $E \triangle F = (E \cup F) \setminus (E \cap F)$ denotes the symmetric difference between sets E and F. For the sake of readability, we simply speak of k-editing of G. We say that F is a k-completion (resp. k-deletion) when $H = (V, E \cup F)$ (resp. $H = (V, E \setminus F)$) is trivially perfect. A vertex is *affected* by a k-editing F if it is contained in some pair of F and *unaffected* otherwise.

Packing, anti-matching and blow-up We now define some structures and operators that will be useful for our kernelization algorithm. We assume in the remaining of this section that we are given a graph $G = (V, E)$. The notion of *r-packing* will be used in reduction rules to ensure the existence of unaffected vertices in ordered sets of critical cliques or of trivially perfect modules.

Definition 2 (r-packing)**.** *Let $\mathcal{S} = \{C_1, \ldots, C_q\}$ be an ordered collection of pairwise disjoint subsets of V. We say that $\mathcal{C} \subseteq \mathcal{S}$ is a* r-packing *of $\mathcal{S}$ if $\mathcal{C} = \{C_1, \ldots, C_p\}$ for $1 \leqslant p \leqslant q$, $\sum_{i=1}^{p} |C_i| \geqslant r$ and the number of vertices contained in $\mathcal{C}$ is minimum for this property.*

In a slight abuse of notation we use $\mathcal{C}$ to denote both $\{C_1, \ldots, C_p\}$ and the set $\cup_{i=1}^{p} C_i$.

Observation 1. *Let $\mathcal{S} = \{C_1, \ldots, C_q\}$ be an ordered collection of pairwise disjoint subsets of V such that $|C_j| \leqslant c$, for $1 \leqslant j \leqslant q$ and some integer $c > 0$. Let $\mathcal{C} = \{C_1, \ldots, C_p\}$ be a r-packing of $\mathcal{S}$. Then $\sum_{i=1}^{p} |C_i| \leqslant r + (c - 1)$.*

Proof. Since $\sum_{i=1}^{p} |C_i| \geqslant r$ and the number of vertices in $\mathcal{C}$ is minimum for this property we have that $\sum_{i=1}^{p-1} |C_i| \leqslant r - 1$. The result follows from the fact that $|C_p| \leqslant c$. □

Definition 3 (Anti-matching)**.** *An* anti-matching *of G is a set of pairwise disjoint pairs $\{u, v\}$ of vertices of G such that $\{u, v\} \notin E$.*

In a slight abuse of notation we denote by $V(D)$ the set of vertices contained in pairs of an anti-matching D.

Observation 2. *Let (G, k) be a* Yes*-instance of* Trivially Perfect Editing *and M be a module containing a $(k+1)$-sized anti-matching. Let F be a k-editing of G and $H = G \triangle F$. Then $N_G(M)$ is a clique in H.*

Proof. Let $D = \{\{u_i, v_i\} \mid 1 \leqslant i \leqslant k+1\}$ be a $(k+1)$-sized anti-matching of M. Assume for a contradiction that $N_G(M)$ is not a clique in H and let $\{u, v\}$ be a non-edge of H with $u, v \in N_G(M)$. By the pigeonhole principle, since $|F| \leqslant k$ there exists $1 \leqslant j \leqslant k+1$ such that $\{u_j, v_j\} \notin F$ and for every $x \in V(G) \setminus M$, $\{u_j, x\}, \{v_j, x\} \notin F$. Hence $\{u_j, u, v_j, v\}$ induces a C_4 in H, a contradiction. □

We conclude this section by introducing a gluing operation on trivially perfect graphs, namely *blow-up*, that will ease the design of some reduction rules.

Definition 4 (Blow-up)**.** *Let u be a vertex of $G = (V, E)$ and $H = (V_H, E_H)$ be any graph. The* blow-up of G by H at u, *denoted $G(u \to H)$ is the graph obtained by replacing u by H in G. More formally:*

$$G(u \to H) = \big((V \setminus \{u\}) \cup V_H, E(G \setminus \{u\}) \cup E_H \cup (V_H \times N_G(u)\big)$$

Proposition 1. *Assume that G is trivially perfect and let u be a vertex of G such that $N_G[u]$ is a clique. For any trivially perfect graph H, the graph $G(u \to H)$ is trivially perfect.*

Proof. Let $S \subseteq V \setminus \{u\}$ be any maximal clique of G containing $N_G(u)$. We apply the forward direction of Lemma 1 on S to obtain components $\{K_1, \ldots, K_r\}$ that are modules such that $G[S \cup K_i]$ is trivially perfect for every $1 \leqslant i \leqslant r$ and $\bigcup_{1 \leqslant i \leqslant r}\{N_G(K_i)\}$ is a nested family. Note that by construction and w.l.o.g., we may assume $K_1 = \{u\}$. The result then directly follows from the reverse direction of Lemma 1 by replacing K_1 by H. □

3 Reduction rules

In the remaining of this section we assume that we are given an instance $(G = (V, E), k)$ of Trivially Perfect Editing.

3.1 Standard reduction rules

We first describe some well-known reduction rules [2, 3, 14, 16] that are essential to obtain a vertex-kernel using Theorem 1 [16]. We will assume in the remaining of this work that the instance at hand is reduced under Rules 1 and 2, meaning that none of them applies to the instance.

Rule 1. *Let $C \subseteq V$ be a connected component of G such that $G[C]$ is trivially perfect. Remove C from G.*

Rule 2. *Let $K \subseteq V$ be a critical clique of G such that $|K| > k + 1$. Remove $|K| - (k + 1)$ arbitrary vertices in K from G.*

Lemma 2 (Folklore, [2, 14])**.** *Rules 1 and 2 are safe and can be applied in polynomial time.*

3.2 An $O(k)$ bound on the size of trivially perfect modules

Using an additional reduction rule bounding the size of independent sets in any trivially perfect module by $O(k)$, Drange and Pilipczuk [14] proved that such modules can be reduced to $O(k^2)$ vertices. We strengthen this result by proving that trivially perfect modules can further be reduced to $O(k)$ vertices. We first deal with modules that contain a large anti-matching.

Rule 3. *Let $M \subseteq V$ be a trivially perfect module of G. If $G[M]$ contains a $(k+1)$-sized anti-matching D, then remove the vertices contained in $M \setminus V(D)$.*

Lemma 3. *Rule 3 is safe.*

Proof. Let $G' = (V', E')$ be the graph obtained after application of Rule 3. We need to prove that $(G = (V, E), k)$ is a YES-instance if and only if $(G' = (V', E'), k)$ is a YES-instance. The forward direction is straightforward since G' is an induced subgraph of G and trivially perfect graphs are hereditary. We now consider the reverse direction. Let $M' = V(D)$ denote the set of vertices kept by Rule 3. Moreover, let F' be a k-editing of G' and $H' = G' \triangle F'$. We will construct a k-editing F^* of G. Note that since the pairs contained in an anti-matching are disjoint (Definition 3), $|M'| \geqslant 2(k+1)$. Moreover, since $|F'| \leqslant k$ there are at most $2k$ affected vertices. Hence let u be an unaffected vertex of M'. By Observation 2 and since M' contains a $(k+1)$-sized anti-matching we have that $N_{G'}(M')$ is a clique in H'. The graph $H_u = H' \setminus (M' \setminus \{u\})$ is trivially perfect by heredity and $N_{H_u}(u) = N_{G'}(M')$. It follows that $N_{H_u}(u)$ is a clique and Proposition 1 implies that the graph $H = H_u(u \to M)$ is trivially perfect. Let F^* be the editing such that $H = G \triangle F^*$. Since u is unaffected by F' and $u \in M$ we have $N_{H_u}(u) = N_G(M)$. Hence, since M is a module in G we have that $N_H(v) = N_G(v)$ for every vertex $v \in M$, implying that $F^* \subseteq F'$. This concludes the proof. □

In order to bound the size of any trivially perfect module by $O(k)$, we actually prove a more general reduction rule that will be useful for the rest of our kernelization algorithm. This rule operates on a more intricate structure, so-called *comb* [16], that induces a trivially perfect graph but not necessarily a module.

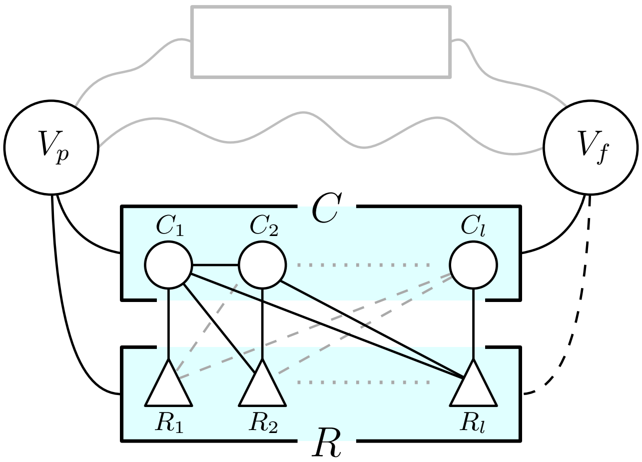

Figure 1: A comb of a graph $G = (V, E)$ with shaft C and teeth R. Each set C_i is a critical clique while each set R_i induces a (possibly disconnected) trivially perfect module, $1 \leqslant i \leqslant l$. Notice that the sets V_p and V_f might be adjacent to some other vertices of the graph.

Definition 5 (Comb [16])**.** *A pair (C, R) of disjoint subsets of V is a comb of G if:*

- *$G[C]$ is a clique that can be partitioned into l critical cliques $\{C_1, ..., C_l\}$*
- *R can be partitioned into l non-empty non-adjacent trivially perfect modules $\{R_1, ..., R_l\}$*
- *$N_G(C_i) \cap R = \bigcup_{j=i}^{l} R_j$ and $N_G(R_i) \cap C = \bigcup_{j=1}^{i} C_j$ for $1 \leqslant i \leqslant l$*
- *there exist two (possibly empty) subsets of vertices $V_f, V_p \subseteq V(G) \backslash \{C \cup R\}$ such that:*
  - *$\forall x \in C,\ N_G(x) \backslash (C \cup R) = V_p \cup V_f$ and*
  - *$\forall y \in R,\ N_G(y) \backslash (C \cup R) = V_p$.*

Given a comb (C, R), C is called the *shaft* of the comb and R the *teeth* of the comb. See Fig. 1 for an illustration of Definition 5. Recall that we assume that the graph is reduced under Rule 2, which means that $|C_i| \leqslant k+1$ for $1 \leqslant i \leqslant l$. Dumas *et al.* [16] showed the following proposition on the structure of combs.

Proposition 2 ([16])**.** *Given a comb (C, R) of G, the subgraph $G[C \cup R]$ is trivially perfect. Moreover the sets V_p and V_f, and the ordered partitions $(C_1, \dots, C_l)$ of C and $(R_1, \dots, R_l)$ of R are uniquely determined.*

In the following we assume that any comb (C, R) is given with the ordered partitions $(C_1, \dots, C_l)$ of C and $(R_1, \dots, R_l)$ of R. We note here that Definition 5 slightly differs from the one given in [16] where the set V_f was required to be non-empty for technical reasons. Dropping this constraint will ease the presentation of our reduction rules.

We now give several observations that will help understand Definition 5, in particular its relation to trivially perfect modules. Given a trivially perfect graph $G = (V, E)$ and its UCD $\mathcal{T}_G = (T, \mathcal{B})$, one can construct a comb (C, R) of G by simply taking a path P from a node v_1 of T to one of its descendent v_l. The shaft C are the vertices in bags of this path, the teeth R are the bags of subtrees rooted in the children (not on P) of any node on the path P. We can observe that in this case, V_p corresponds to vertices in the bags on the path from the parent of v_1 to the root of T and that V_f is empty.

In particular, the vertex set of any connected trivially perfect graph can be partitioned into a comb (C, R) by taking a path from the root of its UCD to one of its leaves. This means that when $V_p = V_f = \emptyset$, Definition 5 corresponds to a connected trivially perfect graph. Similarly, if only the set V_f is empty then Definition 5 corresponds to a *connected trivially perfect module* since $N_G(R) \setminus C = N_G(C) \setminus R = V_p$.

The following directly comes from the definition of a comb and is verified whether sets V_p and V_f are empty or not.

Observation 3. *The set of vertices C (resp. R) is a module of $G \setminus R$ (resp. $G \setminus C$).*

We will show that combs can be safely reduced to $O(k)$ vertices. We first focus on combs having a large shaft, which will allow us to reduce trivially perfect modules with small anti-matching to $O(k)$ vertices (Lemma 6). Then we turn our attention to combs with many vertices in the teeth to bound the size of every comb to $O(k)$ vertices (Lemma 8).

Combs with large shafts Dumas *et al.* [16] showed that the length of a comb (*i.e.* the number l of different critical cliques in the shaft) can be reduced linearly in k. However, as critical cliques contain $O(k)$ vertices by Rule 2, it only allowed the authors to bound the number of vertices in shafts of combs to $O(k^2)$. Rule 4 presented in this section keeps two sets $\mathcal{C}_a$ and $\mathcal{C}_b$ containing a linear number (in k) of vertices at the beginning and at the end of the shaft, allowing to bound its size linearly in k. The two sets $\mathcal{C}_a$ and $\mathcal{C}_b$ will be large enough to ensure the existence of two vertices that will be unaffected by a given k-editing of the graph. We will use such vertices to prove that there exists a k-editing of the graph that does not affect any vertex in the shaft lying between $\mathcal{C}_a$ and $\mathcal{C}_b$, implying the safeness of the rule.

Rule 4. *Let (C, R) be a comb of G such that there exist* disjoint $(2k+1)$-*packings $\mathcal{C}_a$ of $\{C_1, \dots, C_l\}$ and $\mathcal{C}_b$ of $\{C_l, C_{l-1}, \dots, C_1\}$. Remove $C' = C \setminus (\mathcal{C}_a \cup \mathcal{C}_b)$ from G.*

Lemma 4. *Rule 4 is safe.*

Proof. Let $G' = G \setminus C'$ be the graph obtained after application of Rule 4. Since G' is an induced subgraph of G and since trivially perfect graphs are hereditary, any k-editing of G is a k-editing of G'.

For the reverse direction, let F' be a k-editing of G' and $H' = G' \triangle F'$. We will construct a k-editing F^* of G. Let c_a and c_b be unaffected vertices in $\mathcal{C}_a$ and $\mathcal{C}_b$, respectively. Note that both sets contain at least $2k+1$ vertices and that F' affects at most $2k$ vertices, hence c_a and c_b are well-defined. Let C_a and C_b be the critical cliques of C containing c_a and c_b, $1 \leqslant a < b \leqslant l$. Moreover, let $C_\circ = C_{a+1} \cup \ldots \cup C_{b-1}$ and $R_\circ = R_a \cup \ldots \cup R_{b-1}$. Similarly, let $C_< = C_1 \cup \ldots \cup C_a$, $C_> = C_b \cup \ldots \cup C_l$ and $R_> = R_b \cup \ldots \cup R_l$ These sets are depicted Figure 2. Finally, let $G_\circ = G \setminus C_\circ$ and $H_\circ = H' \setminus C_\circ$. Notice in particular that $H_\circ$ is trivially perfect and that $C' \subseteq C_\circ$.

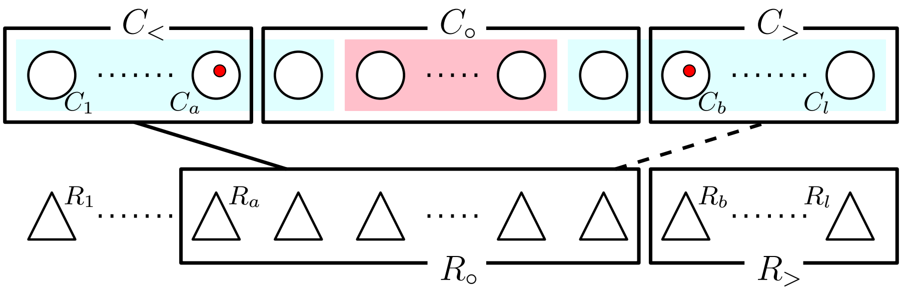

Figure 2: Illustration of the comb and the sets used in the proof of Lemma 4. The circles are critical cliques of the shaft and the triangles are teeth. The red vertices correspond to c_a and c_b, the light blue rectangles correspond to sets $\mathcal{C}_a$ and $\mathcal{C}_b$ and the light red rectangle corresponds to C', which is removed by Rule 4.

Let $F_\circ \subseteq F'$ be the k-editing such that $H_\circ = G_\circ \triangle F_\circ$ and $S_\circ$ be a maximal clique of $H_\circ$ containing $\{c_a, c_b\}$. Notice that since c_a and c_b are unaffected, $S_\circ$ is included in $N_{G_\circ}(\{c_a, c_b\}) = C \cup V_p \cup V_f \cup R_>$. We use Lemma 1 on $S_\circ$ to obtain a set of connected components $\{K_1, \ldots, K_r\}$ of $H_\circ \setminus S_\circ$ such that $\{K_1, \ldots, K_r\}$ are modules in $H_\circ$ whose (possibly empty) neighborhoods in $S_\circ$ form a nested family. We first modify $F_\circ$ to obtain a k-editing of $G_\circ$ where vertices of $R_\circ$ are affected uniformly.

Claim 1. *There exists a k-editing F^* of $G_\circ$ such that, in $H^* = G_\circ \triangle F^*$, $R_\circ$ is a module and $H^*[R_\circ] = G[R_\circ]$.*

Proof. We begin with several useful observations. First, $R_\circ$ is a module in $G_\circ$ since $R \supset R_\circ$ is a module in $G \setminus C$ (Observation 3) and vertices of $R_\circ$ are adjacent to $C_<$ and non adjacent to $C_>$. Next, since any component K_i is a module in $H_\circ$, $1 \leqslant i \leqslant r$, and since c_a and c_b are unaffected by $F_\circ$, we have $N_{H_\circ}(K_i) \cap \{c_a, c_b\} = N_{G_\circ}(K_i) \cap \{c_a, c_b\}$. In other words, vertices in a same component K_i must have the same adjacency with $\{c_a, c_b\}$ in $G_\circ$ and in $H_\circ$. Similarly, no vertex $v \in R_\circ$ belongs to $S_\circ$ since $N_{G_\circ}(v) \cap \{c_b\} = \emptyset$. Moreover, the only vertices of $G_\circ$ that are adjacent to c_a but not c_b are exactly those of $R_\circ$. Hence for any vertex $v_\circ \in R_\circ$ it holds that $N_{H_\circ}(v_\circ) \subseteq S_\circ \cup R_\circ$.

Assume now that $R_\circ$ is not a module in $H_\circ$ and let $v_\circ \in R_\circ$ be a vertex contained in the least number of pairs of $F_\circ$ with the other element in $S_\circ$. Consider the graph $\tilde{H} = H_\circ \setminus (R_\circ \setminus \{v_\circ\})$, which is trivially perfect by heredity. Since $N_{H_\circ}(v_\circ) \subseteq S_\circ \cup R_\circ$, it follows that $N_{\tilde{H}}(v_\circ) \subseteq S_\circ$ is a clique. Hence Proposition 1 implies that the graph $H^* = \tilde{H}(v_\circ \to G[R_\circ])$ is trivially perfect. Let F^* be the editing such that $H^* = G_\circ \triangle F^*$. By the choice of $v_\circ$ we have $|F^*| \leqslant |F_\circ|$. It follows that F^* is a desired k-editing, concluding the proof of Claim 1. ◇

We henceforth consider $H^* = G_\circ \triangle F^*$ where F^* is the k-editing from Claim 1. Note that the components around $S_\circ$ may be different in $H_\circ \setminus S_\circ$ and $H^* \setminus S_\circ$. In a slight abuse of notation, we still define these components by $\{K_1, \ldots, K_r\}$. Recall that $\{K_1, \ldots, K_r\}$ are modules in H^* whose (possibly empty) neighborhoods in $S_\circ$ form a nested family.

Claim 2. *The graph $H = G \triangle F^*$ is trivially perfect.*

Proof. The graph H corresponds to H^* where vertices of $C_\circ$ have been added with the same neighborhood as in G. Let us first observe that $S = S_\circ \cup C_\circ$ is a maximal clique in H. Indeed, $C_\circ$ is a clique by definition and $S_\circ \subseteq \big(C \cup V_p \cup V_f \cup R_>\big) \subseteq N_H(C_\circ) = N_G(C_\circ)$ (recall that C is adjacent to $V_p \cup V_f$ by Definition 5 and that vertices of $C_\circ$ are adjacent to every vertex of $R_>$). Hence components $\{K_1, \ldots, K_r\}$ defined in $H^* \setminus S_\circ$ are the same in $H \setminus S$ and their neighborhoods are nested in $S_\circ$. We split $\{K_1, \ldots, K_r\}$ into three types components w.r.t their adjacencies with $\{c_a, c_b\}$, namely:

1. α-components that are non-adjacent to both c_a and c_b
2. β-components that are adjacent to c_a but not c_b
3. δ-components that are adjacent to both c_a and c_b

In what follows we let K_α, K_β and K_δ denote any α-, β- and δ-component, respectively. Note that $N_{H^*}(K_\alpha) \subseteq N_{H^*}(K_\beta) \subseteq N_{H^*}(K_\delta) \subseteq S_\circ$ holds by construction. Recall that since c_a and c_b are unaffected by F^*, $N_G(K_i) \cap \{c_a, c_b\} = N_H(K_i) \cap \{c_a, c_b\}$ for any K_i. We claim that $\{N_H(K_i) \mid 1 \leqslant i \leqslant r\}$ is a nested family. Note that Lemma 1 will imply the result since S is a maximal clique in H. To sustain this claim, recall that the neighborhoods of vertices of $C_\circ$ are identical in G and H. Moreover, $N_H[c_a] \subseteq N_H[C_\circ] \subseteq N_H[c_b]$ holds as these vertices are unaffected by F^*. It follows that α-components (resp. δ-components) are non-adjacent (resp. adjacent) to every vertices of $C_\circ$ in H. This means in particular that the neighborhoods of both α- and δ-components are nested in S. Moreover we can observe that vertices of β-components are exactly the ones of $R_\circ$ since they are the only ones that are adjacent to c_a but not c_b in G. Hence, in H, we still have:

$$N_H(K_\alpha) \subseteq N_H(K_\beta) \subseteq N_H(K_\delta)$$

It remains to prove that the neighborhoods of β-components are nested in S. Let w.l.o.g. $\{K_1, \ldots, K_p\}$, $1 \leqslant p \leqslant r$ be the β-components. By definition of a comb, the β-components (which are also $R_\circ$) can be ordered w.r.t. the inclusion of their neighborhood in $G[C_\circ]$. We can assume w.l.o.g. that the ordering is $N_{G[C_\circ]}(K_1) \subseteq \ldots \subseteq N_{G[C_\circ]}(K_p)$. Moreover we can observe that for any β-component K_i we have $N_{G[C_\circ]}(K_i) = N_{H[C_\circ]}(K_i)$, $1 \leqslant i \leqslant p$. Since $R_\circ$ is a module in H^* by Claim 1 and since vertices of β-components are exactly those of $R_\circ$, it follows that the neighborhoods of β-components are nested. Hence $\{N_H(K_i) \mid 1 \leqslant i \leqslant r\}$ is a nested family and H is a trivially perfect graph by Lemma 1. ◇

By Claim 2 the graph $H = G \triangle F^*$ is trivially perfect and as $|F^*| \leqslant k$, it follows that F^* is a k-editing of G, concluding the proof of Lemma 4. □

Observation 4. *Assume that the instance (G, k) is reduced under Rules 2 and 4. For any comb (C, R) of G it holds that $|C| \leqslant 6k + 2$.*

Proof. Since G is reduced under Rule 2 every critical clique C_i of the shaft contains at most $k+1$ vertices, $1 \leqslant i \leqslant l$. By Observation 1, any $(2k+1)$-packing of $\{C_1, \ldots, C_l\}$ (resp. $\{C_l, \ldots, C_1\}$) contains at most $3k+1$ vertices. It follows that $|C| \leqslant 6k+2$ since otherwise one could find two *disjoint* $(2k+1)$-packings of $\{C_1, \ldots, C_l\}$ and of $\{C_l, \ldots, C_1\}$ and Rule 4 would apply. □

We are now ready to show how to reduce the size of any trivially perfect module. We need a combinatorial result that will be useful to obtain the claimed bound.

Lemma 5. *Let $G = (V, E)$ be a connected trivially perfect graph and α be the size of a maximum anti-matching of G. There exists a comb (C, R) of G such that $V = C \cup R$ and $|R| \leqslant 4\alpha$. Moreover, such a comb can be computed in polynomial time.*

Proof. We provide a constructive proof that will directly imply the last part of the result. Recall that any trivially perfect graph contains a universal vertex and let $U_1 \subseteq G$ be the universal clique of G. Let $R^1_1, \dots, R^1_{p_1}$ denote the connected components of $G \setminus U_1$. Since G does not contain any $(\alpha+1)$-sized anti-matching, there is at most one set R^1_i, $1 \leqslant i \leqslant p_1$ such that $|R^1_i| > \alpha$ (as there is no edge between R^1_i and R^1_j, $1 \leqslant i < j \leqslant p_1$).

Assume without loss of generality that $|R^1_1| > \alpha$. We add all vertices of $\cup_{i=2}^{p_1} R^1_i$ to some set $R_<$ and we will repeat this process on $G[R^1_1]$ until every connected component is smaller than α. More formally, at step $j > 1$, for the trivially perfect graph $G_j = G[R^{j-1}_1]$, let U_j be its universal clique and $R^j_1, \dots, R^j_{p_j}$ be the connected components of $G_j \setminus U_j$. Let R^j_1 be the one of size greater than α if it exists, if it does not, stop the process and let l be the last step. In particular, $|R^l_i| \leqslant \alpha$, $1 \leqslant i \leqslant p_l$. Let $R_< = \bigcup_{j=1}^{l-1} \bigcup_{i=2}^{p_j} R^j_i$ and $R_> = R^l_1 \cup \dots \cup R^l_{p_l}$.

Recall that $|R^{l-1}_1| > \alpha$ by construction. This implies that $|R_<| \leqslant \alpha$ since otherwise $G[R_< \cup R^{l-1}_1]$ would contain a $(\alpha+1)$-sized anti-matching. We claim that $|R_>| \leqslant 3\alpha$. To support this claim, let us consider the $(\alpha+1)$-packing $\{R^l_1, \dots, R^l_q\}$ of $\{R^l_1, \dots, R^l_{p_l}\}$ and let $R' = \bigcup_{i=1}^{q} R^l_i$ be its vertices. Let $R'' = R_> \setminus R'$. Recall that l is the last step of the process and $|R^l_i| \leqslant \alpha$ for $1 \leqslant i \leqslant p_l$. Hence by Observation 1 it holds that $|R'| \leqslant 2\alpha$. Thus, we have that $|R''| \leqslant \alpha$ since otherwise $G[R' \cup R'']$ would contain a $(\alpha+1)$-sized anti-matching, a contradiction. Hence $|R_>| = |R'| + |R''| \leqslant 3\alpha$.

To obtain a comb for G we consider the set $C = \{U_1, \dots, U_l\}$ as the shaft (recall that U_1 is the universal clique of G and that U_j denotes the universal clique of $G[R^{j-1}_1]$ at every step $1 < j \leqslant l$). Moreover, for every $1 \leqslant j < l$, the tooth R_j is defined as $\{R^j_2, \dots, R^j_{p_j}\}$, the last tooth R_l being $R_>$. By construction $(C, R = \bigcup_{j=1}^{l} R_j)$ is a comb of G such that $|R| = |R_<| + |R_>| \leqslant 4\alpha$. This concludes the proof. □

Lemma 6. *Assume that the instance (G, k) is reduced under Rules 1 to 4 and let M be a trivially perfect module of G. Then M contains at most $11k + 2$ vertices.*

Proof. Observe that if M contains an anti-matching of size more than k, then it is reduced under Rule 3 and contains $2k + 2$ vertices. Hence, suppose that M does not contain a $(k+1)$-sized anti-matching. Assume first that $G[M]$ is connected. Let (C, R) be a comb obtained through Lemma 5, such that $C \cup R = M$ and $|R| \leqslant 4k$. By Observation 4 we have that $|C| \leqslant 6k + 2$. It follows that $|M| \leqslant |C| + |R| \leqslant 10k + 2$.

To conclude it remains to deal with the case where $G[M]$ is disconnected. Let $\{M_1, \dots, M_p\}$ denote the connected components of $G[M]$. As M does not contain a $(k+1)$-sized anti-matching, at most one of its connected component has size greater than k, we may assume w.l.o.g. that it is M_1, if existent. Let $\mathcal{C}$ be the $(k+1)$-packing of $\{M_1, \dots, M_p\}$. As $|M_1| \leqslant 10k + 2$ and $|M_i| \leqslant k$ for $2 \leqslant i \leqslant p$, we have that $|\mathcal{C}| \leqslant 10k + 2$. Moreover, since M does not contain any $(k+1)$-sized anti-matching, $|M \setminus \mathcal{C}| \leqslant k$ and thus $|M| \leqslant 11k + 2$. This concludes the proof. □

3.3 Combs with large teeth

We now turn our attention to the case were a given comb contains many vertices in its teeth. The arguments are somewhat symmetric to the ones used in the proof of Lemma 4. The main difference lies in the fact that the information provided by unaffected vertices differ when they are contained in the teeth rather than in the shaft.

Rule 5. *Let (C, R) be a comb of G such that there exist three disjoint sets $\mathcal{R}_a, \mathcal{R}_b$ and $\mathcal{R}_c$ where:*

- *$\mathcal{R}_a$ is a $(2k+1)$-packing of $\{R_1, \ldots, R_l\}$,*
- *$\mathcal{R}_c = \{R_l, \ldots, R_q\}$ is a $(2k+1)$-packing of $\{R_l, \ldots, R_1\}$,*
- *$\mathcal{R}_b$ is a $(2k+1)$-packing of $\{R_{q-1}, \ldots, R_1\}$,*

Remove $R' = R \setminus (\mathcal{R}_a \cup \mathcal{R}_b \cup \mathcal{R}_c)$ from G.

Lemma 7. *Rule 5 is safe.*

Proof. Let $G' = G \setminus R'$ be the graph obtained after application of Rule 5. Since G' is an induced subgraph of G and since trivially perfect graphs are hereditary, any k-editing of G is a k-editing of G'.

For the reverse direction, let F' be a k-editing of G' and $H' = G' \triangle F'$. We will construct a k-editing F^* of G. Let r_a, r_b and r_c be unaffected vertices in $\mathcal{R}_a$, $\mathcal{R}_b$ and $\mathcal{R}_c$, respectively. Note that these vertices exist as these sets contain at least $2k+1$ vertices and F' affects at most $2k$ vertices. Let R_a, R_b and R_c, $1 \leqslant a < b < c \leqslant l$, be the teeth of R containing r_a, r_b and r_b, respectively (these sets are well-defined since the packings $\mathcal{R}_a$, $\mathcal{R}_b$ and $\mathcal{R}_c$ are disjoint). Moreover, since r_a, r_b and r_c are unaffected by F' their neighborhoods are equal in G' and H' and hence $\big(N_{H'}(r_a) \setminus R_a\big) \subseteq \big(N_{H'}(r_b) \setminus R_b\big) \subseteq \big(N_{H'}(r_c) \setminus R_c\big)$.

Claim 3. *The set $N_{H'}(r_b) \setminus R_b$ is a clique in H'.*

Proof. Assume for a contradiction that $N_{H'}(r_b) \setminus R_b$ contains a non-edge $\{u, v\}$. Recall that there is no edge between R_b and R_c. Hence, since $\big(N_{H'}(r_b) \setminus R_b\big) \subseteq \big(N_{H'}(r_c) \setminus R_c\big)$ we have that the set $\{r_b, u, v, r_c\}$ induces a C_4 in H', a contradiction. ◇

Let $R_\circ = R_{a+1} \cup \ldots \cup R_{b-1}$ and $C_\circ = C_{a+1} \cup \ldots \cup C_b$. Similarly, let $C_< = C_1 \cup \ldots \cup C_a$, $R_< = R_1 \cup \ldots \cup R_a$ and $R_> = R_b \cup \ldots \cup R_l$. Finally, let $G_\circ = G \setminus R_\circ$ and $H_\circ = H' \setminus R_\circ$. These sets are depicted Figure 3. Notice in particular that $H_\circ$ is trivially perfect and that $R' \subseteq R_\circ$. Let $F_\circ \subseteq F'$ be the k-editing such that $H_\circ = G_\circ \triangle F_\circ$. We first modify $F_\circ$ to obtain a k-editing of $G_\circ$ where every vertex of $C_\circ$ is affected uniformly.

Claim 4. *There exists a k-editing F^* of $G_\circ$ such that, in $H^* = G_\circ \triangle F^*$, $C_\circ$ is a clique module.*

Proof. Note that $C_\circ$ is a critical clique in $G_\circ$ since $C \supset C_\circ$ is a module in $G \setminus R$ (Observation 3) and vertices of $C_\circ$ are non-adjacent to vertices of $R_<$ and adjacent to vertices of $R_>$. Assume now that $C_\circ$ is not a clique module in $H_\circ$ and let $v_\circ \in C_\circ$ be a vertex contained in the least number of pairs of $F_\circ$. Consider the graph $H'_\circ = H_\circ \setminus (C_\circ \setminus \{v_\circ\})$, which is trivially perfect by heredity, and let H^* be the graph obtained from $H'_\circ$ by adding vertices of $C_\circ \setminus \{v_\circ\}$ as true twins of $v_\circ$. Let F^* be the editing such that $H^* = G_\circ \triangle F^*$. The graph H^* is trivially perfect as the class of trivially perfect graphs is closed under true twin addition. It follows from construction that $C_\circ$ is a clique module in H^* and by the choice of $v_\circ$, $|F^*| \leqslant |F_\circ|$. ◇

We henceforth consider $H^* = G_\circ \triangle F^*$ where F^* is the editing from Claim 4. We now show that vertices of $R_\circ$ can be added into H^* while ensuring it remains trivially perfect.

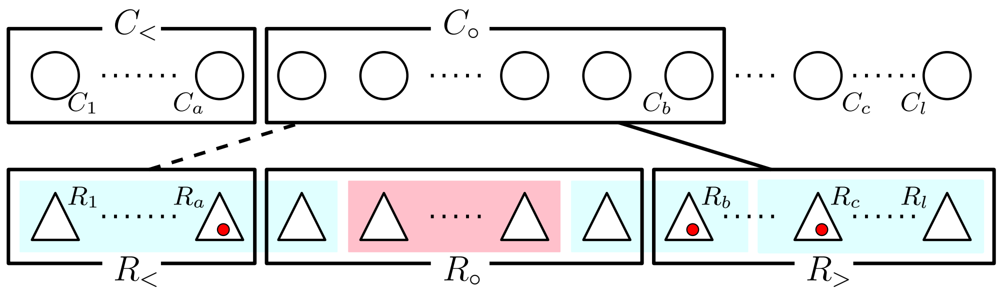

Figure 3: Illustration of the comb and the sets used in the proof of Lemma 7. The circles are critical cliques of the shaft and the triangles are teeth. The red vertices correspond to r_a, r_b and r_c, the light blue rectangles correspond to sets $\mathcal{R}_a$, $\mathcal{R}_b$ and $\mathcal{R}_c$ and the light red rectangle corresponds to R', which is removed by Rule 5.

Claim 5. *The graph $H = G \triangle F^*$ is trivially perfect.*

Proof. We start by removing the vertices of $R_b \setminus \{r_b\}$ from H^*, which will give us more control on the neighborhood of r_b and ease some arguments. Let $\tilde{H} = H^* \setminus (R_b \setminus \{r_b\})$, this graph is trivially perfect by heredity. Let S be a maximal clique of $\tilde{H}$ containing r_b. By Claim 3, $N_{\tilde{H}}(r_b)$ is a clique and since r_b is unaffected by F^* we have that $S = N_{\tilde{H}}[r_b] = C_< \cup C_\circ \cup V_p \cup \{r_b\}$. We use Lemma 1 on S to obtain a set of connected components $\{K_1, \ldots, K_r\}$ of $\tilde{H} \setminus S$ such that $\{K_1, \ldots, K_r\}$ are modules in $\tilde{H}$ whose (possibly empty) neighborhoods in S form a nested family. We further split components $\{K_1, \ldots, K_r\}$ into two types: K_i is an α-component if $N_{\tilde{H}}(K_i) \subseteq \big(N_{\tilde{H}}(r_a) \cap S\big)$ and a β-component otherwise, $1 \leqslant i \leqslant r$. Since $N_{\tilde{H}}(r_a) \cap S = V_p \cup C_<$ we have that, for any α-component K_α, $N_{\tilde{H}}(K_\alpha) \subseteq V_p \cup C_<$. Moreover, since $S = N_{\tilde{H}}[r_b]$ and since $C_\circ$ is a clique module in $\tilde{H}$ by Claim 4, every β-component K_β satisfies $N_{\tilde{H}}(K_\beta) = V_p \cup C_< \cup C_\circ = S \setminus \{r_b\}$.

Observe now that $(V_p \cup C_<) \subseteq N_G(R_\circ) \subseteq S \setminus \{r_b\}$. In other words, the neighborhood of any tooth of $R_\circ$ contains the neighborhood of any α-component and is contained in the neighborhood of any β-component. Moreover the neighborhoods of the teeth of $R_\circ$ are nested in G by definition of a comb. It follows that the vertices of $R_\circ$ can be safely added to $\tilde{H}$ with the same neighborhood as they have in G, ensuring that the resulting graph H_b is trivially perfect. It remains to add the vertices of R_b back into the graph. By Claim 3 and Proposition 1, the graph $H = H_b(r_b \to G[R_b])$ is trivially perfect. ◇

By Claim 5 the graph $H = G \triangle F^*$ is trivially perfect and as $|F^*| \leqslant k$, it follows that F^* is a k-editing of G, concluding the proof of Lemma 7. □

Lemma 8. *Assume that the instance (G, k) is reduced under Rules 1 to 5. Let (C, R) be a comb of G. Then $|C \cup R| = O(k)$.*

Proof. First, note that $|C| \leqslant 6k + 2$ thanks to Observation 4. We proceed in the same fashion to bound the size of R. As the teeth of a comb are trivially perfect modules, Lemma 6 implies that $|R_i| \leqslant 11k + 2$, $1 \leqslant i \leqslant l$. Hence by Observation 1 any $(2k+1)$-packing of $\{R_1, \ldots, R_l\}$ requires at most $13k + 2$ vertices. It follows that $|R| \leqslant 39k + 6$ since otherwise one could find three disjoint $(2k+1)$-packings of R that meet the requirements of Rule 5. Altogether we obtain that $|C \cup R| \leqslant 45k + 8$ which concludes the proof. □

3.4 Reducing the graph exhaustively

We conclude this section by showing that the graph can be reduced in polynomial time.

Lemma 9. *There is a polynomial time algorithm that outputs an instance $G' = (V', E')$ such that none of Rules 1 to 5 applies.*

Proof. First, Rules 1 and 2 can be applied in polynomial time thanks to Lemma 2. We now need to apply the other rules on trivially perfect modules and combs. For the modules, it is sufficient to reduce *strong modules*, which are modules that do not overlap with other modules. We can enumerate strong modules in linear time [34]. For each strong module M we can check in polynomial time if it is trivially perfect. We can moreover check if M contains a $(k+1)$-sized anti-matching by finding a maximum matching in the complement graph $\overline{G[M]}$, for instance using Edmonds' algorithm [17]. If M has a large anti-matching, then we can apply Rule 3. Otherwise, if $|M| \geqslant 11k+2$ then it can be reduced using Rule 4. Indeed, $G[M]$ contains in this case at most one connected component M' with more than k vertices, such that $|M \setminus M'| \leqslant k$ (since otherwise M would contain a $(k+1)$-sized anti-matching). We compute a comb (C, R) through Lemma 5 in $G[M']$, with $|R| \leqslant 4k$. It follows that $|C| > 6k + 2$ and Observation 4 implies that Rule 3 applies.

It remains to show that the combs not included in a trivially perfect module can be reduced in polynomial time. In order to do this Dumas *et al.* [16] showed that so-called *critical combs* can be enumerated in polynomial time, a critical comb being an inclusion-wise maximal comb where $V_f \neq \emptyset$ and $R \cup C \cup V_f$ does not induce a trivially perfect module. In particular, critical combs contain every comb not included in a trivially perfect module. Hence it is sufficient to only reduce these combs. Given a critical comb, Rules 4 and 5 can be applied in polynomial time. This concludes the proof. □

Combining Theorem 1 and Lemmas 6, 8 and 9 we obtain the main result of this work.

Theorem 2. Trivially Perfect Editing *admits a kernel with $O(k^2)$ vertices.*

4 The deletion variant

As mentioned in the introduction, a quadratic vertex-kernel is known to exist for Trivially Perfect Completion [1]. The results presented Section 3 can be adapted to prove that Trivially Perfect Deletion also admits a quadratic vertex-kernel by simply replacing any mention of "editing" by "deletion" (this also work with completion).

More precisely, one can see that in order to prove the safeness of Rules 3 to 5, the k-editing F^* for the original graph that is derived from a k-editing F' for the reduced instance only uses operations that were done by F'. In particular, if F' only contains non-edges then so does F^*, meaning that it is a valid solution. Together with the fact that Rules 1 and 2 are safe for the deletion variant, we obtain the following.

Theorem 3. Trivially Perfect Deletion *admit a kernel with $O(k^2)$ vertices.*

5 Conclusion

In this work we improved known kernelization algorithms for the Trivially Perfect Editing and Trivially Perfect Deletion problems, providing a quadratic vertex-kernel for both of them. This matches the best known bound for the completion variant [1]. Improving upon these bounds is an appealing challenge that may require a novel approach. On the other hand, it would be interesting to develop lower bounds for kernelization on such problems. Finally, even if the use of unaffected vertices in the design of reduction rules is common, its combination with the structural properties of trivially perfect graphs in terms of their maximal cliques allowed us to design stronger reduction rules. We hope that the approach presented in this work may lead to finding or improving kernelization algorithms for some related problems. Let us for instance mention the cubic vertex-kernel for Proper Interval Completion [3] and the quartic one for Ptolemaic Completion [10] that might be appropriate candidates.

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