Algebraic Graph Theory

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1 Overview of the Field

Algebraic graph theory comprises both the study of algebraic objects arising in connection with graphs, for example, automorphism groups of graphs along with the use of algebraic tools to establish interesting properties of combinatorial objects.

One of the oldest themes in the area is the investigation of the relation between properties of a graph and the spectrum of its adjacency matrix.

A central topic and important source of tools is the theory of association schemes. An *association scheme* is, roughly speaking, a collection of graphs on a common vertex set which fit together in a highly regular fashion. These arise regularly in connection with extremal structures: such structures often have an unexpected degree of regularity and, because of this, often give rise to an association scheme. This in turn leads to a semisimple commutative algebra and the representation theory of this algebra provides useful restrictions on the underlying combinatorial object. Thus in coding theory we look for codes that are as large as possible, since such codes are most effective in transmitting information over noisy channels. The theory of association schemes provides the most effective means for determining just how large is actually possible; this theory rests on Delsarte's thesis [4], which showed how to use schemes to translate the problem into a question that be solved by linear programming.

2 Recent Developments and Open Problems

Brouwer, Haemers and Cioabă have recently shown how information on the spectrum of a graph can be used to proved that certain classes of graphs must contain perfect matchings. Brouwer and others have also investigated the connectivity of strongly-regular and distance-regular graphs. This is an old question, but much remains to be done. Recently Brouwer and Koolen [2] proved that the vertex connectivity of a distance-regular graph is equal to its valency. Haemers and Van Dam have worked on extensively on the question of which graphs are characterized by the spectrum of their adjacency matrix. They consider both general graphs and special classes, such as distance-regular graphs. One very significant and unexpected outcome of this work was the construction, by Koolen and Van Dam [10], of a new family of distance-regular graphs with the same parameters as the Grassmann graphs. (The vertices of these graphs are the k-dimensional subspaces of a vector space of dimension v over the finite field $GF(q)$; two vertices are adjacent if their intersection has dimension k_1. The graphs are q-analog of the Johnson graphs, which play a role in design theory.) These graphs showed that the widely held belief that we knew all distance-regular graphs of "large diameter" was false, and they indicate that the classification of distance-regular graphs will be more complex (and more interesting?) than we expected.

Association schemes have long been applied to problems in extremal set theory and coding theory. In his (very) recent thesis, Vanhove [14] has demonstrated that they can also provide many interesting results in finite geometry.

Recent work by Schrijver and others [13] showed how schemes could used in combination with semidefinite programming to provide significant improvements to the best known bounds. However these methods are difficult to use, we do not yet have a feel for we might most usefully apply them and their underlying theory is imperfectly understood.

Work in Quantum Information theory is leading to a wide range of questions which can be successfully studied using ideas and tools from Algebraic Graph Theory. Methods from finite geometry provide the most effective means of constructing mutually unbiased bases, which play a role in quantum information theory and in certain cryptographic protocols. One important question is to determine the maximum size of a set of mutually unbiased bases in d-dimensional complex space. If d is a prime power the geometric methods just mentioned provide sets of size $d + 1$, which is the largest possible. But if d is twice an odd integer then in most cases no set larger than three has been found. Whether larger sets exist is an important open problem.

3 Presentation Highlights

The talks mostly fitted into one of four areas, which we discuss separately.

3.1 Spectra

Willem Haemers spoke on universal adjacency matrices with only two distinct eigenvalues. Such matrices are linear combinations of I, J, D and A (where D is the diagonal matrix of vertex degrees and A the usual adjacency matrix). Any matrix usually considered in spectral graph theory has this form, but Willem is considering these matrices in general. His talk focussed on the graphs for which some universal adjacency matrix has only two eigenvalues. With Omidi he has proved that such a graph must either be strong (its Seidel matrix has only two eigenvalues) or it has exactly two different vertex degrees and the subgraph induced by the vertices of a given degree must be regular.

Brouwer formulated a conjecture on the minimum size of a subset S of the vertices of a strongly-regular graph X such that no component of $X \setminus S$ was a single vertex. Cioabă spoke on his recent work with Jack Koolen on this conjecture. They proved that it is false, and there are four infinite families of counterexamples.

3.2 Physics

As noted above, algebraic graph theory has many applications and potential applications to problems in quantum computing, although the connection has become apparent only very recently. A number of talks were related to this connection.

One important problem in quantum computing is whether there is a quantum algorithm for the graph isomorphism problem that would be faster than the classical approaches. Currently the situation is quite open. Martin Roetteler's talk described recent work [1] on this problem. For our workshop's viewpoint, one surprising feature is that the work made use of the Bose-Mesner algebra of a related association scheme; this connection had not been made before. Severini discussed quantum applications of what is known as the *Lovász theta-function* of a graph. This function can be viewed as an eigenvalue bound and is closely related to both the LP bound of Delsarte and the Delsarte-Hoffman bound on the size of an independent set in a regular graph. Severini's work shows that Lovász's theta-function provides a bound on the capacity of a certain channel arising in quantum communication theory

Work in quantum information theory has lead to interest in complex Hadamard matrices — these are $d \times d$ complex matrices H such that all entries of H have the same absolute value and $HH^* = dI$. Both Chan and Szöllősi dealt with these in their talks.

Aidan Roy spoke on complex spherical designs. Real spherical designs were much studied by Seidel and his coworkers, because of their many applications in combinatorics and other areas. The complex case languished because there were no apparent applications, but now we have learnt that these manifest themselves in quantum information theory under acronyms such as MUBs and SIC-POVMs. Roy's talk focussed on a recent 45 page paper with Suda [12], where (among other things) they showed that extremal complex designs gave rise to association schemes. One feature of this work is that the matrices in their schemes are not symmetric, which is surprising because we have very few interesting examples of non-symmetric schemes that do not arise as conjugacy class schemes of finite groups.

3.3 Extremal Set Theory

Coherent configurations are a non-commutative extension of association schemes. They have played a significant role in work on the graph isomorphism problem but, in comparison with association schemes, they have provided much less information about interesting extremal structures. The work presented by Hobart and Williford may improve matters, since they have been able to extend and use some of the standard bounds from the theory of schemes.

Delsarte [4] showed how association schemes could be used to derive linear programs, whose values provided strong upper bounds on the size of codes. Association schemes have both a combinatorial structure and an algebraic structure and these two structures are in some sense dual to one another. In Delsarte's work, both the combinatorial and the algebraic structure had a natural linear ordering (the schemes are both metric and cometric) and this played an important role in his work. Martin explained how this linearity constraint could be relaxed. This work is important since it could lead to new bounds, and also provide a better understanding of duality.

One of Rick Wilson's many important contributions to combinatorics was his use of association schemes to prove a sharp form of the Erdős-Ko-Rado theorem [15]. The Erdős-Ko-Rado theorem itself ([5]) can certainly be called a seminal result, and by now there are many analogs and extensions of it which have been derived by a range of methods. More recently it has been realized that most of these extensions can be derived in a very natural way using the theory of association schemes. Karen Meagher presented recent joint work (with Godsil, and with Spiga, [8, 11]) on the case where the subsets in the Erdős-Ko-Rado theorem are replaced by permutations. It has long been known that there is an interesting association scheme on permutations, but this scheme is much less manageable than the schemes used by Delsarte and, prior to the work presented by Meagher, no useful combinatorial information had been obtained from it.

Chowdhury presented her recent work on a conjecture of Frankl and Füredi. This concerns families $\mathcal{F}$ of m-subsets of a set X such that any two distinct elements of have exactly λ elements in common. Frankl and Füredi conjectured that the m-sets in any such family contain at least $\binom{m}{2}$ pairs of elements of X. Chowdhury verified this conjecture in a number of cases; she used classical combinatorial techniques and it remains to see whether algebraic methods can yield any leverage in problems of this type.

3.4 Finite Geometry

Eric Moorhouse spoke on questions concerning automorphism groups of projective planes, focussing on connections between the finite and infinite case. Thus for a group acting on a finite plane, the number of orbits on points must be equal to the number of orbits on lines. It is not known if this must be true for planes of infinite order. Is there an infinite plane such that for each positive integer k, the automorphism group has only finitely many orbits on k-tuples? This question is open even for $k = 4$.

Simeon Ball considered the structure of subsets S of a k-dimensional vector space over a field of order q such that each d-subset of S is a basis. The canonical examples arise by adding a point at infinity to the point set of a rational normal curve. These sets arise in coding theory as maximum distance separable codes and in matroid theory, in the study of the representability of uniform matroids (to mention just two applications). It is conjectured that, if $k \leq q - 1$ then $|S| \leq q + 1$ unless q is even and $k = 3$ or $k = q - 1$, in which case $|S| \leq q + 2$. Simeon presented a proof of this theorem when q is a prime and commented on the general case. He developed a connection to Segre's classical characterization of conics in planes of odd order, as sets of $q + 1$ points such that no three are collinear.

There are many analogs between finite geometry and extremal set theory; questions about the geometry of subspaces can often be viewed as q-analogs of questions in extremal set theory. So the EKR-problem, which concerns characterizations of intersecting families of k-subsets of a fixed set, leads naturally to a study of intersecting families of k-subspaces of a finite vector space. In terms of association schemes this means we move from the Johnson scheme to the Grassmann scheme. This is fairly well understood, with the basic results obtained by Frankl and Wilson [6]. But in finite geometry, polar spaces form an important topic. Roughly speaking the object here is to study the families of subspaces that are isotropic relative to some form, for example the subspaces that lie on a smooth quadric. In group theoretic terms we are now dealing with symplectic, orthogonal and unitary groups. There are related association schemes on the isotropic subspaces of maximum dimension. Vanhove spoke on important work from his Ph. D. thesis, where he investigated the appropriate versions of the EKR problem in these schemes.

4 Outcome of the Meeting

It is too early to offer much in the way of concrete evidence of impact. Matt DeVos observed that a conjecture of Brouwer on the vertex connectivity of graphs in an association scheme was wrong, in a quite simple way. This indicates that the question is more complex than expected, and quite possibly more interesting. That this observation was made testifies to the scope of the meeting.

On a broader level, one of the successes of the meeting was the wide variety of seemingly disparate topics that were able to come together; the ideas of algebraic graph theory touch a number of things that would at first glance seem neither algebraic nor graph theoretical. There was a lively interaction between researchers from different domains.

The proportion of post-docs and graduate students was relatively high. This had a positive impact on the level of excitement and interaction at the meeting. The combination of expert and beginning researchers created a lively atmosphere for mathematical discussion.

References

[1] A. Ambainis, L. Magnin, M. Roetteler, J. Roland. Symmetry-assisted adversaries for quantum state generation, arXiv 1012.2112, 35 pp.

[2] A. E. Brouwer, J. H. Koolen. The vertex connectivity of a distance-regular graph. *European J. Combinatorics* 30 (2009), 668–673.

[3] A.E. Brouwer, D.M. Mesner. The connectivity of strongly regular graphs. *European J. Combinatorics*, 6 (1985), 215–216.

[4] P. Delsarte. *An algebraic approach to the association schemes of coding theory.* Philips Res. Rep. Suppl., (10):vi+97, 1973.

[5] P. Erdős, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12 (1961), 313–320.

[6] P. Frankl, R. M. Wilson. The Erdős-Ko-Rado theorem for vector spaces. *J. Combinatorial Theory, Series A*, 43 (1986), 228–236.

[7] D. Gijswijt, A. Schrijver, H. Tanaka. New upper bounds for nonbinary codes based on the Terwilliger algebra and semidefinite programming. *J. Combinatorial Theory, Series A*, 113 (2006), 1719–1731.

[8] C. D. Godsil, K. Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. arXiv 0710.2109, 18 pp.

[9] C. D. Godsil, G. F. Royle. *Algebraic Graph Theory*, Springer-Verlag, (New York), 2001.

[10] J. H. Koolen, E. R. van Dam. A new family of distance-regular graphs with unbounded diameter. *Inventiones Mathematicae*, 162 (2005), 189-193.

[11] K. Meagher, P. Spiga. An Erdos-Ko-Rado theorem for the derangement graph of PGL(2,q) acting on the projective line. arXiv 0910.3193, 17 pp.

[12] A. P. Roy, S. Suda. Complex spherical Codes and designs, (2011), arXiv 1104.4692, 45 pp.

[13] A. Schrijver. New code upper bounds from the Terwilliger algebra and semidefinite programming. *IEEE Transactions on Information Theory* 51 (2005), 2859–2866.

[14] F. Vanhove. *Incidence geometry from an algebraic graph theory point of view*. Ph. D. Thesis, Gent 2011.

[15] R. M. Wilson. The exact bound in the Erds-Ko-Rado theorem. *Combinatorica*, 4 (1984), 247–257.